CLASSIFICATION OF WEIL-PETERSSON ISOMETRIES

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Abstract. This paper contains two main results. The first is the existence of an equivariant Weil-Petersson geodesic in Teichmüller space for any choice of pseudo-Anosov mapping class. As a consequence one obtains a classification of the elements of the mapping class group as Weil-Petersson isometries which is parallel to the Thurston classification. The second result concerns the asymptotic behavior of these geodesics. It is shown that geodesics that are equivariant with respect to independent pseudo-Anosov's diverge. It follows that subgroups of the mapping class group which contain independent pseudo-Anosov's act in a reductive manner with respect to the Weil-Petersson geometry. This implies an existence theorem for equivariant harmonic maps to the metric completion.

1. Introduction. The Thurston classification of surface diffeomorphisms has strong analogies with the classification of isometries of complete nonpositively curved manifolds. If one regards mapping classes as isometries with respect to the Teichmüller metric then this point of view is made precise by Bers' theorem on the existence of Teichmüller geodesics that are equivariant with respect to irreducible, nonperiodic diffeomorphisms [B]. However, the Teichmüller metric is known not to have nonpositive curvature [M1], so the analogy is not quite germane.

The Weil-Petersson metric does have negative curvature, but it is not complete. Nevertheless, Wolpert showed that any two points can be joined by a Weil-Petersson geodesic [W4], and it then follows on general principles that the metric completion of Teichmüller space is a nonpositively curved length space in the sense of Alexandrov (an *NPC* or *CAT*(0) space). It is therefore natural and desirable to recover the classification of mapping classes from the perspective of Weil-Petersson isometries, and that is the first goal of this paper. Note that by a recent result of Masur and Wolf, the mapping class group essentially accounts for all Weil-Petersson isometries [MW].

Let $\Gamma_{g,n}$ denote the mapping class group of a closed, compact, oriented surface Σ_g of genus g with n marked points. Let $\mathbf{T}_{g,n}$ denote the Teichmüller space of Σ_g with its marked points, and equip $\mathbf{T}_{g,n}$ with the Weil-Petersson metric. Then $\Gamma_{g,n}$ acts isometrically and properly discontinuously on $\mathbf{T}_{g,n}$ with quotient $\mathbf{M}_{g,n}$, the Riemann moduli space of curves. We will denote the metric completion of $\mathbf{T}_{g,n}$ by

$\overline{\mathbf{T}}_{g,n}$. The extended action of $\Gamma_{g,n}$ on $\overline{\mathbf{T}}_{g,n}$ has quotient $\overline{\mathbf{M}}_{g,n}$, the Deligne-Mumford compactification of $\mathbf{M}_{g,n}$. The first main result of this paper is the following:

THEOREM 1.1. *If $\gamma \in \Gamma_{g,n}$ is pseudo-Anosov there is a unique γ-equivariant complete Weil-Petersson geodesic in $\mathbf{T}_{g,n}$.*

By analogy with hyperbolic geometry, we will refer to this geodesic as the **axis** of γ, and we will denote it by A_γ. The projection of A_γ to $\mathbf{M}_{g,n}$ is then a closed geodesic representative of the class of γ in the (orbifold) fundamental group of $\mathbf{M}_{g,n}$.

Let us briefly outline the proof of Theorem 1.1: the first step is a preliminary statement about geodesics in $\overline{\mathbf{T}}_{g,n}$. More precisely, we prove in Proposition 3.5 that a γ-equivariant geodesic in $\overline{\mathbf{T}}_{g,n}$ must actually lie in $\mathbf{T}_{g,n}$, provided γ is an irreducible mapping class. This result is a combination of two ideas. The first, Theorem 3.6, is that any geodesic from a point in $\mathbf{T}_{g,n}$ to the boundary $\partial \mathbf{T}_{g,n}$ intersects the boundary only at its endpoint. The second, Theorem 3.7, is that the different strata of the boundary intersect transversely. For example, a geodesic between different components of the top dimensional boundary stratum of $\overline{\mathbf{T}}_{g,n}$ must pass through the interior $\mathbf{T}_{g,n}$ (see also Remark 3.8). The details are given in Section 3.

We then consider a length minimizing sequence of equivariant paths $u_j \colon \mathbb{R} \to \mathbf{T}_{g,n}$ with uniform modulus of continuity, and let v_j denote the quotients maps to $\mathbf{M}_{g,n}$. After passing to a subsequence, the v_j converge to a limiting path $v_\infty \colon \mathbb{R} \to \overline{\mathbf{M}}_{g,n}$. This follows from the compactness of $\overline{\mathbf{M}}_{g,n}$. It is an important fact that the space $\overline{\mathbf{T}}_{g,n}$ is not even *locally* compact near points on the boundary $\partial \mathbf{T}_{g,n}$, so extracting a convergent subsequence directly in $\mathbf{T}_{g,n}$ (or $\overline{\mathbf{T}}_{g,n}$) is difficult. The second important step in the proof is to show that v_∞ is the quotient of an equivariant path $\tilde{v}_\infty$ to $\overline{\mathbf{T}}_{g,n}$. This is rather technical, and the proof occupies all of Section 4. The idea is to inductively lift v_∞ from the lower dimensional strata ε-close to the path u_j. The parameter ε is chosen sufficiently small with respect to the injectivity radius in a compact piece of the lower dimensional stratum so that continuity of the lift from stratum to stratum is preserved. The argument also shows that the lift is equivariant. The initial sequence can be taken to be piecewise geodesic, and the geodesic convexity of $\mathbf{T}_{g,n}$ is used repeatedly.

Theorem 1.1 leads to the desired classification via the **translation length** $\mathfrak{L}_{\mathrm{WP}}(\gamma)$ of a mapping class $\gamma \in \Gamma_{g,n}$, regarded as an isometry of $\mathbf{T}_{g,n}$ with respect to the Weil-Petersson metric (see (3.7)). We refer to [BGS] for a discussion of how this works for general NPC manifolds and to [Ab] for the case of the Teichmüller metric. As in those cases, **semisimple** isometries, i.e., those for which the translation length is attained at some point, play an important role. The four possibilities—semisimple or not, $\mathfrak{L}_{\mathrm{WP}}$ equal to zero or not—fit in conveniently with the Thurston classification of surface diffeomorphisms. According to Thurston, a mapping class is either **periodic**, **reducible**, or **pseudo-Anosov** (see Section 2 below). Those reducible diffeomorphisms that are periodic on the com-

Table 1. Classification of Weil-Petersson isometries.

	semisimple	not semisimple
$\mathfrak{L}_{\mathrm{WP}} = 0$	periodic	strictly pseudoperiodic
$\mathfrak{L}_{\mathrm{WP}} \neq 0$	pseudo-Anosov	reducible but not pseudoperiodic

ponent pieces are called **pseudoperiodic**. With this terminology, the classification may be summarized as in Table 1 (see Theorem 5.1 for the proof).

The second goal of this paper is to continue the study, initiated in [DKW] of harmonic maps to Teichmüller space. The equivariant harmonic map problem to complete nonpositively curved manifolds (and metric spaces) has been analyzed in great detail (cf. [DO, D1, C, La, JY, KS2]). Unlike the Dirichlet problem, or the problem of minimizing in a given homotopy class of maps between compact manifolds, some condition is generally required of an isometric action on the target manifold in order to guarantee the existence of energy minimizing equivariant maps. There are somewhat different notions in various papers; in the case of a symmetric space target, for example, the action should be *reductive*.

When it comes to Teichmüller space, the salient competing properties of the Weil-Petersson metric are its negative curvature and its noncompleteness. Taken together, these two facts are a reflection of important subtleties in the structure of the mapping class group. Nevertheless, in light of Theorem 1.1, and more generally, of the action of $\Gamma_{g,n}$ on the space of measured foliations, the condition that the action contain two independent pseudo-Anosovs (i.e., the image subgroup is **sufficiently large**, see Definition 2.1) emerges as a natural candidate for the analogue of the reductive hypothesis just mentioned (cf. [DKW]).

The argument justifying this point of view depends on an understanding of the asymptotic geometry of Weil-Petersson geodesics. More precisely, one needs to show that the axes for independent pseudo-Anosov's diverge, in an appropriate sense, with respect to the Weil-Petersson distance (see Theorem 6.2). This involves Thurston's compactification of $\mathbf{T}_{g,n}$ by projective measured foliations in a crucial way.

To state the second main result, we recall a key notion in the theory of equivariant harmonic maps (cf. [KS2, §2]): A finitely generated subgroup of $\Gamma_{g,n}$ is called **proper** if there is a set of generators $\gamma_1, \ldots, \gamma_k$ of the subgroup such that the sublevel sets of the displacement function

$$\delta(\sigma) = \max\{\mathrm{d}_{\mathrm{WP}}(\sigma, \gamma_i \sigma) : i = 1, \ldots, k\}$$

are bounded (see Definition 6.5). We apply the same notion to homomorphisms of groups into $\Gamma_{g,n}$ according to the image subgroup. In Section 6 we prove:

THEOREM 1.2. *For finitely generated subgroups of $\Gamma_{g,n}$, sufficiently large $\Longrightarrow$ proper.*

This result is important for the harmonic map problem. In the context of the metric completion, the Sobolev theory of finite energy maps from Riemannian

domains to $\overline{\mathbf{T}}_{g,n}$ has been developed by Korevaar and Schoen [KS1] (see also Jost [J]). The important result here is that energy minimizing (harmonic) maps for both the Dirichlet and equivariant problems are Lipschitz continuous. The following is an immediate consequence of Theorem 1.2 and [KS2, Theorem 2.1.3 and Remark 2.1.5]:

COROLLARY 1.3. *Let M be a finite volume complete Riemannian manifold with universal cover $\widetilde{M}$ and ρ: $\pi_1(M) \to \Gamma_{g,n}$ a homomorphism. Assume that $\pi_1(M)$ is finitely generated and that there exists a ρ-equivariant map $\widetilde{M} \to \overline{\mathbf{T}}_{g,n}$ with finite energy (i.e. as a map $M \to \overline{\mathbf{M}}_{g,n}$). Then if ρ is sufficiently large there exists a finite energy ρ-equivariant harmonic map u: $\widetilde{M} \to \overline{\mathbf{T}}_{g,n}$.*

The sufficiently large condition is flexible enough to provide a wide range of examples so that harmonic maps may be a useful tool in the study of homomorphisms into mapping class groups. For instance, by [McP, Theorem 4.6] any irreducible subgroup of $\Gamma_{g,n}$ that is not sufficiently large is either finite or virtually cyclic. It then follows from Corollary 1.3 that if M is a compact Riemannian manifold and ρ: $\pi_1(M) \to \Gamma_{g,n}$ is any homomorphism, there exists a ρ'-equivariant harmonic map $\widetilde{M} \to \overline{\mathbf{T}}_{g,n}$, where ρ' is the restriction of ρ to some finite index subgroup of $\pi_1(M)$. Possible applications along these lines are proofs of certain rigidity results for homomorphisms of lattices in Lie groups to mapping class groups and existence results for harmonic representatives of the classifying maps associated to symplectic Lefschetz pencils. In order to carry out this program, a regularity result for harmonic maps to $\overline{\mathbf{T}}_{g,n}$ is required. This issue will be considered in a future paper.

2. Mapping class groups and measured foliations. In this section, we review some basic facts about measured foliations and mapping class groups that will be required throughout the paper. A good reference for much of this material is [FLP]. As in the Introduction, Σ_g denotes a closed, compact, oriented surface of genus g. We will denote a choice of distinct points $\{p_1, \ldots, p_n\} \subset \Sigma_g$ by $\Sigma_{g,n}$. The **mapping class group** $\Gamma_{g,n}$ is the group of isotopy classes of orientation preserving homeomorphisms of $\Sigma_{g,n}$ that permute the points $\{p_1, \ldots, p_n\}$. The equivalence is through isotopies fixing the points. We will usually assume that $3g - 3 + n \geq 0$.

Let $\mathcal{C}(\Sigma_{g,n})$ denote the set of isotopy classes of simple closed essential nonperipheral curves on $\Sigma_{g,n}$. In addition, for each integer $k \geq 1$, we let $\mathcal{C}_k(\Sigma_{g,n})$ denote the set of collections $\mathbf{c}_{(k)} = \{c_1, \ldots, c_k\}$ of k distinct isotopy classes of simple closed curves in $\Sigma_{g,n}$ that are each essential and nonperipheral and which have representatives that are mutually disjoint. Given c and c' in $\mathcal{C}(\Sigma_{g,n})$, let $i(c, c')$ denote the geometric intersection number between them, and for $c \in \mathcal{C}(\Sigma_{g,n})$ and $\sigma \in \mathbf{T}_{g,n}$, let $\ell_c(\sigma)$ denote the length of a geodesic representative of c with respect to the complete hyperbolic metric on $\Sigma_{g,n}$ in the class σ.

Let $\mathcal{MF}$ denote the space of measured foliations on $\Sigma_{g,n}$ modulo isotopy and Whitehead equivalence. By an abuse of notation, we shall typically denote a measured foliation and its underlying foliation by the same letter, e.g. $F \in \mathcal{MF}$, assuming that the transverse measure on F is understood. We do not consider the "zero measure" to be an element of $\mathcal{MF}$. We can scale the measure on a foliation F by a real number $r > 0$ to obtain a new measured foliation, which we denote by rF. We say that F and F' are **projectively equivalent** if $F' = rF$ for some r. The projective equivalence class of a measured foliation F will be denoted by $[F]$, and the set of all projective equivalence classes will be denoted by $\mathcal{PMF}$.

For $F \in \mathcal{MF}$ and $c \in \mathcal{C}(\Sigma_{g,n})$, we have an intersection number $i(F, c)$, which by definition is the infimum of the total measures of representatives of c that are quasi-transverse to F. Unbounded sequences $\sigma_j \in \mathbf{T}_{g,n}$ have subsequences converging in $\mathcal{PMF}$ in the following sense: there are positive numbers $r_j \to 0$ and $F \in \mathcal{MF}$ such that for each $c \in \mathcal{C}(\Sigma_{g,n})$, $r_j\ell_c(\sigma_j) \to i(F, c)$. Any other sequence r_j' for which $r_j'\ell_c(\sigma_j)$ converges on all c (and not identically to zero) gives rise to a measured foliation that is projectively equivalent to F. We shall refer to this convergence as the **Thurston topology** and denote it by $\sigma_j \to [F]$. When we want to emphasize convergence in $\mathcal{MF}$, we write $r_j\sigma_j \to F$.

For any $c \in \mathcal{C}(\Sigma_{g,n})$ and $r > 0$, there is an associated $rc \in \mathcal{MF}$ such that $i(rc, c') = ri(c, c')$ for all $c' \in \mathcal{C}(\Sigma_{g,n})$. An important fact is that the intersection number for curves and between curves and measured foliations extends continuously to a nonnegative function $i(\cdot, \cdot)$ on $\mathcal{MF} \times \mathcal{MF}$ such that $i(rF, r'F') = rr'i(F, F')$. By analogy with curves, a pair $F, F' \in \mathcal{MF}$ is called **transverse** if $i(F, F') \neq 0$. A measured foliation F is called **minimal** if $i(F, c) > 0$ for every $c \in \mathcal{C}(\Sigma_{g,n})$. Let $\mathcal{F}_{\min} \subset \mathcal{MF}$ denote the subset of minimal foliations. Since the condition is independent of the projective factor, we also obtain a subset $\mathcal{PF}_{\min} \subset \mathcal{PMF}$ of projective minimal foliations. For any $[F] \in \mathcal{PMF}$, we let

(2.1) $Top[F] = \{[G] \in \mathcal{PMF} : i(F, G) = 0\}$

(2.2) $Gr[F] = \{[G] \in \mathcal{PMF} : i(G, c) = 0 \iff i(F, c) = 0 \text{ for all } c \in \mathcal{C}(\Sigma_{g,n})\}$

Note that these definitions are independent of the choice of representatives of the classes $[F]$ and $[G]$. The sets $Gr[F]$ give a countable partition of $\mathcal{PMF} \setminus \mathcal{PF}_{\min}$, while for $[F] \in \mathcal{PF}_{\min}$, the condition defining $Top[F]$ is an equivalence relation.

Indeed, the set *Top*[*F*] consists precisely of the projective equivalence classes of those measured foliations whose underlying foliations are topologically equivalent to that of *F*, but whose measures may be different.

Thurston's classification of surface diffeomorphisms may be described in terms of the natural action of $\Gamma_{g,n}$ on $\mathcal{MF}$ and $\mathcal{PMF}$: An element $\gamma \in \Gamma_{g,n}$ is called **reducible** if γ fixes some collection $\mathbf{c}_{(k)} \in \mathcal{C}_k(\Sigma_{g,n})$. It is called **pseudo-Anosov** if there is $r > 1$ and transverse measured foliations F_+, F_- on $\Sigma_{g,n}$ such that $\gamma F_+ \simeq r F_+$, and $\gamma F_- \simeq r^{-1} F_-$. F_+ and F_- are called the **stable** and **unstable** foliations of γ, respectively. The classification states that any $\gamma \in \Gamma_{g,n}$ is either periodic (i.e. finite order), infinite order and reducible, or pseudo-Anosov. Moreover, these are mutually exclusive possibilities.

The stable and unstable foliations of a pseudo-Anosov are both minimal and carry a unique transverse measure, up to projective equivalence. We will need the following facts:

(2.3) γ pseudo-Anosov and $[F] \notin \mathcal{PF}_{\min} \Longrightarrow [\gamma F] \notin Gr[F]$.

(2.4) γ pseudo-Anosov and $[F] \in \mathcal{PF}_{\min} \setminus \{[F_+], [F_-]\} \Longrightarrow [\gamma F] \notin Top[F]$.

Finally, a pseudo-Anosov element fixes *precisely* two points in $\mathcal{PMF}$; namely the points $[F_+]$ and $[F_-]$ represented by the stable and unstable foliations. We say that pseudo-Anosov's are **independent** if their fixed point sets in $\mathcal{PMF}$ do not coincide.

With this background we are now prepared to make the following important

Definition 2.1. ([McP, p. 142]) A subgroup of $\Gamma_{g,n}$ is **sufficiently large** if it contains two independent pseudo-Anosov's.

3. The Weil-Petersson metric.   Let $\mathbf{T}_{g,n}$ denote the Teichmüller space of $\Sigma_{g,n}$, and $\mathbf{M}_{g,n} = \mathbf{T}_{g,n}/\Gamma_{g,n}$ the Riemann moduli space of curves. The important metric on $\mathbf{T}_{g,n}$ for this paper is the Weil-Petersson metric, which we denote by d_{WP}. We will also have occasion to use the Teichmüller metric, which we denote by d_{T}. This is a complete Finsler metric with respect to which $\Gamma_{g,n}$ also acts by isometries.

Let us briefly recall the definitions: the cotangent space $T_\sigma^* \mathbf{T}_{g,n}$ is identified with the space of meromorphic quadratic differentials on $(\Sigma_{g,n}, \sigma)$ that have at most simple poles at the marked points $\{p_1, \ldots, p_n\}$. Here we abuse notation slightly, and regard the point $\sigma \in \mathbf{T}_{g,n}$ as giving a conformal structure on $\Sigma_{g,n}$ with its marked points. The complete hyperbolic metric on $(\Sigma_{g,n}, \sigma)$ can be expressed in local conformal coordinates as $ds^2 = \rho(z)|dz|^2$. Similarly, a quadratic differential has a local expression $\Phi = \varphi(z)dz^2$. Then for $\Phi \in T_\sigma^* \mathbf{T}_{g,n}$, the **Weil-Petersson**

cometric is given by

$$(3.1) \qquad \|\Phi\|_{WP}^2 = \int_\Sigma |\varphi(z)|^2 \rho(z)^{-1} \, dx \, dy.$$

The **Teichmüller cometric** is given by

$$(3.2) \qquad \|\Phi\|_T = \int_\Sigma |\varphi(z)| \, dx \, dy.$$

For a point $\sigma \in \mathbf{T}_{g,n}$ and a subset $A \subset \mathbf{T}_{g,n}$, we set

$$\mathrm{dist}_{WP}(\sigma, A) = \inf\{\mathrm{d}_{WP}(\sigma, \sigma') : \sigma' \in A\}.$$

We record the following important facts:

PROPOSITION 3.1. (cf. [Ah, W1, W3, W4, Chu, R, Tr])
   (1) *The Weil-Petersson metric is a noncomplete Kähler metric with negative sectional curvature.*
   (2) *Any two points in $\mathbf{T}_{g,n}$ may be joined by a unique Weil-Petersson geodesic in $\mathbf{T}_{g,n}$.*
   (3) *Any length function ℓ_c is strictly convex.*
   (4) *There is a strictly convex exhaustion function on $\mathbf{T}_{g,n}$.*

Let $\overline{\mathbf{T}}_{g,n}$ denote the metric completion of $\mathbf{T}_{g,n}$ with respect to the Weil-Petersson metric. Further notation: let $\partial \mathbf{T}_{g,n} = \overline{\mathbf{T}}_{g,n} \backslash \mathbf{T}_{g,n}$ and $\partial \mathbf{M}_{g,n} = \overline{\mathbf{M}}_{g,n} \backslash \mathbf{M}_{g,n}$. The completion has the following local description (cf. [M2]): $\partial \mathbf{T}_{g,n}$ is a disjoint union of smooth connected strata $D(\mathbf{c}_{(k)})$. These are described by choosing a collection $\mathbf{c}_{(k)} = \{c_1, \dots, c_k\}$ of disjoint simple closed essential and nonperipheral curves on $\Sigma_{g,n}$ and forming a nodal surface by collapsing each of the c_j to points. Associated to a nodal surface is another Teichmüller space which is by definition the set of equivalence classes of conformal structures on the normalized (possibly disconnected) surface, with the preimages of the nodes as additional marked points. It is therefore naturally isomorphic to a product of lower dimensional Teichmüller spaces. The statement is that the stratum $D(\mathbf{c}_{(k)})$ with its induced metric is isometric to this product of lower dimensional Teichmüller spaces with the product Weil-Petersson metric.

To describe this in more detail we define an incomplete metric space

$$(3.3) \qquad \mathbb{H} = \{(\theta, \xi) \in \mathbb{R}^2 : \xi > 0\}, \qquad ds_\mathbb{H}^2 = 4d\xi^2 + \xi^6 d\theta^2 \ .$$

Let $\overline{\mathbb{H}}$ denote the metric completion of $\mathbb{H}$, obtained by adding a single point $\partial\mathbb{H}$ corresponding to the entire real axis $\xi = 0$, and denote the distance function on $\overline{\mathbb{H}}$ by $d_{\overline{\mathbb{H}}}$. The completion is then an NPC space which is, however, not locally compact. We note the following simple:

LEMMA 3.2. *The geodesic* $w \colon [0,1] \to \overline{\mathbb{H}}$ *from* $\partial\mathbb{H}$ *to any point* $(\theta_1, \xi_1) \in \mathbb{H}$ *is given by* $w(x) = (\theta_1, x\xi_1)$.

The importance of $\overline{\mathbb{H}}$ is that it is a model for the normal space to the boundary strata. More precisely, let $\sigma \in D(\mathbf{c}_{(k)})$. Then the curves in $\mathbf{c}_{(k)}$ divide $\Sigma_{g,n}$ into a disjoint union of surfaces $\Sigma_{g_1,n_1}, \ldots, \Sigma_{g_N,n_N}$, where the boundary components corresponding to the curves are collapsed to additional marked points. Set

$$(3.4) \qquad\qquad \widehat{\mathbf{T}} = \mathbf{T}_{g_1,n_1} \times \cdots \times \mathbf{T}_{g_N,n_N} .$$

Then σ corresponds to a point in $\widehat{\mathbf{T}}$, which we will also denote by σ. There is an open neighborhood U of σ in $\overline{\mathbf{T}}_{g,n}$ that is homeomorphic to an open neighborhood in the product $\overline{\mathbb{H}}^k \times \widehat{\mathbf{T}}$, where σ is mapped to the point $(\partial\mathbb{H}, \ldots, \partial\mathbb{H}, \sigma)$. We will call such a U a **model neighborhood** of σ.

Next, we describe the asymptotic behavior of the Weil-Petersson metric near the boundary using a model neighborhood U. We choose complex coordinates $\tau_{k+1}, \ldots, \tau_{3g-3+n}$ for $\widehat{\mathbf{T}}$ near σ. Let (θ_i, ξ_i), $i = 1, \ldots, k$ denote the coordinates for each factor of $\overline{\mathbb{H}}$ in the model neighborhood. With respect to these coordinates, $U \cap D(\mathbf{c}_{(k)})$ is given by the equations $\xi_1 = \cdots = \xi_k = 0$. As $\xi = (\xi_1, \ldots, \xi_k) \to 0$, the Weil-Petersson metric has an expansion:

$$
\begin{aligned}
(3.5) \qquad ds^2_{\mathrm{WP}} = {} & \sum \left(G_{i\bar{j}} + \sum_\ell O\left(\xi_\ell^4\right) \right) d\tau_i \otimes d\bar{\tau}_j \\
& + \sum O\left(\xi_j^3\right) \times \left[d\tau_i \otimes d\xi_j \ \text{or} \ d\bar{\tau}_i \otimes d\xi_j \right] \\
& + \sum O\left(\xi_j^6\right) \times \left[d\tau_i \otimes d\theta_j \ \text{or} \ d\bar{\tau}_i \otimes d\theta_j \right] \\
& + \sum \left(B_i + \sum_\ell O\left(\xi_\ell^4\right) \right) 4d\xi_i^2 \\
& + \sum_{i \neq j} O\left(\xi_i^3 \xi_j^3\right) d\xi_i \otimes d\xi_j \\
& + \sum \left(B_i + \sum_\ell O\left(\xi_\ell^4\right) \right) \xi_i^6 \, d\theta_i^2 \\
& + \sum_{i \neq j} O(\xi_i^6 \xi_j^6) \, d\theta_i \otimes d\theta_j,
\end{aligned}
$$

where $G_{i\bar{j}}$ denotes the Weil-Petersson metric on $\widehat{\mathbf{T}}$, and the coefficients $B_i \neq 0$ are continuous functions of $\tau_{k+1}, \ldots, \tau_{3g-3+n}$ defined in a neighborhood (which we assume is the model neighborhood) of $\sigma \subset \widehat{\mathbf{T}}$. In particular, the induced metrics on the factors $\overline{\mathbb{H}}$ are asymptotic to metrics that are uniformly quasi-isometric to the model (3.3). This expansion is due to Yamada [Y1]. For the sake of completeness, we provide a brief justification in the Appendix below.

Remark 3.3. As noted in [MW, Remarks in §1.3], $\overline{\mathbf{T}}_{g,n}$ is a complete non-positively curved length space in the sense of Alexandrov, i.e. $\overline{\mathbf{T}}_{g,n}$ is an NPC (or equivalently, a CAT(0)) space with an isometric action of $\Gamma_{g,n}$ extending the action on $\mathbf{T}_{g,n}$ (cf. [BH, Corollary 3.11]). Moreover, the strata $D(\mathbf{c}_{(k)}) \subset \partial\mathbf{T}_{g,n}$ are totally geodesically embedded.

Remark 3.4. Since $\overline{\mathbf{T}}_{g,n}$ is a length space, between any two points σ_0, σ_1 we can find a rectifiable path u: $[0, 1] \to \overline{\mathbf{T}}_{g,n}$ with

$$(*) \quad \begin{cases} u(0) = \sigma_0 \\ u(1) = \sigma_1 \end{cases}$$

whose length $L(u)$ coincides with the distance $\mathrm{d}_{\mathrm{WP}}(\sigma_0, \sigma_1)$. We will often regard u as an energy minimizing path (in the sense of [KS1]) with respect to the boundary conditions $(*)$. We will denote the energy of such a path by $E(u)$.

Before proceeding, let us introduce some convenient notation regarding $\overline{\mathbf{T}}_{g,n}$: For a fixed $k \geq 1$, let $\dot{\Delta}_k$ denote the union of strata $D(\mathbf{c}_{(k)})$ for all possible collections $\mathbf{c}_{(k)} \in \mathcal{C}_k(\Sigma_{g,n})$, and let Δ_k denote its closure in the completion $\overline{\mathbf{T}}_{g,n}$. Then we have

(3.6) $$\partial\mathbf{T}_{g,n} = \bigcup_{k=1,\dots,k_{\max}} \dot{\Delta}_k, \quad \dot{\Delta}_k = \Delta_k \backslash \Delta_{k+1}.$$

Here, $k_{\max} = 3g - 3 + n$ is the maximal number of disjoint simple closed essential and nonperipheral curves on $\Sigma_{g,n}$. Note that with this notation, $\partial\mathbf{T}_{g,n} = \Delta_1$. It will also be useful to set $\dot{\Delta}_0 = \mathbf{T}_{g,n}$.

To state the main result of this section, we clarify some terminology that we have already used in the statement of Theorem 1.1: Let $\gamma \in \Gamma_{g,n}$ and u: $\mathbb{R} \to \mathbf{T}_{g,n}$. We will say that u is **equivariant** with respect to γ if $u(x+L) = \gamma u(x)$ for all $x \in \mathbb{R}$ and some fixed $L > 0$. By convention, the length of u will then be taken to mean the length of u restricted to $[0, L]$. Unless otherwise stated, we will parametrize so that $L = 1$. Also, for $\gamma \in \Gamma_{g,n}$, we define the **Weil-Petersson translation length** of γ by

(3.7) $$\mathfrak{L}_{\mathrm{WP}}(\gamma) = \inf\{\mathrm{d}_{\mathrm{WP}}(\sigma, \gamma\sigma) : \sigma \in \mathbf{T}_{g,n}\}.$$

The following characterization of Weil-Petersson geodesics in terms of the completion will be used in the next section:

PROPOSITION 3.5. *Suppose $\gamma \in \Gamma_{g,n}$ is irreducible. Let u: $\mathbb{R} \to \overline{\mathbf{T}}_{g,n}$ be a rectifiable γ-equivariant path, and suppose that $L(u) \leq \mathfrak{L}_{\mathrm{WP}}(\gamma)$. Then u is a Weil-Petersson geodesic and has image in $\mathbf{T}_{g,n}$.*

The remainder of this section is devoted to the proof of this result. It will follow from two properties of the geometry of the Weil-Petersson completion, Theo-

rems 3.6 and 3.7, that were first stated by Yamada [Y1]. The Harnack type estimate in [Y1] does not, however, appear to hold in the generality needed. The proofs given below will proceed along a different line of reasoning. Specifically, we use a rescaling argument. The main result is the statement that geodesics in $\overline{\mathbf{T}}_{g,n}$ between points in different boundary components must pass through the interior $\mathbf{T}_{g,n}$. We formulate this in Theorem 3.7 below. First we show that geodesics from points in Teichmüller space to the boundary touch the boundary only at their endpoints.

THEOREM 3.6. *Let* u: $[0,1] \rightarrow \overline{\mathbf{T}}_{g,n}$ *be a geodesic with* $u(0) = \sigma_0 \in \partial\mathbf{T}_{g,n}$ *and* $u(1) = \sigma_1 \in \mathbf{T}_{g,n}$. *Then* $u((0,1]) \subset \mathbf{T}_{g,n}$. *Similarly, if* $\sigma_0 \in \partial D(\mathbf{c}_{(k)})$ *and* $\sigma_1 \in D(\mathbf{c}_{(k)})$, *then* $u((0,1]) \subset D(\mathbf{c}_{(k)})$.

Proof. First consider the case where $\sigma_0 \in \partial\mathbf{T}_{g,n}$ and $\sigma_1 \in \mathbf{T}_{g,n}$. Let $\hat{x} = \sup\{x : u(x) \in \partial\mathbf{T}_{g,n}\}$. We wish to prove that $\hat{x} = 0$. Suppose not, and set $\sigma_{\hat{x}} = u(\hat{x})$. Then there is a collection $\mathbf{c}_{(k)} \in \mathcal{C}_k(\Sigma_{g,n})$ such that $\sigma_{\hat{x}} \in D(\mathbf{c}_{(k)})$. By changing the endpoints we may assume that the image of u is contained in a model neighborhood of $\sigma_{\hat{x}}$. $\square$

Claim 1. $u([0,\hat{x}]) \subset D(\mathbf{c}_{(k)})$.

Proof. Let $c \in \mathbf{c}_{(k)}$, and suppose that $\ell_c u(x) \neq 0$ for some $0 \leq x < \hat{x}$. We may find $\gamma \in \mathcal{C}(\Sigma_{g,n})$ such that $i(\gamma, c) \neq 0$, but $i(\gamma, c') = 0$ for any $c' \in \mathbf{c}_{(k)}$, $c' \neq c$. Let $\tilde{u}_j$: $[x, 1] \rightarrow \mathbf{T}_{g,n}$ be a sequence of geodesics converging in $\overline{\mathbf{T}}_{g,n}$ to $u|_{[x,1]}$. By Proposition 3.1 (3), $f_j = \ell_\gamma \tilde{u}_j$ is a sequence of convex functions on $[x, 1]$, and since $u(1) \in \mathbf{T}_{g,n}$ and $\ell_c u(x) \neq 0$, $f_j(1)$ and $f_j(x)$ are uniformly bounded. Hence, $f_j(\hat{x})$ is also uniformly bounded. But this contradicts the fact that $\ell_c \tilde{u}_j(\hat{x}) \rightarrow 0$ and $i(\gamma, c) \neq 0$. $\square$

By definition, $u((\hat{x}, 1]) \subset \mathbf{T}_{g,n}$. Thus, after reparametrizing, we have the following situation: u: $[-1, 1] \rightarrow \overline{\mathbb{H}}^k \times \widehat{\mathbf{T}}$ is energy minimizing with respect to its boundary conditions. Furthermore,

$$(3.8) \qquad u([-1, 0]) \subset (\partial\mathbb{H})^k \times \widehat{\mathbf{T}} , \text{ and } u((0, 1]) \subset \mathbb{H}^k \times \widehat{\mathbf{T}}.$$

We shall derive a contradiction to the existence of a geodesic satisfying (3.8). The strategy is to use a rescaling argument to compare Weil-Petersson geodesics to geodesics in $\overline{\mathbb{H}}^k \times \widehat{\mathbf{T}}$ – where we take the product of the model metric on $\overline{\mathbb{H}}^k$ and the Weil-Petersson metric on $\widehat{\mathbf{T}}$ – and thus derive a contradiction to Lemma 3.2. Without loss of generality, we assume that $B_i(\sigma_{\hat{x}}) = 1$, $i = 1, \ldots, k$, in the asymptotic expression (3.5).

Let u' denote the projection of u onto the first factor $\overline{\mathbb{H}}^k$. We also choose a sequence of rescalings $\varepsilon_j \downarrow 0$, and set $u_j(x) = u(\varepsilon_j x)$, $u'_j(x) = u'(\varepsilon_j x)$. Note that

each u_j' satisfies:

$$u_j'(x) = (\partial\mathbb{H})^k \qquad \text{for all } x \in [-1, 0].\tag{3.9}$$

Let w_j denote the map $[-1, 1] \to \overline{\mathbb{H}}^k$ with boundary conditions

$$w_j(-1) = u_j'(-1) = (\partial\mathbb{H})^k, \qquad w_j(1) = u_j'(1) \in \mathbb{H}^k,\tag{3.10}$$

which is harmonic with respect to the model metric (see (3.3) and Lemma 3.2). Note that since w_j is a nonconstant energy minimizer, $0 \neq E_{\overline{\mathbb{H}}^k}(w_j) \leq E_{\overline{\mathbb{H}}^k}(u_j')$ for all j. Here, $E_{\overline{\mathbb{H}}^k}(u)$ denotes the energy of a path in $\overline{\mathbb{H}}^k$ with respect to the product of the model metrics $ds_{\mathbb{H}}^2$.

Claim 2. There is a constant C independent of j such that

$$\max_{x\in[-1,1]} d_{\overline{\mathbb{H}}^k}(u_j'(x), (\partial\mathbb{H})^k) \leq C E_{\overline{\mathbb{H}}^k}^{1/2}(u_j'),$$

and

$$\max_{x\in[-1,1]} d_{\overline{\mathbb{H}}^k}(w_j(x), (\partial\mathbb{H})^k) = d_{\overline{\mathbb{H}}^k}(w_j(1), (\partial\mathbb{H})^k) \leq C E_{\overline{\mathbb{H}}^k}^{1/2}(u_j').$$

Proof. From the estimate (3.5) the Weil-Petersson metric on $\overline{\mathbb{H}}^k \times \widehat{\mathbf{T}}$ is quasi-isometric to the product of the model metric on $\overline{\mathbb{H}}^k$ and the Weil-Petersson metric on a lower dimensional Teichmüller space. Let $\mathcal{B} = (\partial\mathbb{H})^k \times \widehat{\mathbf{T}}$. Then the quasi-isometry implies that there is a constant C such that for all j and all $x \in [-1, 1]$,

$$C^{-1} \, d_{\mathrm{WP}}(u_j(x), \mathcal{B}) \leq d_{\overline{\mathbb{H}}^k}(u_j'(x), (\partial\mathbb{H})^k) \leq C \, d_{\mathrm{WP}}(u_j(x), \mathcal{B}).\tag{3.11}$$

Since $\mathcal{B}$ is convex, u_j is a geodesic, and $\overline{\mathbf{T}}_{g,n}$ is NPC, $x \mapsto d_{\mathrm{WP}}(u_j(x), \mathcal{B})$ is a convex function. Hence,

$$\max_{x\in[-1,1]} d_{\mathrm{WP}}(u_j(x), \mathcal{B}) = d_{\mathrm{WP}}(u_j(1), \mathcal{B}).\tag{3.12}$$

Combining (3.11) with (3.12), we have

$$\max_{x\in[-1,1]} d_{\overline{\mathbb{H}}^k}(u_j'(x), (\partial\mathbb{H})^k) \leq C d_{\overline{\mathbb{H}}^k}(u_j'(1), (\partial\mathbb{H})^k) \leq C L(u_j') \leq C E_{\overline{\mathbb{H}}^k}^{1/2}(u_j'),$$

which is the first inequality in the claim. (Here we followed the standard practice of using C to denote a generic constant whose value may vary from line to line.) The second inequality in the claim is immediate from the fact that w_j is a geodesic and $E_{\overline{\mathbb{H}}^k}(w_j) \leq E_{\overline{\mathbb{H}}^k}(u_j')$. $\qquad\square$

Claim 3. There are sequences μ_j and η_j such that $\mu_j \to 1$ and $\eta_j E_{\overline{\mathbb{H}}^k}^{-1}(u_j') \to 0$ as $j \to \infty$, and

$$0 \neq E_{\overline{\mathbb{H}}^k}(w_j) \leq E_{\overline{\mathbb{H}}^k}(u_j') \leq \mu_j E_{\overline{\mathbb{H}}^k}(w_j) + \eta_j.$$

Proof. Let u_j'' denote the projection of u_j onto the $\widehat{\mathbf{T}}$ factor of the product $\overline{\mathbb{H}}^k \times \widehat{\mathbf{T}}$. The Weil-Petersson energy may be bounded

$$(3.13) \quad E_{\mathcal{B}}(u_j'') + \delta_j^{-1} E_{\overline{\mathbb{H}}^k}(u_j') - E(u_j', u_j'') \leq E(u_j)$$
$$\leq E_{\mathcal{B}}(u_j'') + \delta_j E_{\overline{\mathbb{H}}^k}(u_j') + E(u_j', u_j''),$$

where $E(u_j', u_j'') \geq 0$ is a cross term reflecting the error of the Weil-Petersson metric from the product metric, and $E_{\mathcal{B}}(u_j'')$ denotes the energy of u_j'' with respect to the metric on $\mathcal{B} = (\partial \mathbb{H})^k \times \widehat{\mathbf{T}}$ induced from the Weil-Petersson metric by the inclusion in $\overline{\mathbf{T}}_{g,n}$. Such an estimate follows from (3.5). The constants $\delta_j \geq 1$ arise because the induced metric on $\overline{\mathbb{H}}^k$ is only quasi-isometric to the product of (3.3). Because the functions B_i in (3.5) corresponding to the curves $c_1, \ldots, c_k$ are continuous, and we have assumed $B_i(\sigma_{\hat{x}}) = 1$, we conclude that $\delta_j \to 1$ as $j \to \infty$. Consider the map $\hat{u}_j$ with coordinates (w_j, u_j''). This has the same boundary conditions as u_j. Using (3.13) applied to $\hat{u}_j$, we have

$$(3.14) \qquad E(\hat{u}_j) \leq E_{\mathcal{B}}(u_j'') + \delta_j E_{\overline{\mathbb{H}}^k}(w_j) + E(w_j, u_j'').$$

The left-hand side of (3.13) is:

$$(3.15) \qquad E_{\mathcal{B}}(u_j'') + \delta_j^{-1} E_{\overline{\mathbb{H}}^k}(u_j') - E(u_j', u_j'') \leq E(u_j).$$

On the other hand, since u_j is energy minimizing, $E(u_j) \leq E(\hat{u}_j)$. Using this fact and equations (3.14) and (3.15), we have:

$$(3.16) \qquad E_{\overline{\mathbb{H}}^k}(u_j') \leq \mu_j E_{\overline{\mathbb{H}}^k}(w_j) + \delta_j(E(w_j, u_j'') + E(u_j', u_j'')),$$

where $\mu_j = \delta_j^2 \to 1$.

By the estimate (3.5) the cross terms in the expansion of the Weil-Petersson metric near $\partial \mathbf{T}_{g,n}$ depend on terms involving $(u_j)_{\xi_i}^3$, $i = 1, \ldots, k$. Moreover, the uniform Lipschitz bound on u implies that $|(d/dx)(u_j')_{\xi_i}|$ and $(u_j')_{\xi_i}^3 |(d/dx)(u_j')_{\theta_i}|$ are bounded. Similarly, the norm of $(d/dx)u_j''$ with respect to the induced metric is also uniformly bounded. We then have an estimate:

$$E(u_j', u_j'') \leq C \left(\max_{x \in [-1,1]} d_{\overline{\mathbb{H}}^k}(u_j'(x), (\partial \mathbb{H})^k) \right)^3,$$
$$E(w_j, u_j'') \leq C \left(\max_{x \in [-1,1]} d_{\overline{\mathbb{H}}^k}(w_j(x), (\partial \mathbb{H})^k) \right)^3,$$

for some constant C independent of j. Applying Claim 2 we have:

$$E(u_j', u_j'') + E(w_j, u_j'') \leq C E_{\overline{\mathbb{H}}^k}^{3/2}(u_j').$$

Plugging this into (3.16) proves the claim.   □

We now rescale the metric on $\overline{\mathbb{H}}^k$ by $E_{\overline{\mathbb{H}}^k}^{-1/2}(u_j')$ to obtain metrics $d_{\overline{\mathbb{H}}^k}^{(j)}$. We shall denote the energies with respect to these new metrics by $E_{\overline{\mathbb{H}}^k}^{(j)}$. Note that by definition of this rescaling and Claim 3, we have:

(3.17) $$E_{\overline{\mathbb{H}}^k}^{(j)}(u_j') = 1 \quad \text{for all } j,$$

(3.18) $$\lim_{j \to \infty} E_{\overline{\mathbb{H}}^k}^{(j)}(w_j) = 1.$$

Claim 4. $\lim_{j \to \infty} d_{\overline{\mathbb{H}}^k}^{(j)}(u_j'(x), w_j(x)) = 0$, uniformly for $x \in [-1, 1]$.

Proof. Let $\hat{w}_j(x)$ be the midpoint along the geodesic from $u_j'(x)$ to $w_j(x)$ in $\overline{\mathbb{H}}^k$ (with the metric $d_{\overline{\mathbb{H}}^k}^{(j)}$). By [KS1, proof of Theorem 2.2] it follows that the map $x \mapsto \hat{w}_j(x)$ is an admissible competitor to w_j with the same boundary conditions. Moreover, we have [KS1, eq. (2.2iv)]:

$$2E_{\overline{\mathbb{H}}^k}^{(j)}(\hat{w}_j) \leq E_{\overline{\mathbb{H}}^k}^{(j)}(u_j') + E_{\overline{\mathbb{H}}^k}^{(j)}(w_j) - \frac{1}{2} \int_{-1}^{1} \left| \frac{d}{ds} d_{\overline{\mathbb{H}}^k}^{(j)}(u_j'(s), w_j(s)) \right|^2 ds.$$

By (3.17) and the fact that w_j is a minimizer:

$$2E_{\overline{\mathbb{H}}^k}^{(j)}(w_j) \leq 2E_{\overline{\mathbb{H}}^k}^{(j)}(\hat{w}_j) \leq 1 + E_{\overline{\mathbb{H}}^k}^{(j)}(w_j) - \frac{1}{2} \int_{-1}^{1} \left| \frac{d}{ds} d_{\overline{\mathbb{H}}^k}^{(j)}(u_j'(s), w_j(s)) \right|^2 ds$$

$$\frac{1}{2} \int_{-1}^{1} \left| \frac{d}{ds} d_{\overline{\mathbb{H}}^k}^{(j)}(u_j'(s), w_j(s)) \right|^2 ds \leq 1 - E_{\overline{\mathbb{H}}^k}^{(j)}(w_j).$$

By (3.18), it follows that

$$\lim_{j \to \infty} \int_{-1}^{1} \left| \frac{d}{ds} d_{\overline{\mathbb{H}}^k}^{(j)}(u_j'(s), w_j(s)) \right|^2 ds = 0.$$

By (3.10) we have for any $x \in [-1, 1]$,

$$d_{\overline{\mathbb{H}}^k}^{(j)}(u_j'(x), w_j(x)) \leq \int_{-1}^{x} \left| \frac{d}{ds} d_{\overline{\mathbb{H}}^k}^{(j)}(u_j'(s), w_j(s)) \right| ds$$

$$\leq \sqrt{2} \left\{ \int_{-1}^{1} \left| \frac{d}{ds} \, d_{\underline{\mathbb{H}}^k}^{(j)}(u_j'(s), w_j(s)) \right|^2 ds \right\}^{1/2}.$$

The claim now follows.   □

We continue with the proof of Theorem 3.6. First, note that since the w_j are constant speed geodesics, we may assume that the pull-back distance functions $w_j^* d_{\underline{\mathbb{H}}^k}^{(j)}$ converge to a nonzero multiple of the euclidean distance on $[-1, 1]$. In particular, there is a $c \neq 0$ such that for any $x \in [-1, 1]$,

$$d_{\underline{\mathbb{H}}^k}^{(j)}(w_j(x), w_j(-1)) \to c|x + 1|.$$

By Claim 4 it follows that

$$d_{\underline{\mathbb{H}}^k}^{(j)}(u_j'(x), w_j(-1)) \to c|x + 1|,$$

for all x. But by (3.9) and (3.10),

$$d_{\underline{\mathbb{H}}^k}^{(j)}(u_j'(x), w_j(-1)) = d_{\underline{\mathbb{H}}^k}^{(j)}(u_j'(x), (\partial \mathbb{H})^k) = 0,$$

for all $x \in [-1, 0]$ and all j. This contradiction rules out the existence of a geodesic satisfying (3.8), and therefore completes the proof of the Theorem for the first case. Since $D(\mathbf{c}_{(k)}) \hookrightarrow \overline{\mathbf{T}}_{g,n}$ is totally geodesic, the proof for the case where $\sigma_0 \in \partial D(\mathbf{c}_{(k)})$ and $\sigma_1 \in D(\mathbf{c}_{(k)})$ is identical.   □

Next we show that the different strata of the boundary intersect transversely:

THEOREM 3.7. *Let* u: $[0, 1] \to \overline{\mathbf{T}}_{g,n}$ *be a geodesic. Suppose* u *is contained in some* Δ_k, *but is not completely contained in* Δ_{k+1}. *Then* u *is contained in the closure of a single connected component of* $\dot{\Delta}_k$.

Remark 3.8. To clarify the statement of this result, let us give an example. Consider disjoint nonisotopic simple closed essential curves $\{c_1, c_2\}$ on a closed compact surface Σ_g. Let σ_1 denote a point in the boundary component $D(c_1)$ of $\overline{\mathbf{T}}_g$ corresponding to pinching c_1. Similarly, let σ_2 denote a point in the boundary component $D(c_2)$ of $\overline{\mathbf{T}}_g$ corresponding to pinching c_2. Since c_1 and c_2 are disjoint, the intersection of the closures $\overline{D}(c_1) \cap \overline{D}(c_2)$ is nonempty, and in fact contains $D(c_1, c_2)$. In particular, there is a path in $\overline{\mathbf{T}}_g$ from σ_1 to σ_2, lying completely in the boundary, which corresponds to first pinching c_2, and then "opening up" c_1. Theorem 3.7 states that this path has a "corner" at its intersection with $D(c_1, c_2)$, and is therefore not length minimizing. In fact, the geodesic from σ_1 to σ_2 intersects the boundary of $\overline{\mathbf{T}}_g$ only in its endpoints.

Proof of Theorem 3.7. The proof of this result may be modelled on that of Theorem 3.6 above. We outline the approach here. Assume, with the intent of arriving at a contradiction, that $u\colon [0,1] \to \Delta_k$ is a geodesic with $u(0) \in D(\mathbf{c}_{(k)})$, $u(1) \in D(\mathbf{c}'_{(k')})$, $\mathbf{c}'_{(k')} \in \mathcal{C}_{k'}(\Sigma_{g,n})$, with $u(1) \notin \overline{D}(\mathbf{c}_{(k)})$, and $k' \geq k$. By changing endpoints, we may assume u is contained in $\overline{D}(\mathbf{c}_{(k)}) \cup \overline{D}(\mathbf{c}'_{(k')})$. It then follows from Theorem 3.6 that there is some $\hat{x} \in (0,1)$ such that

$$u([0,\hat{x})) \subset D(\mathbf{c}_{(k)}), \ \ u((\hat{x},1]) \subset D(\mathbf{c}'_{(k')}), \ \ u(\hat{x}) \in D(\mathbf{c}_{(\ell)}) \subset \overline{D}(\mathbf{c}_{(k)}) \cap \overline{D}(\mathbf{c}'_{(k')}).$$

After renumbering, and by the assumption that $u(1) \notin \overline{D}(\mathbf{c}_{(k)})$, we may write

$$\mathbf{c}_{(k)} = \{c_1, \ldots, c_{k_1}, c_{k_1+1}, \ldots, c_k\}$$
$$\mathbf{c}'_{(k')} = \{c_1, \ldots, c_{k_1}, c'_{k_1+1}, \ldots, c'_{k'}\},$$

for some $0 \leq k_1 < k$, and $c_i \neq c'_j$ for any $i,j \geq k_1 + 1$. Then by the exact same argument as was used in Claim 1 above, it follows that

$$\mathbf{c}_{(\ell)} = \{c_1, \ldots, c_{k_1}, c_{k_1+1}, \ldots, c_k, c'_{k_1+1}, \ldots, c'_{k'}\}.$$

Thus, after reparametrization and forgetting about the first k_1 factors, we have produced a geodesic

$$u\colon [-1,1] \to \overline{\mathbb{H}}^{k-k_1} \times \overline{\mathbb{H}}^{k'-k_1} \times \widehat{\mathbf{T}}$$

satisfying:

$$
\begin{aligned}
(3.19) \qquad & u([-1,0)) \subset (\partial\mathbb{H})^{k-k_1} \times \mathbb{H}^{k'-k_1} \times \widehat{\mathbf{T}} \\
& u((0,1]) \subset \mathbb{H}^{k-k_1} \times (\partial\mathbb{H})^{k'-k_1} \times \widehat{\mathbf{T}} \\
& u(0) \in (\partial\mathbb{H})^{k-k_1} \times (\partial\mathbb{H})^{k'-k_1} \times \widehat{\mathbf{T}}
\end{aligned}
$$

Using a rescaling argument as in the proof of Theorem 3.6, we compare this geodesic to one in the model metric:

$$w\colon [-1,1] \to \overline{\mathbb{H}}^{k-k_1} \times \overline{\mathbb{H}}^{k'-k_1}$$
$$w(-1) \in (\partial\mathbb{H})^{k-k_1} \times \mathbb{H}^{k'-k_1}, \ w(1) \in \mathbb{H}^{k-k_1} \times (\partial\mathbb{H})^{k'-k_1}.$$

Any such geodesic with length bounded away from zero will keep $w(0)$ bounded away from $(\partial\mathbb{H})^{k-k_1} \times (\partial\mathbb{H})^{k'-k_1}$. As in the proof of Theorem 3.6, this leads to a contradiction to (3.19). $\qquad\square$

We are now prepared to complete the following:

Proof of Proposition 3.5. Since $\overline{\mathbf{T}}_{g,n}$ is a length space, u is a geodesic. If u has a point in $\mathbf{T}_{g,n}$, then by Proposition 3.1 (2), u is entirely contained in $\mathbf{T}_{g,n}$. Suppose u is entirely contained in $\partial\mathbf{T}_{g,n}$. Then there is some $k \geq 1$ such that u is contained in Δ_k but not Δ_{k+1}. Therefore, $u(x) \in D(\mathbf{c}_{(k)})$ for some x and some $\mathbf{c}_{(k)} \in \mathcal{C}_k(\Sigma_{g,n})$. By equivariance, it follows that $u(x + 1) = \gamma u(x) \in \dot{\Delta}_k$ as well. By Theorem 3.7, we must have that $\gamma u(x) \in D(\mathbf{c}_{(k)})$. But then γ fixes $\mathbf{c}_{(k)}$, contradicting the assumption of irreducibility. $\qquad\square$

4. Proof of Theorem 1.1.   Since γ is pseudo-Anosov it follows that $\mathcal{L}_{\mathrm{WP}}(\gamma)$ > 0. For if not we could find a sequence $\sigma_i \in \mathbf{T}_{g,n}$ such that $\mathrm{d}_{\mathrm{WP}}(\sigma_i, \gamma\sigma_i) \to 0$. By compactness we may assume that the corresponding sequence $[\sigma_i] \in \mathbf{M}_{g,n}$ converges to $[\sigma]$ in $\overline{\mathbf{M}}_{g,n}$. If $[\sigma] \in \mathbf{M}_{g,n}$, then by proper discontinuity of the action of $\Gamma_{g,n}$, γ is periodic, which is a contradiction. If $[\sigma] \in \partial\mathbf{M}_{g,n}$, then for i large, γ fixes the collection of short geodesics on σ_i. This implies that γ is reducible, which is again a contradiction.

Note that it suffices to prove the result for some power γ^k of the pseudo-Anosov: If A_{γ^k} is the unique γ^k-equivariant geodesic, then since γA_{γ^k} is also a geodesic left invariant by γ^k, we must have $\gamma A_{\gamma^k} = A_{\gamma^k}$, so that this is also a γ-equivariant geodesic. With this understood, we may assume without loss of generality that γ is in some finite index subgroup $\Gamma'_{g,n} \subset \Gamma_{g,n}$. We will choose $\Gamma'_{g,n}$ to consist of **pure** elements (cf. [Iv, p. 3 and Corollary 1.8]): these are mapping classes that are either pseudo-Anosov or reducible. Furthermore, the reducible pure mapping classes have representatives that fix a collection of simple closed curves pointwise, leave invariant the components of the complement, and on each component they are isotopic to either the identity or a pseudo-Anosov. This turns out to be a very useful construction, mainly for the following reason:

LEMMA 4.1. *Let $\Gamma'_{g,n}$ be a finite index subgroup of pure mapping classes. Let $\mathbf{M}'_{g,n}$, $\overline{\mathbf{M}}'_{g,n}$, and $[D(\mathbf{c}_{(k)})]'$ denote the quotients of $\mathbf{T}_{g,n}$, $\overline{\mathbf{T}}_{g,n}$, and $D(\mathbf{c}_{(k)})$ by $\Gamma'_{g,n}$. Then $\overline{\mathbf{M}}'_{g,n}$ is compact, $p': \mathbf{T}_{g,n} \to \mathbf{M}'_{g,n}$ is a covering space, and for each stratum $p': D(\mathbf{c}_{(k)}) \subset \partial\mathbf{T}_{g,n}$, the map $D(\mathbf{c}_{(k)}) \to [D(\mathbf{c}_{(k)})]'$ is also a covering.*

Proof. Compactness of $\overline{\mathbf{M}}'_{g,n}$ follows from the fact that $\overline{\mathbf{M}}_{g,n}$ is compact and $\Gamma'_{g,n}$ has finite index. Moreover, $\Gamma'_{g,n}$ acts properly discontinuously on $\mathbf{T}_{g,n}$ since $\Gamma_{g,n}$ does. The isotropy subgroups in $\Gamma'_{g,n}$ are trivial since $\Gamma'_{g,n}$ contains no periodic elements. Hence, $\mathbf{T}_{g,n} \to \mathbf{M}'_{g,n}$ is a covering space. The argument for the strata of $\partial\mathbf{T}_{g,n}$ is similar. $\qquad\square$

Choose a length minimizing sequence $u_j: \mathbb{R} \to \mathbf{T}_{g,n}$ of smooth γ-equivariant paths with uniform modulus of continuity. By convexity we have that $L(u_j) \to \mathcal{L}_{\mathrm{WP}}(\gamma)$. Let $v_j = p' \circ u_j$ denote the quotient map $[0, 1] \to \mathbf{M}'_{g,n}$. By Ascoli's theorem and the compactness of $\overline{\mathbf{M}}'_{g,n}$, we may assume that after passing to a subsequence the v_j converge uniformly to a Lipschitz map $v_\infty: [0, 1] \to \overline{\mathbf{M}}'_{g,n}$.

We again point out that there is no à priori reason for the existence of a convergent subsequence in $\overline{\mathbf{T}}_{g,n}$. For the sake of clarity, we will start by considering the case where v_∞ does not lie entirely in the boundary. Hence, there is a point x_0 such that $v_\infty(x_0) \in \mathbf{M}'_{g,n}$. Without loss of generality, we reparametrize so that $x_0 = 0$.

We now modify the sequence by replacing each u_j on $[0, 1]$ by the *geodesic* from $u_j(0)$ to $u_j(1) = \gamma u_j(0)$. We then extend this modified sequence equivariantly to $\mathbb{R}$. We shall continue to denote it by u_j and its projection to $\mathbf{M}'_{g,n}$ by v_j. Note that since the replacement decreases length, u_j is still a length minimizing sequence and still has a uniform modulus of continuity. We may assume, after again passing to a subsequence, that v_j converges uniformly to a Lipschitz map, again denoted by $v_\infty \colon [0,1] \to \overline{\mathbf{M}}'_{g,n}$ with $v_\infty(0) = [\sigma_0]' \in \mathbf{M}'_{g,n}$. By lower semicontinuity of lengths (cf. [BH, Proposition 1.20]), $L(v_\infty) \leq \mathfrak{L}_{\mathrm{WP}}(\gamma)$.

By Proposition 3.5, it then suffices to show that v_∞ is the projection of a γ-equivariant path $\tilde{v}_\infty \colon \mathbb{R} \to \overline{\mathbf{T}}_{g,n}$. To do this, set $S = v_\infty^{-1}(\partial \mathbf{M}'_{g,n})$. Since by Lemma 4.1, $\mathbf{T}_{g,n} \to \mathbf{M}'_{g,n}$ is a covering map, if S is empty the desired lift can be found. So the closed set $S \subset (0, 1)$ is the only possible obstruction to lifting, and we therefore assume it is nonempty. A posteriori, of course, the conclusion of Proposition 3.5 implies that this possibility does not, in fact, occur.

For simplicity, we will first also assume that the image of S is contained in the top dimensional stratum of $\partial \mathbf{M}'_{g,n}$, i.e. $v_\infty[0, 1] \subset \mathbf{M}'_{g,n} \cup [\dot{\Delta}_1]'$, where $[\dot{\Delta}_1]'$ denotes the quotient of $\dot{\Delta}_1 \subset \partial \mathbf{T}_{g,n}$ by $\Gamma'_{g,n}$.

Associated to the compact set $v_\infty(S) \subset [\dot{\Delta}_1]'$ we may choose $\kappa > 0$ such that:

$(*)$ The injectivity radius in $[\dot{\Delta}_1]'$ on each component of $v_\infty(S)$ is $\gg \kappa$;

$(**)$ Let $\widetilde{v_\infty(S)}$ denote the preimage of $v_\infty(S)$ in $\overline{\mathbf{T}}_{g,n}$. For any two nonisotopic simple closed curves $c, c' \in \mathcal{C}(\Sigma_{g,n})$, the distance in $\overline{\mathbf{T}}_{g,n}$ between $\widetilde{v_\infty(S)} \cap D(c)$ and $\widetilde{v_\infty(S)} \cap D(c')$ is $\gg \kappa$.

Both statements follow from the compactness of $v_\infty(S)$.

Recall that $[D(c)]'$ denotes the quotient of $D(c) \subset \partial \mathbf{T}_{g,n}$ by $\Gamma'_{g,n}$. In particular, $[\dot{\Delta}_1]'$ is the union of connected components $[D(c)]'$ over isotopy classes of curves $c \in \mathcal{C}(\Sigma_{g,n})$. Given $\delta > 0$, we define a neighborhood $U_{c,\delta}$ of $[D(c)]'$ by the following condition: $[\sigma]' \in U_{c,\delta}$ if $\ell_c(\sigma) < \delta$ for some representative σ of $[\sigma]'$. Also, let $U_\delta = \cup\{U_{c,\delta} : c \in \mathcal{C}(\Sigma_{g,n})\}$. We will also need the following:

LEMMA 4.2. *Given any $\varepsilon > 0$ there is a $\delta > 0$ such that if $[\sigma]' \in U_{c,\delta}$, then $[\sigma]'$ is within ε of $[D(c)]'$. Moreover, there is a constant $\varepsilon_1 > 0$, depending only on the choice of κ, with the following significance: if ε is sufficiently small compared to κ, if $[\sigma]' \in U_{c,\delta}$ is within ε_1 of $v_\infty(S) \cap [D(c)]'$, and if σ_1, σ_2 are any two lifts of $[\sigma]'$ to $\overline{\mathbf{T}}_{g,n}$, then either $d_{\mathrm{WP}}(\sigma_1, \sigma_2) \geq \kappa$, or $d_{\mathrm{WP}}(\sigma_1, \sigma_2) \leq 2\varepsilon$.*

Proof. The first statement follows from continuity of the length function with respect to the Weil-Petersson metric. Since $\Gamma'_{g,n}$ acts continuously and consists of pure mapping classes, we may choose ε_1 sufficiently small so that if σ is within

ε_1 of $\widetilde{v_\infty(S)} \cap D(c)$, and $\gamma \in \Gamma'_{g,n}$ satisfies $\gamma B_\kappa(\sigma) \cap B_\kappa(\sigma) \neq \emptyset$, then γ acts trivially on $D(c)$.

With this choice of ε_1, let $[\sigma]'$ satisfy the conditions of the lemma. Then by the first statement there is some $[\hat{\sigma}]' \in [D(c)]'$ such that $d_{WP}([\sigma]', [\hat{\sigma}]') \leq \varepsilon$. Given any lift σ_1 of $[\sigma]'$, then since $\varepsilon \ll \kappa$, $(*)$ guarantees that $[\hat{\sigma}]'$ has a *unique* lift $\hat{\sigma}$ in a κ-neighborhood of σ_1; moreover, $d_{WP}(\sigma_1, \hat{\sigma}) \leq \varepsilon$. As discussed above, if σ_2 is another lift of $[\sigma]'$ in this neighborhood, i.e. $\sigma_2 = \gamma \sigma_1$ for some $\gamma \in \Gamma'_{g,n}$, then γ acts trivially on $D(c)$. In particular, $\gamma \hat{\sigma} = \hat{\sigma}$. Hence, $d_{WP}(\sigma_2, \hat{\sigma}) \leq \varepsilon$ as well. The result follows. ∎

Claim 1. Let $\mathcal{O} = [0,1] \backslash S$. Then $\mathcal{O}$ is a **finite** union of relatively open intervals I_β.

Proof. If this were not the case, there would be a connected component $I_\beta = (x_\beta, y_\beta)$ of $\mathcal{O}$ with $v_\infty(I_\beta)$ arbitrarily short. This follows from the uniform modulus of continuity. We may assume that the length of $v_\infty(I_\beta)$ is so small that $v_\infty(I_\beta)$ is contained in a neighborhood of radius $\kappa/2$ of $v_\infty(S) \cap [D(c)]'$ for some c. For j large, it follows that $v_j(I_\beta)$ is contained in a κ-neighborhood of $v_\infty(S) \cap [D(c)]'$. By $(**)$ it follows that there exist simple closed curves c_j, related to c by elements of $\Gamma'_{g,n}$, such that $\ell_{c_j} u_j(x_\beta)$ and $\ell_{c_j} u_j(y_\beta)$ both tend to zero as $j \to \infty$. Since u_j is a geodesic on I_β, it follows by Proposition 3.1 (3) that $\ell_{c_j} u_j(z) \to 0$ for any $z \in I_\beta$. But then $v_\infty(I_\beta) \subset \partial \mathbf{M}'_{g,n}$, which is a contradiction. ∎

Claim 2. For $\delta > 0$ sufficiently small and every β, where $I_\beta = (x_\beta, y_\beta)$, there are curves c_{x_β} and c_{y_β} and points $x_\beta \leq w_\beta < z_\beta \leq y_\beta$ such that:
  (1) $v_\infty([x_\beta, w_\beta]) \subset \overline{U}_{c_{x_\beta}, \delta}$,
  (2) $v_\infty([z_\beta, y_\beta]) \subset \overline{U}_{c_{y_\beta}, \delta}$,
  (3) $v_\infty([w_\beta, z_\beta]) \subset \mathbf{M}'_{g,n} \backslash U_\delta$.
If I_β is of the form $[0, y_\beta)$ (resp. $(x_\beta, 1]$), then we have:
  (4) $v_\infty([z_\beta, y_\beta]) \subset \overline{U}_{c_{y_\beta}, \delta}$, (resp. $v_\infty([x_\beta, w_\beta]) \subset \overline{U}_{c_{x_\beta}, \delta}$),
  (5) $v_\infty([0, z_\beta]) \subset \mathbf{M}'_{g,n} \backslash U_\delta$ (resp. $v_\infty([w_\beta, 1]) \subset \mathbf{M}'_{g,n} \backslash U_\delta$.

Proof. Since $v_\infty(I_\beta) \subset \mathbf{M}'_{g,n}$, we may find a δ satisfying (3) by compactness. By choosing δ even smaller, and using the uniform modulus of continuity of v_∞, we may assume that $v_\infty(x_\beta, w_\beta)$ and $v_\infty(z_\beta, y_\beta)$ have very small length compared to κ and that $v_\infty(w_\beta)$ and $v_\infty(z_\beta)$ are on the boundary of U_δ. Thus, by the same argument as in the proof of Claim 1, there exist c_j, related to some c_{x_β} by elements of $\Gamma'_{g,n}$, such that $\ell_{c_j} u_j(x_\beta) \to 0$, and $\ell_{c_j} u_j(w_\beta) \to \delta$ as $j \to \infty$. By convexity, it follows that for any point $z \in (x_\beta, w_\beta)$, $\limsup_{j \to \infty} \ell_{c_j} u_j(z) \leq \delta$. It follows that $v_\infty(z) \in \overline{U}_{c_{x_\beta}, \delta}$. The other parts follow similarly. ∎

Claim 3. For $\varepsilon_2 > 0$ sufficiently small and j sufficiently large there is a lift $\tilde{v}_{\infty, \beta}$ of v_∞ on each I_β such that $\sup_{x \in I_\beta} d_{WP}(\tilde{v}_{\infty, \beta}(x), u_j(x)) \leq 2\varepsilon_2$. Moreover, if

$\delta > 0$ and x_β, y_β are the endpoints of I_β, we can arrange that $v_j(x_\beta)$ and $v_j(y_\beta)$ are in U_δ.

Proof. Fix ε_2 small relative to κ and ε_1. Choose δ small relative to ε_2 so that the conclusion of Lemma 4.2 (with ε_2 as ε) holds. We may choose δ possibly smaller so that the conclusion of Claim 2 holds. On the complement of U_δ there is a uniform lower bound on the injectivity radius of $\mathbf{M}'_{g,n}$. Choose $\varepsilon \leq \varepsilon_2$ small compared to this. For j sufficiently large v_j is within ε of v_∞ uniformly on $[0,1]$. Thus, for such a j, v_∞ lifts on I_β to a path $\tilde{v}_{\infty,\beta}$ that is within ε of u_j at some point, and hence every point in (w_β, z_β). Now by Lemma 4.2, any two lifts of v_∞ on (x_β, w_β) are either uniformly within $2\varepsilon_2$ of each other, or they are separated by a distance at least κ. Since we assume $\varepsilon \leq \varepsilon_2 \ll \kappa$, we see that the distance between the lift $\tilde{v}_{\infty,\beta}$ and u_j on (x_β, w_β) is at most $2\varepsilon_2$. The same reasoning applies to (z_β, y_β). Finally, since by Claim 1 there are only finitely many intervals I_β, we may choose j uniformly. In particular, we can guarantee the second statement. $\square$

Recall that $\mathcal{O}$ is the finite union of intervals I_β. For any $\varepsilon_2 > 0$ sufficiently small we have defined a lift $\tilde{v}_{\infty,\mathcal{O}} : \mathcal{O} \to \mathbf{T}_{g,n}$ of v_∞ such that $\sup_{x \in I_\beta} d_{\mathrm{WP}}(\tilde{v}_{\infty,\beta}(x), u_j(x)) \leq 2\varepsilon_2$, where j is chosen large according to ε_2. We first point out that:

Claim 4. $\tilde{v}_{\infty,\mathcal{O}}(1) = \gamma \tilde{v}_{\infty,\mathcal{O}}(0)$.

Proof. The two points are both lifts of $[\sigma_0]' \in \mathbf{M}'_{g,n}$, which lies in the complement of U_δ. By the equivariance of u_j and the construction in Claim 3, they are also within 2ε of each other. Since ε has been chosen much smaller than the injectivity radius of $\mathbf{M}'_{g,n}$ on the complement of U_δ, the points must agree. $\square$

By the uniform modulus of continuity, $\tilde{v}_{\infty,\mathcal{O}}$ extends to a map $\tilde{v}_{\infty,\overline{\mathcal{O}}} : \overline{\mathcal{O}} \to \overline{\mathbf{T}}_{g,n}$ which is still a lift of v_∞.

Claim 5. $\tilde{v}_{\infty,\overline{\mathcal{O}}}$ is continuous.

Proof. Consider intervals $I_\beta = (x_\beta, y_\beta)$ (or $[0, y_\beta)$) and $I_{\beta'} = (x_{\beta'}, y_{\beta'})$ (or $(x_{\beta'}, 1]$) with $y_\beta = x_{\beta'}$. Then $\tilde{v}_{\infty,\overline{\mathcal{O}}}(y_\beta)$ and $\tilde{v}_{\infty,\overline{\mathcal{O}}}(x_{\beta'})$ are two lifts of $v_\infty(y_\beta)$. Moreover, since both points are within $2\varepsilon_2$ of $u_j(y_\beta)$, it follows that they are within $4\varepsilon_2$ of each other. On the other hand, they are lifts of points in $v_\infty(S)$, and since ε_2 is small compared to κ, they must coincide by $(*)$. $\square$

Now for each interval $J = (a,b) \subset [0,1] \backslash \overline{\mathcal{O}}$, we have $v_\infty(J) \subset [D(c)]'$ for some c. Since $D(c) \to [D(c)]'$ is a covering, there is a unique lift $\tilde{v}_{\infty,J}$ of v_∞ on J such that $\lim_{x \downarrow a} \tilde{v}_{\infty,J}(x) = \tilde{v}_{\infty,\overline{\mathcal{O}}}(a)$.

Claim 6. $\sup_{x \in J} d_{\mathrm{WP}}(\tilde{v}_{\infty,J}(x), u_j(x)) \leq 2\varepsilon_2$.

Proof. By $(**)$, the fact that v_j is uniformly within $\varepsilon \ll \kappa$ of v_∞, and the second statement in Claim 3, it follows that there is some representative c_j of c

such that $\ell_{c_j} u_j(a)$ and $\ell_{c_j} u_j(b)$ are both less than δ. By convexity, this holds on the entire interval J. Hence, u_j is within ε_2 of $\widetilde{v_\infty(S)} \cap D(c_j)$. As in the proof of Lemma 4.2, a lift $\tilde{v}_{\infty,J}$ is either uniformly within ε_2 of u_j or uniformly a distance at least κ away. Since the latter case does not hold at the initial point a, the result follows. $\qquad\square$

Given Claim 6, it follows exactly as in Claim 5 that $\lim_{x \uparrow b} \tilde{v}_{\infty,I}(x) = \tilde{v}_{\infty,\overline{\mathcal{O}}}(b)$. Continuing this construction for each connected component J of $[0,1] \backslash \overline{\mathcal{O}}$, we finally end up with a lift $\tilde{v}_\infty$ of v_∞ on $[0,1]$. By Claim 4, we can extend $\tilde{v}_\infty$ to a continuous equivariant path $\tilde{v}_\infty \colon \mathbb{R} \to \overline{\mathbf{T}}_{g,n}$. By Claim 1 it follows that $L(\tilde{v}_\infty) = L(v_\infty) \leq \mathfrak{L}_{\mathrm{WP}}(\gamma)$. By Proposition 3.5, $\tilde{v}_\infty$ is a γ-equivariant geodesic in $\mathbf{T}_{g,n}$, and this completes the existence part of the proof of Theorem 1.1 in the case where $v_\infty([0,1]) \subset \mathbf{M}'_{g,n} \cup [\dot{\Delta}_1]'$.

The general case will follow from an inductive argument. First, note that the case where $v_\infty([0,1]) \subset [\dot{\Delta}_s]' \cup [\dot{\Delta}_{s+1}]'$, $s \geq 1$, follows exactly as above. We also point out that in the argument given above we may assume without loss of generality that $v_\infty(0)$ lies in $[\dot{\Delta}_s]'$. This is because the argument may be easily adapted to a minimizing sequence of once-broken geodesics rather than geodesics. Suppose now that

$$v_\infty([0,1]) \subset [\dot{\Delta}_s]' \cup [\dot{\Delta}_{s+1}]' \cup \cdots \cup [\dot{\Delta}_k]',$$

where $0 \leq s \leq k$. We induct on $k - s$. More precisely, we prove the following statement:

$(* * *)$ *Given a path* $v_\infty = \lim v_j$ *with* $v_j = p' \circ u_j \colon [0,1] \to \overline{\mathbf{M}}'_{g,n}$, *where* $u_j \colon [0,1] \to \overline{\mathbf{T}}_{g,n}$ *are geodesics with uniform modulus of continuity such that*

$$v_\infty([0,1]) \subset [\dot{\Delta}_s]' \cup [\dot{\Delta}_{s+1}]' \cup \cdots \cup [\dot{\Delta}_k]', \ v_\infty(0) \in [\dot{\Delta}_s]',$$

then for $\varepsilon > 0$ *sufficiently small there is a lift* $\tilde{v}_\infty$ *of* v_∞ *such that*

$$\max_{x \in [0,1]} \mathrm{d}_{\mathrm{WP}}(u_j(x), \tilde{v}_\infty(x)) < \varepsilon.$$

Furthermore, if the sequence satisfies $u_j(1) = \gamma u_j(0)$ *for some* $\gamma \in \Gamma_{g,n}$, *then the* $\tilde{v}_\infty$ *can be chosen so that* $\tilde{v}_\infty(1) = \gamma \tilde{v}_\infty(0)$.

The statement $(* * *)$ has already be proven in the case $k - s = 1$. Inductively assume the statement to be true for all integers $< k - s$, and let v_∞ be as in $(* * *)$. Let $S = v_\infty^{-1}[\dot{\Delta}_k]'$. If $S = \emptyset$ the statement holds by the inductive hypothesis. Thus we may assume that S is a nonempty closed subset of $(0,1)$. Next, associated to the compact set $v_\infty(S) \subset [\dot{\Delta}_k]'$ we can choose $\kappa > 0$ such that:

$(*_k)$ The injectivity radius in $[\dot{\Delta}_k]'$ on each component of $v_\infty(S)$ is $\gg \kappa$;

$(**_k)$ Let $\widetilde{v_\infty(S)}$ denote the preimage of $v_\infty(S)$ in $\overline{\mathbf{T}}_{g,n}$. For any two non-isotopic $\mathbf{c}_{(k)}, \mathbf{c}'_{(k)} \in \mathcal{C}_k(\Sigma_{g,n})$, the distance in $\overline{\mathbf{T}}_{g,n}$ between $\widetilde{v_\infty(S)} \cap D(\mathbf{c}_{(k)})$ and $\widetilde{v_\infty(S)} \cap D(\mathbf{c}'_{(k)})$ is $\gg \kappa$.

As before, both statements can be satisfied by the compactness of $v_\infty(S)$.

Given a collection $\mathbf{c}_{(k)} = \{c_1, \ldots, c_k\} \in \mathcal{C}_k(\Sigma_{g,n})$ we define a neighborhood $U_{\mathbf{c}_{(k)},\delta} \subset [\Delta_s]'$ of $[D(\mathbf{c}_{(k)})]'$ by the following condition: $[\sigma]' \in U_{\mathbf{c}_{(k)},\delta}$ if $\ell_{c_i}(\sigma) < \delta$, $i = 1, \ldots, k$, for some representative σ of $[\sigma]'$. Also, let $U_\delta = \cup\{U_{\mathbf{c}_{(k)},\delta} : \mathbf{c}_{(k)} \in \mathcal{C}_k(\Sigma_{g,n})\}$. Then a version of Lemma 4.2 holds in this situation as well, *mutatis mutandi*. Also, if we set $\mathcal{O} = [0,1]\backslash S$, then $\mathcal{O}$ is a finite union of intervals as in Claim 1. The proof of Claim 2 also goes through with no change. For convenience, we state the appropriate version:

Claim 2 (*k*). For $\delta > 0$ sufficiently small and every β, where $I_\beta = (x_\beta, y_\beta)$, there are curves $\mathbf{c}_{(k)_{x_\beta}}$ and $\mathbf{c}_{(k)_{y_\beta}}$ and points $x_\beta \leq w_\beta < z_\beta \leq y_\beta$ such that:

(1) $v_\infty([x_\beta, w_\beta]) \subset \overline{U}_{\mathbf{c}_{(k)_{x_\beta}},\delta}$

(2) $v_\infty([z_\beta, y_\beta]) \subset \overline{U}_{\mathbf{c}_{(k)_{y_\beta}},\delta}$

(3) $v_\infty([w_\beta, z_\beta]) \subset [\dot{\Delta}_s]' \cup [\dot{\Delta}_{s+1}]' \cup \cdots \cup [\dot{\Delta}_{k-1}]'\backslash U_\delta$

If I_β is of the form $[0, y_\beta)$ (resp. $(x_\beta, 1]$), then we have

(4) $v_\infty([z_\beta, y_\beta]) \subset \overline{U}_{\mathbf{c}_{(k)_{y_\beta}},\delta}$ (resp. $v_\infty([x_\beta, w_\beta]) \subset \overline{U}_{\mathbf{c}_{(k)_{x_\beta}},\delta}$)

(5) $v_\infty([0, z_\beta]) \subset \mathbf{M}'_{g,n}\backslash U_\delta$ (resp. $v_\infty([w_\beta, 1]) \subset [\dot{\Delta}_s]' \cup [\dot{\Delta}_{s+1}]' \cup \cdots \cup [\dot{\Delta}_{k-1}]'\backslash U_\delta$

Claim 3 (*k*). For $\varepsilon_2 > 0$ sufficiently small and j sufficiently large there is a lift $\tilde{v}_{\infty,\beta}$ of v_∞ on each I_β such that $\sup_{x \in I_\beta} d_{\mathrm{WP}}(\tilde{v}_{\infty,\beta}(x), u_j(x)) \leq 2\varepsilon_2$. Moreover, if $\delta > 0$ and x_β, y_β are the endpoints of I_β, we can arrange that $v_j(x_\beta)$ and $v_j(y_\beta)$ are in U_δ.

Proof. Again choose ε_2 and δ as in Claim 3. Then choose $\varepsilon \leq \varepsilon_2$ less than the injectivity radius of the strata $\dot{\Delta}_{k'}\backslash U_\delta$, $s \leq k' < k$. By the inductive hypothesis, if j is sufficiently large there is a lift $\tilde{v}_{\infty,\beta}$ of v_∞ on I_β that is within ε of u_j at every point in (w_β, z_β). Now by the analogue of Lemma 4.2, any two lifts of v_∞ on (x_β, w_β) or (z_β, y_β) are either uniformly within $2\varepsilon_2$ of each other, or they are separated by a distance at least κ. Since we assume $\varepsilon \leq \varepsilon_2 \ll \kappa$, we see that the distance between the lift $\tilde{v}_{\infty,\beta}$ and u_j on (x_β, w_β) is at most $2\varepsilon_2$. As before, since there are only finitely many intervals I_β, we may choose j uniformly, and the second statement follows. $\qquad\square$

In this way we have constructed a lift $\tilde{v}_{\infty,\mathcal{O}}: \mathcal{O} \to \overline{\mathbf{T}}_{g,n}$. Note that because of the choice of ε in Claim 3 (*k*), the exact same statement Claim 4 applies to $\tilde{v}_{\infty,\mathcal{O}}$ as well. Now the remaining parts of the argument, namely, Claims 5 and 6, follow exactly as before to give the desired lift $\tilde{v}_\infty$. This proves $(***)$ for

$k - s$, which by induction then holds in general. By Proposition 3.5, $\tilde{v}_\infty$ is a γ-equivariant geodesic in $\mathbf{T}_{g,n}$, and this completes the existence part of the proof of Theorem 1.1. Uniqueness follows from the geodesic convexity and negative curvature of $\mathbf{T}_{g,n}$.

5. Classification of Weil-Petersson isometries.

The existence of equivariant geodesics for pseudo-Anosov mapping classes allows for the precise classification of Weil-Petersson isometries in terms of translation length (3.7) that we have given in Table 1. First, let us clarify the terminology used there: $\gamma \in \Gamma_{g,n}$ is **pseudoperiodic** if it is either periodic, or it is reducible and periodic on the reduced components; it is **strictly pseudoperiodic** if it is pseudoperiodic but not periodic. Furthermore, we say that γ is **semisimple** if there is $\sigma \in \mathbf{T}_{g,n}$ such that $\mathfrak{L}_{\mathrm{WP}}(\gamma) = \mathrm{d}_{\mathrm{WP}}(\sigma, \gamma\sigma)$.

THEOREM 5.1. *An infinite order mapping class is semisimple if and only if it is irreducible. More precisely, any $\gamma \in \Gamma_{g,n}$ belongs to exactly one of the four classes characterized in Table 1.*

A slightly different picture emerges if one considers the action of $\Gamma_{g,n}$ on the completion $\overline{\mathbf{T}}_{g,n}$ of $\mathbf{T}_{g,n}$. There, every $\gamma \in \Gamma_{g,n}$ is semisimple, although the action of $\Gamma_{g,n}$ on $\overline{\mathbf{T}}_{g,n}$ is, of course, no longer properly discontinuous (indeed, according to [KL] it cannot be). It is worth mentioning that while we have defined pseudo-Anosov's in the usual way in terms of measured foliations, the only property that we use in the proofs of Theorems 1.1 and 5.1 is that these mapping classes have infinite order and are irreducible. In particular, the description given in Table 1 is independent of Thurston's classification provided we replace the entry "pseudo-Anosov" by "infinite irreducible."

To begin the proof, note that the following is a consequence of the proper discontinuity of the action of the mapping class group that we have already implicitly used in the proof of Lemma 4.2:

LEMMA 5.2. *Let $\sigma \in \overline{\mathbf{T}}_{g,n}$. Then there is a neighborhood U of σ such that if $\gamma \in \Gamma_{g,n}$ is such that $\gamma U \cap U \neq \emptyset$, then γ is pseudoperiodic.*

LEMMA 5.3. *Let $\gamma \in \Gamma_{g,n}$. Then $\mathfrak{L}_{\mathrm{WP}}(\gamma) = 0$ if and only if γ is pseudoperiodic.*

Proof. Clearly, we need only show that $\mathfrak{L}_{\mathrm{WP}}(\gamma) = 0$ implies pseudoperiodic, the other direction following from the fact that a pseudoperiodic element has a fixed point in $\overline{\mathbf{T}}_{g,n}$. Let $\sigma_j \in \mathbf{T}_{g,n}$ be a sequence such that $\mathrm{d}_{\mathrm{WP}}(\sigma_j, \gamma\sigma_j) \to 0$. We may assume that $[\sigma_j]$ converges to some $[\sigma] \in \overline{\mathbf{M}}_{g,n}$. Hence, we can find $h_j \in \Gamma_{g,n}$ such that $h_j\sigma_j$ converges to some lift $\sigma \in \overline{\mathbf{T}}_{g,n}$ of $[\sigma]$. Let U be a neighborhood of σ as in Lemma 5.2. For j large, $h_j\sigma_j$ and $h_j\gamma\sigma_j$ are in U. Then by the conclusion of the lemma, $h_j\gamma h_j^{-1}$ is pseudoperiodic; hence, so is γ. $\qquad\square$

LEMMA 5.4. *If there is a complete, nonconstant Weil-Petersson geodesic in $\mathbf{T}_{g,n}$ that is equivariant with respect to a mapping class $\gamma \in \Gamma_{g,n}$, then γ is pseudo-Anosov.*

Proof. By geodesic convexity and negative curvature, γ cannot be periodic. We claim that γ cannot be reducible either. To prove the claim, let $c \in \mathcal{C}(\Sigma_{g,n})$. By Wolpert's result Proposition 3.1 (3), if $u \colon \mathbb{R} \to \mathbf{T}_{g,n}$ is a Weil-Petersson geodesic, then $f(x) = \ell_c(u(x))$ is a strictly convex function. On the other hand, if $\gamma(c) = c$ up to isotopy, and if u is equivariant with respect to γ, then $f(x)$ would also be periodic, which is absurd. $\square$

Proof of Theorem 5.1. The first row of Table 1 follows from Lemma 5.3 and the fact that γ has a fixed point in $\mathbf{T}_{g,n}$ if and only if γ is periodic. If γ is pseudo-Anosov, then as a consequence of Theorem 1.1, $\mathfrak{L}_{\mathrm{WP}}(\gamma)$ is attained along an equivariant geodesic, so pseudo-Anosov's are semisimple. Conversely, suppose the infimum is attained at $\sigma \in \mathbf{T}_{g,n}$, but $\gamma\sigma \neq \sigma$. Then we argue as in Bers [B] (see also, [BGS, p. 81]): by Proposition 3.1 (2) there is a Weil-Petersson geodesic u from σ and $\gamma\sigma$. Similarly, there is a geodesic β from σ to $\gamma^2\sigma$. We claim that $\beta = u \cup \gamma u$. For if not, and if σ' denotes the point on u half the distance from σ to $\gamma\sigma$, then the distance from σ' to $\gamma\sigma'$ would be strictly less than $\mathrm{d}_{\mathrm{WP}}(\sigma, \gamma\sigma) = \mathfrak{L}_{\mathrm{WP}}(\gamma)$, which is a contradiction. In this way, we see that u may be extended to a complete γ-equivariant geodesic. Lemma 5.4 then implies that γ is pseudo-Anosov. A process of elimination fills in the remaining item of the table. $\square$

6. Proof of Theorem 1.2.   We begin this section by studying the asymptotic behavior of Weil-Petersson geodesics. The guiding idea is that the ideal boundary of $\mathbf{T}_{g,n}$ with respect to the Weil-Petersson metric should not be too different from the Thurston boundary of projective measured foliations. While we shall not attempt a full description here, the result we obtain in this direction nevertheless indicates a significant link between the Weil-Petersson geometry of the action of $\Gamma_{g,n}$ on $\mathbf{T}_{g,n}$ on the one hand, and the purely topological action of $\Gamma_{g,n}$ on $\mathcal{PMF}$ on the other.

Definition 6.1. Let $\alpha, \alpha' \colon \mathbb{R} \to \mathbf{T}_{g,n}$ be paths parametrized by arc length. We say that α and α' **diverge** if the function $(t,s) \mapsto \mathrm{d}_{\mathrm{WP}}(\alpha(t), \alpha'(s))$ is proper.

Our goal is to prove the following:

THEOREM 6.2. *Let A_γ and $A_{\gamma'}$ be the axes for independent pseudo-Anosov mapping classes γ and γ'. Then A_γ and $A_{\gamma'}$ diverge.*

As a first step, we have:

LEMMA 6.3. *Let A_γ be the axis of a pseudo-Anosov mapping class γ, and let $\{F_+, F_-\}$ denote the stable and unstable foliations of γ. If $A_\gamma(t_j) \to [F]$ in the Thurston topology for some sequence $t_j \to \infty$, then $[F] \in \{[F_+], [F_-]\}$.*

Proof. By definition, there is a fixed length $L = \mathfrak{L}_{\mathrm{WP}}(\gamma)$ such that $\mathrm{d}_{\mathrm{WP}}(A_\gamma(t_j), \gamma A_\gamma(t_j)) = L$ for all j. By equivariance and the parametrization by arc length,

$\gamma A_\gamma(t_j) = A_\gamma(t_j + L)$. Since the projection of A_γ to $\mathbf{M}_{g,n}$ is a closed curve, it lives in a fixed compact subset. Hence, along the path $A_\gamma(t)$ the Weil-Petersson and Teichmüller metrics are uniformly quasi-isometric. Thus, for some constant K, $d_T(A_\gamma(t_j), \gamma A_\gamma(t_j)) \le K$ for all j. For any simple closed curve c on $\Sigma_{g,n}$, it follows from [W2, Lemma 3.1] that

$$(6.1) \qquad e^{-2K} \ell_c(A_\gamma(t_j)) \le \ell_c(\gamma A_\gamma(t_j)) \le e^{2K} \ell_c(A_\gamma(t_j))$$

for all j. The result now follows easily: recall from Section 2 the notion of a minimal foliation and the associated definitions (2.1) and (2.2). If $[F] \notin \mathcal{F}_{\min}$, then it is easily deduced from (6.1) that $i(F, c) = 0 \iff i(\gamma F, c) = 0$. Hence, $[\gamma F] \in Gr[F]$, contradicting (2.3). On the other hand, if $[F] \in \mathcal{F}_{\min}$, then choose a sequence $c_j \in \mathcal{C}(\Sigma_{g,n})$ such that $\ell_{c_j}(A_\gamma(t_j))$ is uniformly bounded for all j. We may assume there are numbers $r_j \le 1$ and $G \in \mathcal{MF}$ so that $r_j c_j \to G$ in the Thurston topology. It then follows as in [DKW, Lemma 2.1] that $i(F, G) = 0$. By (6.1), we also have that $\ell_{c_j}(\gamma A_\gamma(t_j))$ is uniformly bounded, so that by the same argument $i(\gamma F, G) = 0$. Since we assume F is minimal, this implies $[\gamma F] \in Top[F]$. Hence, by (2.4), $[F] \in \{[F_+], [F_-]\}$. □

LEMMA 6.4. *Let σ_j, σ_j' converge in the Thurston topology to projective measured foliations $[F]$ and $[F']$, respectively. Assume that F is minimal and that $[\sigma_j]$ and $[\sigma_j']$ lie in a fixed compact subset of $\mathbf{M}_{g,n}$. Also, we suppose that there is a fixed $D \ge 0$ such that $d_{WP}(\sigma_j, \sigma_j') \le D$ for all j. Then $[F'] \in Top[F]$.*

Proof. Let u_j denote the Weil-Petersson geodesic in $\mathbf{T}_{g,n}$ from σ_j to σ_j', and let v_j denote its projection to $\mathbf{M}_{g,n}$. As in Section 4, we may assume (after passing to a subsequence) that the v_j converge to a piecewise geodesic $v_\infty \colon [0, 1] \to \overline{\mathbf{M}}_{g,n}$. More precisely, there are values $0 = x_0 < x_1 \le x_2 \le \cdots \le x_{N-1} < x_N = 1$ with the following properties: (1) $v_\infty|_{(x_i, x_{i+1})}$ is either a geodesic in $\mathbf{M}_{g,n}$ or is entirely contained in $\partial \mathbf{M}_{g,n}$; and (2) if $v_\infty|_{[x_i, x_{i+1}]} \subset \partial \mathbf{M}_{g,n}$ then for j sufficiently large there is a sequence $c_j^{(i)}$ of simple closed curves on $\Sigma_{g,n}$ whose lengths with respect to both $u_j(x_i)$ and $u_j(x_{i+1})$ converge to zero as $j \to \infty$. We now proceed inductively: For any $x \in [0, x_1)$ and any sequence such that $u_j(x) \to [F_x]$ in the Thurston topology, we claim that F_x is minimal and $i(F, F_x) = 0$. For since $v_\infty|_{[0,x]}$ is contained in a compact subset of $\mathbf{M}_{g,n}$, on which the Weil-Petersson and Teichmüller metrics are therefore uniformly quasi-isometric, the claim follows exactly as in the proof of Lemma 6.3 above. Now choose $x < x_1$ and $x_2 < y$ sufficiently close so that the lengths of $c_j^{(1)}$ with respect to both $u_j(x)$ and $u_j(y)$ are bounded. If we choose $r_j \le 1$ such that $r_j c_j \to G$, then as in the proof of Lemma 6.3 above, $i(F, G) = 0$. Similarly, if $u_j(y) \to [F_y]$, then $i(G, F_y) = 0$, and so $F_y \in Top[F]$. Continuing in this way, we see that $F' \in Top[F]$. □

Proof of Theorem 6.2. Suppose A_γ and $A_{\gamma'}$ do not diverge. Then there is a constant $D \ge 0$ and an unbounded sequence (t_j, s_j) such that $d_{WP}(A_\gamma(t_j), A_{\gamma'}(s_j)) \le$

D. Necessarily, t_j and s_j are both unbounded, so we may assume that $t_j, s_j \to \infty$. Let $\{F_+, F_-\}$ and $\{F'_+, F'_-\}$ denote the stable and unstable foliations of γ and γ', respectively. Since γ and γ' are independent, all four of these foliations are mutually topologically distinct (cf. [McP, Corollary 2.6]). After perhaps passing to a subsequence, we may assume that there are measured foliations F and F' such that $A_\gamma(t_j) \to [F]$ and $A_{\gamma'}(s_j) \to [F']$ in the Thurston topology. By Lemma 6.3, $[F] \in \{[F_+], [F_-]\}$ and $[F'] \in \{[F'_+], [F'_-]\}$. Let $\sigma_j = A_\gamma(t_j)$ and $\sigma'_j = A_{\gamma'}(s_j)$. Then since $\mathrm{d}_{\mathrm{WP}}(\sigma_j, \sigma'_j) \leq D$, and $[\sigma_j], [\sigma'_j]$ lie on the closed geodesics $[A_\gamma]$ and $[A_{\gamma'}]$, and the stable and unstable foliations of a pseudo-Anosov are minimal, the hypotheses of Lemma 6.4 are satisfied. Thus, F and F' are topologically equivalent, which is a contradiction. This completes the proof. $\qquad\square$

Let $H \subset \Gamma_{g,n}$ be a finitely generated subgroup. To each set of generators $\mathcal{G}$ we associate a function on $\mathbf{T}_{g,n}$:

$$(6.2) \qquad \delta_{H,\mathcal{G}}(\sigma) = \max\{\mathrm{d}_{\mathrm{WP}}(\sigma, \gamma\sigma) : \gamma \in \mathcal{G}\}.$$

Definition 6.5. (cf. [KS2, §2]) A finitely generated subgroup $H \subset \Gamma_{g,n}$ is **proper** (or acts **properly** on $\mathbf{T}_{g,n}$) if there exists a generating set $\mathcal{G}$ of H with the property that for every $M \geq 0$, the set $\{\sigma \in \mathbf{T}_{g,n} : \delta_{H,\mathcal{G}}(\sigma) \leq M\}$ is bounded.

Clearly, the condition in the definition of a proper subgroup is independent of the choice of generating set. For complete manifolds of nonpositive curvature, the existence of two hyperbolic isometries with nonasymptotic axes is sufficient to prove that a subgroup of isometries is proper. Theorem 1.2 shows that this works for mapping class groups as well, where hyperbolic is replaced with pseudo-Anosov. The approach taken below to prove this result uses the relationship, demonstrated above, between the axes of pseudo-Anosov's and their corresponding stable and unstable foliations. We first need one more:

LEMMA 6.6. *Let A_γ be the axis of a pseudo-Anosov γ and $\sigma_i \in \mathbf{T}_{g,n}$ any sequence. Then as $i \to \infty$,*

$$\mathrm{dist}_{\mathrm{WP}}(\sigma_i, A_\gamma) \to +\infty \quad \Longrightarrow \quad \mathrm{d}_{\mathrm{WP}}(\sigma_i, \gamma\sigma_i) \to +\infty.$$

Proof. Choose $\sigma \notin A_\gamma$, and let $\sigma_0 \in A_\gamma$ such that $\mathrm{dist}_{\mathrm{WP}}(\sigma, A_\gamma) = \mathrm{d}_{\mathrm{WP}}(\sigma, \sigma_0)$. Let α be the geodesic from σ_0 to σ, parametrized by arc length. Now $f(t) = \mathrm{d}_{\mathrm{WP}}(\alpha(t), \gamma\alpha(t))$ is strictly convex with $f'(0) = 0$, since $f(0) = \mathfrak{L}_{\mathrm{WP}}(\gamma)$ is the minimum of f. Moreover, since A_γ projects to a closed curve in $\mathbf{M}_{g,n}$, there is a neighborhood of A_γ on which the curvature is bounded above by a strictly negative constant. Therefore, for $t_\gamma > 0$ sufficiently small depending only A_γ, there is $\varepsilon_\gamma > 0$, depending only on t_γ and A_γ, such that $f'(t_\gamma) \geq \varepsilon_\gamma$. So if

$L = \mathrm{d}_{\mathrm{WP}}(\sigma, \sigma_0)$, then

$$(6.3) \qquad \mathrm{d}_{\mathrm{WP}}(\sigma, \gamma\sigma) = f(L) \geq \varepsilon_\gamma(L - t_\gamma) = \varepsilon_\gamma(\,\mathrm{d}_{\mathrm{WP}}(\sigma, \sigma_0) - t_\gamma).$$

Applying this to the sequence σ_i completes the proof. □

Proof of Theorem 1.2. Let $H \subset \Gamma_{g,n}$ be sufficiently large. Then by definition H contains two independent pseudo-Anosov's γ and γ'. Since the condition of being proper does not depend on the choice of generating set, we may include the elements γ, γ' in $\mathcal{G}$. If H is not proper then there is a number $M \geq 0$ and an unbounded sequence $\sigma_i \in \mathbf{T}_{g,n}$ such that $\delta_{H,\mathcal{G}}(\sigma_i) \leq M$. In particular, $\mathrm{d}_{\mathrm{WP}}(\sigma_i, \gamma\sigma_i)$ and $\mathrm{d}_{\mathrm{WP}}(\sigma_i, \gamma'\sigma_i)$ are both bounded by M for all i. By Lemma 6.6, it follows that both $\mathrm{dist}_{\mathrm{WP}}(\sigma_i, A_\gamma)$ and $\mathrm{dist}_{\mathrm{WP}}(\sigma_i, A_{\gamma'})$ are bounded. Since σ_i is unbounded in $\mathbf{T}_{g,n}$, this implies that $\mathrm{d}_{\mathrm{WP}}(A_\gamma(t_i), A_{\gamma'}(s_i))$ is bounded along some unbounded sequence (t_i, s_i). But this contradicts Theorem 6.2. □

7. Further results. This section is independent of the rest of the paper. In it we discuss the convergence of the heat equation as a method for finding equivariant geodesics. The result, Theorem 7.4, relies on the existence Theorem 1.1. It would be useful to have a direct proof of convergence that would circumvent the detailed discussion in Section 4, though at present it is not clear how to do this. At the end of this section we also point out additional consequences of the existence of equivariant geodesics.

Let $\gamma \in \Gamma_{g,n}$ and $u \colon \mathbb{R} \to \mathbf{T}_{g,n}$, equivariant with respect to γ. The length of u is, by our convention,

$$(7.1) \qquad L(u) = L\left(u|_{[0,1]}\right) = \int_0^1 \|\dot{u}\|\, dx,$$

where $\|\cdot\|$ denotes the Weil-Petersson norm on tangent vectors to $\mathbf{T}_{g,n}$. We use a similar convention for the energy $E(u)$:

$$(7.2) \qquad E(u) = E\left(u|_{[0,1]}\right) = \int_0^1 \|\dot{u}\|^2\, dx.$$

In particular, $L(u) \leq E^{1/2}(u)$.

We define the heat flow of u as follows (cf. [ES, Ham]). The flow is a time dependent family of γ-equivariant paths $u(t, \cdot) \colon \mathbb{R} \to \mathbf{T}_{g,n}$, where for $t \geq 0$, $u(t,x)$ satisfies the heat equation:

$$(7.3) \qquad \dot{u} = \Delta u, \qquad u(0,x) = u(x).$$

Here, $\dot{u}$ denotes the t derivative of $u(t,x)$, and Δ is the induced Laplacian from the Levi-Città connection of the Weil-Petersson metric on $\mathbf{T}_{g,n}$. The following

is standard:

LEMMA 7.1. *For any initial condition, there is $T > 0$ such that the solution $u(t, x)$ of (7.3) exists and is unique for all $0 \leq t < T$.*

To obtain existence for all time, we need to guarantee that the solution does not run off to the boundary $\partial \overline{\mathbf{T}}_{g,n}$ in finite time. By convexity, it suffices to require this at a single point, as the following lemma demonstrates:

LEMMA 7.2. *Let $u(t, x)$ be a γ-equivariant solution to the heat equation for $0 \leq t < T$. Suppose there is $x_0 \in \mathbb{R}$ and a compact set $K \subset \mathbf{T}_{g,n}$ such that $u(t, x_0) \in K$ for all $t < T$. Then there is $\varepsilon > 0$ such that $u(t, x)$ may be extended to a solution for $0 \leq t < T + \varepsilon$.*

Proof. Let f be the convex exhaustion function on $\mathbf{T}_{g,n}$ from Proposition 3.1 (4). Then by a straightforward calculation using the negative curvature of $\mathbf{T}_{g,n}$, $g(t, x) = f(u(t, x))$ is a subsolution to the heat equation. By equivariance of the path we conclude that for each $N \in \mathbb{Z}$ there is a compact set $K_N \subset \mathbf{T}_{g,n}$ such that $u(t, x_0 + N)$ and $u(t, x_0 - N)$ are contained in K_N for all $t < T$. In particular, $g(t, x_0 + N)$ and $g(t, x_0 - N)$ are uniformly bounded in t. On the other hand, since $g(t, x)$ is a subsolution, an interior maximum on $[x_0 - N, x_0 + N]$ decreases in t. Hence, for each fixed N, $g(t, x)$ is uniformly bounded for all $0 \leq t < T$ and for all $x \in [x_0 - N, x_0 + N]$. So for x in a compact set in $\mathbb{R}$, $u(t, x)$ lies in a compact set in $\mathbf{T}_{g,n}$. It follows that as $t \to T$, $u(t, x)$ converges to a γ-equivariant path $u_T: \mathbb{R} \to \mathbf{T}_{g,n}$. Using u_T as initial conditions in Lemma 7.1, we may find an extension to some $T + \varepsilon$. $\square$

We also have the following criterion for convergence of the flow at infinite time:

LEMMA 7.3. *Let $u(t, x)$ be a γ-equivariant solution to the heat equation. Suppose there is $x_0 \in \mathbb{R}$, $T \geq 0$, and a compact set $K \subset \mathbf{T}_{g,n}$ such that $u(t, x_0) \in K$ for all $t \geq T$. Then $u(t, x)$ converges to a complete γ-equivariant geodesic in $\mathbf{T}_{g,n}$.*

Proof. The proof follows that of Lemma 7.2 verbatim. One obtains convergence of $u(t, x)$ along a subsequence $t_j \to \infty$ by standard methods. Convergence for all t follows from Hartman's lemma [Har]. $\square$

Finally, given the existence Theorem 1.1, we can state a general result for convergence of the heat flow:

THEOREM 7.4. *Let $\gamma \in \Gamma_{g,n}$ be pseudo-Anosov. There is a constant $c(\gamma) > 0$ with the following significance: If $u: \mathbb{R} \to \mathbf{T}_{g,n}$ is a γ-equivariant smooth map satisfying $E^{1/2}(u) \leq \mathfrak{L}_{\mathrm{WP}}(\gamma) + c(\gamma)$, and $u(t, x)$ denotes the heat flow (7.3) with initial condition $u(x)$, then $u(t, x)$ exists for all $t \geq 0$ and converges as $t \to \infty$ to the unique γ-equivariant complete geodesic in $\mathbf{T}_{g,n}$.*

Proof. Let $A_\gamma\colon \mathbb{R} \to \mathbf{T}_{g,n}$ denote the complete γ-equivariant geodesic from Theorem 1.1. By (6.3) it follows that if the length of a γ-equivariant map u is near to $\mathcal{L}_{\mathrm{WP}}(\gamma)$, then u must be uniformly close to A_γ. We can now apply Hartman's distance decreasing property [Har] to conclude that along the flow,

$$\mathrm{d}_{\mathrm{WP}}(u(t,x), A_\gamma(x)) \leq \sup_{x'} \mathrm{d}_{\mathrm{WP}}(u(x'), A_\gamma(x'))$$

for all x and all t in the domain of definition. In particular, if the right-hand side in the above expression is sufficiently small compared to the distance of A_γ to $\partial \overline{\mathbf{T}}_{g,n}$, then the inequality guarantees that $u(t,x)$ stays in a compact set in $\mathbf{T}_{g,n}$. The result then follows from Lemmas 7.2 and 7.3. □

To conclude, it is worth pointing out two additional consequences of the results in this paper. First, we immediately obtain the following well-known fact concerning mapping class groups (cf. [Mc]):

COROLLARY 7.5. *Let $\gamma \in \Gamma_{g,n}$ be pseudo-Anosov and let $\langle\gamma\rangle$ denote the cyclic group generated by γ. Then the centralizer and normalizer of $\langle\gamma\rangle$ in $\Gamma_{g,n}$ are both virtually cyclic.*

Proof. It clearly suffices to prove the statement for the normalizer, which we denote by $N\langle\gamma\rangle$. Recall from Section 4 that there is a finite index subgroup $\Gamma'_{g,n} \subset \Gamma_{g,n}$ which contains no periodic mapping classes. Let $N'\langle\gamma\rangle = N\langle\gamma\rangle \cap \Gamma'_{g,n}$, and let A_γ the axis of γ. Then for $g \in N'\langle\gamma\rangle$, we have $\gamma^k g A_\gamma \subset g A_\gamma$ for some k. But A_γ is also the unique γ^k-equivariant geodesic, so $g A_\gamma = A_\gamma$. In this way, we have a homomorphism $N'\langle\gamma\rangle \to \mathrm{Iso}(A_\gamma) \simeq \mathrm{Iso}(\mathbb{R})$, which, by the assumption of no periodic mapping classes, must be injective. The result follows, since by proper discontinuity of the action of $\Gamma_{g,n}$, the image of $N'\langle\gamma\rangle$ in $\mathrm{Iso}(A_\gamma)$ is also discrete. □

Second, the proof of Theorem 5.1 shows that, just like for the Teichmüller metric, there is a *shortest* length for a closed Weil-Petersson geodesic in the moduli space of curves.

THEOREM 7.6. *There exists a constant $\delta(g,n) > 0$ depending only on (g,n) such that if $\gamma \in \Gamma_{g,n}$ satisfies $\mathcal{L}_{\mathrm{WP}}(\gamma) \neq 0$, then $\mathcal{L}_{\mathrm{WP}}(\gamma) \geq \delta(g,n)$.*

Proof. The proof proceeds by induction on $3g-3+n$. Reducible elements fall into the inductive hypothesis. Hence, we may assume that γ is pseudo-Anosov. Let A_γ be the axis of γ as above. We claim that there is some $\delta_1 > 0$ depending only on (g,n) such that either $\mathcal{L}_{\mathrm{WP}}(\gamma) \geq \delta_1$ or there is some point $\sigma \in A_\gamma$ such that $\mathrm{dist}_{\mathrm{WP}}(\sigma, \partial \mathbf{T}_{g,n}) \geq \delta_1$. This follows from Lemma 5.2 and the compactness of $\overline{\mathbf{M}}_{g,n}$. Next, on the set of points a distance $\geq \delta_1/2$ from $\partial \mathbf{T}_{g,n}$, the Weil-Petersson and Teichmüller metrics are uniformly quasi-isometric. If $\mathcal{L}_{\mathrm{WP}}(\gamma) \leq \delta_1/2$ is very

small, then the translation length of γ with respect to the Teichmüller metric would also be very small. On the other hand, the latter has a uniform lower bound depending only on (g,n) (cf. [Pe]). This completes the proof. □

8. Appendix.   The purpose of this appendix is to prove the asymptotic estimate for the Weil-Petersson metric asserted in (3.5). The idea, due to Yamada [Y1], is to combine the original construction of Masur [M2] with the precise estimates for the hyperbolic metric due to Wolpert [W5]. This requires some background notation which we shall sketch rather quickly. A more detailed discussion may be found in the references cited.

Consider a point $\sigma \in \widehat{\mathbf{T}} = \mathbf{T}_{g_1,n_1} \times \cdots \times \mathbf{T}_{g_N,n_N}$ obtained by collapsing curves $c_1, \ldots, c_k$. Then σ corresponds to a nodal surface: that is, a collection of Riemann surface $\Sigma_{g_1,n_1}, \ldots, \Sigma_{g_N,n_N}$ with identifications that are holomorphically the coordinate lines in $\mathbb{C}^2$ made in a neighborhood of certain of the marked points. More precisely, each curve c_ℓ gives rise to a pair of points p_ℓ, q_ℓ that are identified with marked points in the disjoint union of Σ_{g_j,n_j}. Let z_ℓ, w_ℓ indicate local conformal coordinates at p_ℓ, q_ℓ. Then the (connected) nodal surface Σ_0 is formed via the identification $z_\ell w_\ell = 0$. We construct a degenerating family of Riemann surfaces Σ_t, $t = (t_1, \ldots, t_k)$, by replacing the nodal neighborhoods with annuli $A_{\ell,t}$ given by the equations $z_\ell w_\ell = t_\ell$. We also may deform the conformal structure on the initial nodal surface Σ_0 in regions compactly supported away from the nodes. In the notation of (3.5), this gives rise to deformations $\Sigma_{t,\tau}$ of the family Σ_t corresponding to the coordinates $\tau_{k+1}, \ldots, \tau_{3g-3+n}$. Passing to the infinite abelian covering with coordinates $(\theta_\ell, |t_\ell|)$, these degenerating families parametrize a neighborhood of σ in the model neighborhood. For brevity we will omit the τ parameters from the notation. Since the uniformizing coordinates may be chosen to vary continuously with respect to the τ parameters, this is no loss of generality.

We will denote by Σ^* the surface with boundary which is the complement of a neighborhood of the nodes of Σ_0. The gluing of Σ^* with the annuli $A_{\ell,t}$ takes place on pairs of annuli $B_{\ell,t}$, also supported away from the nodes (see [W5]). The nodal surface Σ_0 carries a complete hyperbolic metric $ds_0^2 = \rho_0(z)|dz|^2$ which induces a hyperbolic metric on the surface Σ^* with boundary. We assume that the conformal coordinates used in the gluing have been chosen so that in a neighborhood of each node, $\rho_0(z_\ell) = (|z_\ell| \log |z_\ell|)^{-2}$. Furthermore, each annulus $A_{\ell,t}$ carries a hyperbolic metric:

$$(8.1) \qquad ds_{\ell,|t|}^2 = ds_{\ell,0}^2 \frac{\Theta_{\ell,|t|}^2}{(\sin \Theta_{\ell,|t|})^2},$$

where

$$ds_{\ell,0}^2 = \frac{|dz_\ell|^2}{|z_\ell|^2(\log |z_\ell|)^2} \quad , \quad \Theta_{\ell,|t|} = \pi \frac{\log |z_\ell|}{\log |t_\ell|}.$$

Following Wolpert, we construct a "grafted" metric $ds_{t,\mathrm{graft}}^2$ on Σ_t as follows: $ds_{t,\mathrm{graft}}^2$ agrees with the hyperbolic metrics $ds_{\ell,|t|}^2$ on each annulus $A_{\ell,t}$, $ds_{t,\mathrm{graft}}^2$ is a smooth family of hyperbolic metrics with boundary on Σ^* converging as $t \to 0$ to the one induced by ds_0^2, and in the regions $B_{\ell,t}$, $ds_{t,\mathrm{graft}}^2$ interpolates between these hyperbolic metrics. The actual hyperbolic metric ds_t^2 on Σ_t will be related to $ds_{t,\mathrm{graft}}^2$ by a conformal factor:

$$\rho_t(z)|dz|^2 = ds_t^2 = \frac{\rho_t(z)}{\rho_{t,\mathrm{graft}}(z)} ds_{t,\mathrm{graft}}^2.$$

Then we have the following uniform estimate on Σ_t (cf. [W5, Expansion 4.2]):

$$(8.2) \qquad \left| \frac{\rho_t(z)}{\rho_{t,\mathrm{graft}}(z)} - 1 \right| = \sum_{\ell=1}^{k} O((\log |t_\ell|)^{-2}).$$

We also note that in each $A_{\ell,t}$ the interpolation satisfies:

$$(8.3) \qquad \left| \frac{\rho_0(z_\ell)}{\rho_{t,\mathrm{graft}}(z_\ell)} - 1 \right| = O(\Theta_{\ell,|t|}^2)$$

(cf. [W5, §3.4.MG]).

Next, we review Masur's estimate of the Weil-Petersson metric. According to [M2], we can find a family of quadratic differentials $\varphi_i(z,t)$, meromorphic on Σ_t with at most simple poles at the n marked points, parametrizing the cotangent bundle of $\mathbf{T}_{g,n}$ at the point represented by Σ_t. Then for $i \geq k+1$, the φ_i converge as $t \to 0$ to quadratic differentials with at most simple poles at the marked points (which now include the nodes). For $i \leq k$, the φ_i represent normal directions to the boundary stratum.

In terms of the local coordinate z_ℓ for the annular region $A_{\ell,t}$ we have the following uniform bounds (cf. [M2], 5.4, 5.5, and note the remark on p. 633 before "Proof of Theorem 1"). Here, C denotes a constant that is independent of t.

(8.4)   For $i,j \geq k+1$, $|\varphi_i(z_\ell,t)\bar\varphi_j(z_\ell,t)| \leq C/|z_\ell|^2$

(8.5)   For $i,j \leq k$, $|\varphi_i(z_\ell,t)\bar\varphi_j(z_\ell,t)|$

$$\leq \begin{cases} C_1|t_\ell|^2/|z_\ell|^4 + C|t_\ell|^2/|z_\ell|^3 & i=j=\ell,\ C_1 \neq 0 \\ C|t_i||t_j|/|z_\ell|^3 & i=\ell, j\neq\ell \text{ or } \textit{vice versa} \\ C|t_i||t_j|/|z_\ell|^2 & i\neq\ell, j\neq\ell \end{cases}$$

(8.6)   For $i \leq k,\ j \geq k+1$, $|\varphi_i(z_\ell,t)\bar\varphi_j(z_\ell,t)| \leq \begin{cases} C|t_i|/|z_\ell|^3 & i=\ell \\ C|t_i|/|z_\ell|^2 & i\neq\ell \end{cases}$

(8.7)   For $i,j \geq k+1$, $|\varphi_i(z_\ell,t)\bar\varphi_j(z_\ell,t) - \varphi_i(z_\ell,0)\bar\varphi_j(z_\ell,0)| \leq C|t_\ell|/|z_\ell|^4.$

Let $G_{i\bar{j}}(t)$ denote the components of the Weil-Petersson metric tensor with respect to this choice of (dual) basis. With this understood, we now state the following:

LEMMA 8.1. (i) *There is a nonzero constant C independent of t such that for* $i \leq k$,

$$|G_{i\bar{i}}(t)| = C|t_i|^{-2}(-\log|t_i|)^{-3}\left(1 + \sum_{\ell=1}^{k} O((-\log|t_\ell|)^{-2})\right).$$

(ii) *For* $i,j \leq k, i \neq j$,

$$|G_{i\bar{j}}(t)| = O(|t_i|^{-1}|t_j|^{-1}(-\log|t_i|)^{-3}(-\log|t_j|)^{-3}).$$

(iii) *For* $i \leq k, j \geq k+1$,

$$|G_{i\bar{j}}(t)| = O(|t_i|^{-1}(-\log|t_i|)^{-3}).$$

(iv) *For* $i,j \geq k+1$,

$$|G_{i\bar{j}}(t) - G_{i\bar{j}}(0)| = \sum_{\ell=1}^{k} O((-\log|t_\ell|)^{-2}).$$

Let us first point out that the estimate (3.5) follows from Lemma 8.1 by making the substitution: $(\theta_i, |t_i|) \mapsto (\theta_i, \xi_i)$, where $\xi_i = (-\log|t_i|)^{-1/2}$. The rest of this section is therefore devoted to the following:

Proof of Lemma 8.1. As in [M2], the result will follow by first estimating the cometric. Let $\langle \varphi_i(t), \varphi_j(t) \rangle$ denote the Weil-Petersson pairing. We will show that:
(i′) There is a nonzero constant C such that for $i \leq k$,

$$|\langle \varphi_i(t), \varphi_i(t) \rangle| = C|t_i|^2(-\log|t_i|)^3\left(1 + \sum_{\ell=1}^{k} O((-\log|t_\ell|)^{-2})\right).$$

(ii′) For $i,j \leq k, i \neq j$, $|\langle \varphi_i(t), \varphi_j(t) \rangle| = O(|t_i||t_j|)$.
(iii′) For $i \leq k, j \geq k+1$, $|\langle \varphi_i(t), \varphi_j(t) \rangle| = O(|t_i|)$.
(iv′) For $i,j \geq k+1$,

$$|\langle \varphi_i(t), \varphi_j(t) \rangle - \langle \varphi_i(0), \varphi_j(0) \rangle| = \sum_{\ell=1}^{k} O((-\log|t_\ell|)^{-2}).$$

Given these estimates, Lemma 8.1 then follows via the cofactor expansion for the inverse as in [M2].

To begin, notice that (ii$'$) and (iii$'$) are precisely the statements (iv) and (vi), respectively, of [M2, p. 634]. To prove (i$'$), we note that for $i \leq k$, $|\varphi_i(t)|^2$ is $O(|t_i|^2)$ uniformly on compact sets away from the pinching region. We therefore need to estimate:

$$(8.8) \qquad \int_{A_{\ell,t}} \frac{|\varphi_i(z_\ell,t)|^2}{\rho_t(z_\ell)} |dz_\ell|^2.$$

Using (8.2) we observe that (8.8) is bounded by

$$(8.9) \qquad \int_{A_{\ell,t}} \frac{|\varphi_i(z_\ell,t)|^2}{\rho_0(z_\ell)} |dz_\ell|^2 + \int_{A_{\ell,t}} \frac{|\varphi_i(z_\ell,t)|^2}{\rho_0(z_\ell)} \left(\frac{\rho_0(z_\ell)}{\rho_t(z_\ell,t)} - 1 \right) |dz_\ell|^2$$

$$\leq \int_{A_{\ell,t}} |\varphi_i(z_\ell,t)|^2 |z_\ell|^2 (\log |z_\ell|)^2 |dz_\ell|^2$$

$$+ C \int_{A_{\ell,t}} |\varphi_i(z_\ell,t)|^2 |z_\ell|^2 (\log |z_\ell|)^2 \left(\frac{\rho_{t,\mathrm{graft}}(z_\ell)}{\rho_t(z_\ell,t)} - 1 \right) |dz_\ell|^2$$

$$+ \int_{A_{\ell,t}} |\varphi_i(z_\ell,t)|^2 |z_\ell|^2 (\log |z_\ell|)^2 \left(\frac{\rho_0(z_\ell)}{\rho_{t,\mathrm{graft}}(z_\ell,t)} - 1 \right) |dz_\ell|^2$$

$$\leq \int_{A_{\ell,t}} |\varphi_i(z_\ell,t)|^2 |z_\ell|^2 (\log |z_\ell|)^2 |dz_\ell|^2$$

$$+ \sum_{j=1}^{k} O\left((\log |t_j|)^{-2} \right) \int_{A_{l,t}} |\varphi_i(z_\ell,t)|^2 |z_\ell|^2 (\log |z_\ell|)^2 |dz_\ell|^2$$

$$+ \int_{A_{\ell,t}} |\varphi_i(z_\ell,t)|^2 |z_\ell|^2 (\log |z_\ell|)^2 \left(\frac{\rho_0(z_\ell)}{\rho_{t,\mathrm{graft}}(z_\ell,t)} - 1 \right) |dz_\ell|^2.$$

Using (8.5) we see that the right hand side of (8.9) is $O(|t_i|^2)$ if $i \neq l$. If $i = l$, then again by (8.5) the first term on the right hand side of (8.9) is $O(|t_i|^2(-\log |t_i|)^3)$, and this contribution is in fact nonzero. The second term is similarly bounded. Finally, the third term is bounded in the same way, after applying (8.3). This proves (i$'$).

To prove (iv$'$), first note that by (8.2) we have an estimate

$$\left| \int_K \frac{\varphi_i(z,t)\bar{\varphi}_j(z,t)}{\rho_t(z)} |dz|^2 - \int_K \frac{\varphi_i(z,0)\bar{\varphi}_j(z,0)}{\rho_0(z)} |dz|^2 \right| = \sum_{\ell=1}^{k} O((\log |t_\ell|)^{-2})$$

where K is a compact set supported away from the pinching region. Note also that since $\varphi_i(z,0)$ have at most simple poles,

$$\left| \int_{|z_\ell| \leq |t_\ell|} \frac{\varphi_i(z_\ell,0)\bar{\varphi}_j(z_\ell,0)}{\rho_0(z_\ell)} |dz_\ell|^2 \right| = O(|t_\ell|).$$

Hence, we need only estimate

$$(8.10) \quad \left| \int_{A_{\ell,t}} \frac{\varphi_i(z_\ell,t)\bar{\varphi}_j(z_\ell,t)}{\rho_t(z_\ell)} |dz_\ell|^2 - \int_{A_{\ell,t}} \frac{\varphi_i(z_\ell,0)\bar{\varphi}_j(z_\ell,0)}{\rho_0(z_\ell)} |dz_\ell|^2 \right|$$

$$\leq \int_{A_{\ell,t}} \frac{|\varphi_i(z_\ell,t)\bar{\varphi}_j(z_\ell,t)|}{\rho_0(z_\ell)} \left| \frac{\rho_0(z_\ell)}{\rho_t(z_\ell,t)} - 1 \right| |dz_\ell|^2$$

$$+ \int_{A_{\ell,t}} |\varphi_i(z_\ell,t)\bar{\varphi}_j(z_\ell,t) - \varphi_i(z_\ell,0)\bar{\varphi}_j(z_\ell,0)| \frac{|dz_\ell|^2}{\rho_0(z_\ell)}$$

$$\leq C \int_{A_{\ell,t}} \frac{|\varphi_i(z_\ell,t)\bar{\varphi}_j(z_\ell,t)|}{\rho_0(z_\ell)} \left| \frac{\rho_{t,\mathrm{graft}}(z_\ell)}{\rho_t(z_\ell,t)} - 1 \right| |dz_\ell|^2$$

$$+ \int_{A_{\ell,t}} \frac{|\varphi_i(z_\ell,t)\bar{\varphi}_j(z_\ell,t)|}{\rho_0(z_\ell)} \left| \frac{\rho_0(z_\ell)}{\rho_{t,\mathrm{graft}}(z_\ell,t)} - 1 \right| |dz_\ell|^2$$

$$+ \int_{A_{\ell,t}} |\varphi_i(z_\ell,t)\bar{\varphi}_j(z_\ell,t) - \varphi_i(z_\ell,0)\bar{\varphi}_j(z_\ell,0)| \frac{|dz_\ell|^2}{\rho_0(z_\ell)}.$$

By (8.2) and (8.4), the first term on the right-hand side of (8.10) is bounded by

$$\sum_{p=1}^{k} O((\log|t_p|)^{-2}) \int_{A_{\ell,t}} (\log|z_\ell|)^2 |dz_\ell|^2 = \sum_{p=1}^{k} O((\log|t_p|)^{-2}).$$

By (8.3), the second term is $O\left((\log|t_\ell|)^{-2}\right)$. Using (8.7), the third term on the right-hand side of (8.10) is bounded by a constant times

$$\int_{A_{\ell,t}} \frac{|t_\ell|}{|z_\ell|^4} |z_\ell|^2 (\log|z_\ell|)^2 |dz_\ell|^2 = O(|t_\ell|(-\log|t_\ell|)^3).$$

Hence, the desired estimate holds on each annular region $A_{l,t}$ as well. This proves (iv$'$) and completes the proof of the lemma. $\qquad\square$

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