HIGHER EULER CHARACTERISTICS:
VARIATIONS ON A THEME OF EULER

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ABSTRACT. We provide a natural interpretation of the secondary Euler characteristic and generalize it to higher Euler characteristics. For a compact oriented manifold of odd dimension, the secondary Euler characteristic recovers the Kervaire semi-characteristic. We prove basic properties of the higher invariants and illustrate their use. We also introduce their motivic variants.

Being trivial is our most dreaded pitfall.
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"Trivial" is relative. Anything grasped as long as two minutes ago seems trivial to a working mathematician.

M. Gromov, *A few recollections*, 2011.

The characteristic introduced by L. Euler [7, 8, 9], (first mentioned in a letter to C. Goldbach dated 14 November 1750) via his celebrated formula

$$V - E + F = 2,$$

is a basic ubiquitous invariant of topological spaces; two extracts from the letter:

...Folgende Proposition aber kann ich nicht recht rigorose demonstriren

$$\cdots H + S = A + 2.$$

...Es nimmt mich Wunder, dass diese allgemeinen proprietates in der Stereometrie noch von Niemand, so viel mir bekannt, sind angemerkt worden; doch viel mehr aber, dass die fürnehmsten davon als theor. 6 et theor. 11 so schwer zu beweisen sind, den ich kann dieselben noch nicht so beweisen, dass ich damit zufrieden bin....

When the Euler characteristic vanishes, other invariants are necessary to study topological spaces. For instance, an odd-dimensional compact oriented manifold has vanishing Euler characteristic; the semi-characteristic of M. Kervaire [16] provides an important invariant of such manifolds.

In recent years, the "secondary" or "derived" Euler characteristic has made its appearance in many disparate fields [2, 10, 14, 4, 5, 18, 3]; in fact, this secondary invariant dates back to 1848 when it was introduced by A. Cayley (in a paper "*A Theory of elimination*", see [12, Corollary 15 on p. 486 and p. 500, Appendix B]). In this short paper, we provide a natural interpretation and generalizations of the "secondary" Euler characteristic. Our initial aim was to understand the appearance of the "secondary" Euler characteristic in formulas for special values of zeta functions [18] (see §2).

As motivation, consider the following questions:

- **Q1** Given a compact manifold M of the form

$$M = N \times \underbrace{S^1 \times S^1 \cdots S^1}_{\text{r factors}}, \qquad r > 0$$

which topological invariant detects the integer $r > 0$? The Euler characteristic of M is always zero: $\chi(M) = \chi(N).\chi(S^1)^r = 0$ independent of r. A related question: given M, how to compute the Euler characteristic of N?
- **Q2** For any ring A, write $K_0(A)$ for the Grothendieck group of the (exact) category Mod_A of finitely generated projective A-modules. Any bounded complex C of finitely generated projective A-modules defines a class $[C] \in K_0(A)$. As the class $[C]$ of an acyclic complex C is zero, one can ask: Are there natural non-trivial invariants of acyclic complexes C? Are there enough to help distinguish an acyclic complex from a tensor product (itself acyclic) of acyclic complexes?

The higher Euler characteristics answer these questions; these invariants are "special values" of the Poincaré polynomial; see Remark 1.8. We show (Lemma 1.2) that the secondary Euler characteristic recovers the semi-characteristic of M. Kervaire [16]. The topological and the K-theoretic versions of the higher Euler characteristics are in the first and third section. The last section indicates certain generalizations in the context of motivic measures and raises related questions. The second section is a gallery of secondary Euler characteristics.

Note the analogy between taking a product with a circle $X \mapsto X \times S^1$ and taking the cone CN of a self-map $N \to N$ (compare part (iii) of Theorems 1.4 and 3.2). J. Rosenberg alerted us to a definition of "higher Euler characteristics" due to R. Geoghegan and A. Nicas [13]; the relations with this paper will be explored in future work.

Notations. A nice topological space is, or is homotopy equivalent to, a finite CW complex.

1. TOPOLOGICAL SETTING

Introduction. Recall that, for any nice topological space M, its Euler characteristic

$$\chi(M) = \sum_i (-1)^i b_i(M)$$

is the alternating sum of the Betti numbers $b_i = b_i(M) = \text{rank}_{\mathbb{Z}} H_i(M, \mathbb{Z})$. The "secondary" Euler characteristic of M is defined as

$$\chi'(M) = \sum_i (-1)^{i-1} i b_i = b_1 - 2b_2 + 3b_3 - \cdots .$$

The topological invariant χ satisfies (and is characterized by) the following properties: it is invariant under homotopy, $\chi(\text{point}) = 1$, and, for nice spaces U and V,

$$\chi(U \times V) = \chi(U).\chi(V),$$
(1)
$$\chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Clearly, χ' cannot satisfy the same properties: χ' is invariant under homotopy, $\chi'(\text{point}) = 0$ but, in general, $\chi'(U \times V) \neq \chi'(U).\chi'(V)$ and χ' does not satisfy (1). For a oriented compact closed manifold M of odd dimension, Poincaré duality implies that $\chi(M) = 0$. Thus, $\chi'(M)$ is the simplest nontrivial natural topological invariant for such manifolds.

Lemma 1.1. *Let M and N be nice topological spaces.*
(i) $\chi'(M \times S^1) = \chi(M)$.
(ii) $\chi'(M \times N) = \chi(M)\chi'(N) + \chi(N).\chi'(M)$.

Proof. (i) This is just direct computation: Let b_i be the Betti numbers of $M \times S^1$ and c_i the Betti numbers of M. By the Künneth theorem, one has $b_{i+1} = c_{i+1} + c_i$. Therefore,

$$\begin{aligned}
\chi'(M \times S^1) &= 0.b_0 + b_1 - 2b_2 + 3b_3 - \cdots \\
&= (c_1 + c_0) - 2(c_2 + c_1) + 3(c_3 + c_2) \cdots \\
&= c_0 - c_1 + c_2 - \cdots \\
&= \chi(M). \quad \square
\end{aligned}$$

(ii) Direct computation. For a conceptual proof, see the proof of part (ii) of Theorem 1.4. $\square$

Kervaire's semi-characteristic. Let M be a compact oriented manifold of odd dimension $2n+1$. Since $\chi(M) = 0$, Kervaire [16] introduced the invariant $K_M = \sum_{i=0}^{i=n}(-1)^i b_i(M) \mod 2$.

Lemma 1.2. $\chi'(M) \equiv K_M \mod 2$.

Proof. Clearly $\chi'(M) = \sum_{i=0}^{i=n} b_{2i+1} \mod 2$. If $n = 2k + 1$, then using $(b_i = b_{4k+3-i})$ we have $\chi'(M) = b_1 + b_3 + \cdots b_n + b_{2k} + b_{2k-2} + \cdots b_0 \mod 2$ So $\chi'(M) = K_M \mod 2$ in this case. If $n = 2k$, then using $b_i = b_{4k+1-i}$, we have $\chi'(M) = b_1 + b_3 + \cdots b_{2k-1} + b_{2k} + b_{2k-2} + \cdots b_0 \mod 2$. So $\chi'(M) = K_M \mod 2$. $\square$

Basic definitions and results.

Definition 1.3. (Higher Euler characteristics) For any nice topological space M and for any integer $j \geq 0$, we define the j'th Euler characteristic of M as

(2)
$$\chi_j(M) = \sum_i (-1)^{i-j} \binom{i}{j} b_i.$$

Clearly, $\chi_0(M) = \chi(M)$ and $\chi_1(M) = \chi'(M)$. If M is a manifold of dimension N, then $\chi_j(M) = 0$ for $j > N$. Note that $\chi_j(S^1) = 0$ for $j \neq 1$ and $\chi_1(S^1) = 1$. The higher Euler characteristics[1] share many of the properties of χ and χ'.

Theorem 1.4. *(i) χ_j is invariant under homotopy; for a disjoint union $U \amalg V$, one has $\chi_j(U \amalg V) = \chi_j(U) + \chi_j(V)$ and $\chi_j(\text{point}) = 0$ for $j > 0$.*
(ii) If $\chi_r(M)$ and $\chi_r(N)$ vanish for $0 \leq r < j$, then

$$\begin{aligned}
\chi_k(M \times N) &= 0, \qquad \text{for } 0 \leq k < 2j \\
\chi_{2j}(M \times N) &= \chi_j(M).\chi_j(N), \\
\chi_{2j+1}(M \times N) &= \chi_j(M).\chi_{j+1}(N) + \chi_{j+1}(M).\chi_j(N).
\end{aligned}$$

(iii) Let $M = N \times (\underbrace{S^1 \times \cdots \times S^1}_{j \text{ factors}})$. Then $\chi_j(M) = \chi(N)$ and $\chi_{k+j}(M) = \chi_k(N)$ for $k \geq 0$,

$$\chi_0(M) = 0, \cdots, \chi_{j-1}(M) = 0.$$

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[1]The "secondary" Euler characteristic is the first higher characteristic.

Remark 1.5. There are at least two natural choices for the definition of the higher Euler characteristics; for $\chi_2(M)$, one could take either $\sum_i (-1)^i i^2 b_i$ or $\sum_i (-1)^i i(i-1) b_i$. More generally, an alternate definition is given by

$$\chi_j(M) = \sum (-1)^{i-j} i^j b_i(X).$$

One has $\chi_0(X) = \chi(X)$ and $\chi_1(X) = \chi'(X)$. □

Our proof of Theorem 1.4 is based on Lemma 1.6.

Lemma 1.6. *For any polynomial* $P(t) = \sum_i b_i t^i \in \mathbb{Z}[t]$, *consider the expansion of* $P(t)$ *about* $t = -1$, *namely, define* $Q(u) = \sum_i a_i u^i \in \mathbb{Z}[u]$ *by*

$$P_M(t) = Q_M(1+t),$$

(3)
$$P(t) = b_0 + b_1 t + b_2 t^2 + \cdots$$
$$= a_0 + a_1(1+t) + a_2(1+t)^2 + \cdots .$$

For any $j \geq 0$, *one has*

$$a_j = \sum_i (-1)^{i-j} \binom{i}{j} b_i.$$

Proof. Evaluating both sides of (3) at $t = -1$ gives

$$a_0 = b_0 - b_1 + \cdots = \sum_i (-1)^i b_i = P(-1).$$

Taking the formal derivative of (3) with respect to t gives

(4)
$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots =$$
$$a_1 + 2a_2(1+t) + 3a_3(1+t)^2 + \cdots .$$

Evaluating at $t = -1$ gives

$$a_1 = b_1 - 2b_2 + 3b_3 - \cdots = \sum_i (-1)^{i-1} i b_i.$$

Applying $\frac{d}{dt}$ to (4) gives

(5)
$$2b_2 + 6b_3 t + \cdots + n(n-1)b_n t^{n-2} + \cdots =$$
$$= 2a_2 + 8a_3(1+t) + \cdots + n(n-1)a_n(1+t)^{n-2} + \cdots .$$

Plugging in $t = -1$ gives

$$2a_2 = 2b_2 - 6b_3 + 12b_4 + \cdots + n(n-1)b_n(-1)^{n-2} + \cdots$$

and so

$$a_2 = \sum_i (-1)^{i-2} \binom{i}{2} b_i.$$

Iterating these steps (apply $\frac{d}{dt}$ and evaluate at $t = -1$) provides the required relation for any a_j. □

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Corollary 1.7. *For any nice topological space M, write the Poincaré polynomial $P_M(t) = \sum_i b_i(M)t^i$ as a function of $u = 1 + t$, i.e., define $Q_M(u) \in \mathbb{Z}[u]$ by $P_M(t) = Q_M(1+t)$. Then,*

$$Q_M(u) = \sum_j \chi_j(M) \, u^j.$$

This shows that the higher Euler characteristics form a natural generalization of the Euler characteristic: $\chi_M = P_M(-1)$ and $\chi_1(M), \chi_2(M), \cdots$ are the coefficients of the Taylor expansion of $P_M(t)$ near $t = -1$.

Proof. of Theorem 1.4. (i) the first statement is clear as the Betti numbers are homotopy invariant. For the second, use $P_{U \amalg V}(t) = P_U(t) + P_V(t)$.

(ii) Since $P_{M \times N}(t) = P_M(t).P_N(t)$ (Künneth), so $Q_{M \times N}(u) = Q_M(u).Q_N(u)$. Now apply Lemma 1.6. We are given that $Q_M(u)$ and $Q_N(u)$ are both divisible by u^j. So $Q_{M \times N}(u)$ is divisible by u^{2j}. As $Q_M(u) = u^j(\chi_j(M) + \chi_{j+1}(M)u + \cdots)$ and $Q_N(u) = u^j(\chi_j(N) + \chi_{j+1}(N)u + \cdots)$, we have $Q_{M \times N}(u) = u^{2j}(\chi_j(M).\chi_j(N) + (\chi_j(M).\chi_{j+1}(N) + \chi_{j+1}(M).\chi_j(N))u + \cdots)$. Now apply Lemma 1.6.

(iii) By Künneth, one has $P_M(t) = P_N(t)(1+t)^j$ and so $Q_M(u) = Q_N(u)u^j$. Now apply Lemma 1.6. $\qquad \square$

Remark 1.8. The higher Euler characteristics are *special values* of the Poincaré polynomial, in the following sense.

(a) For any scheme X of finite type over Spec $\mathbb{Z}$, one introduces the analytic function $\zeta_X(s)$ (the zeta function of X). Conjecturally, there is arithmetic information in the special values of $\zeta_X(s)$ at $s = n \in \mathbb{Z}$; if $\zeta_X(n) = 0$, then one looks at the leading term in the Taylor expansion of $\zeta_X(s)$ about $s = n$. This leading term is called a "special value" of $\zeta_X(s)$ at $s = n$. In our context, Lemma 1.6 tells us that the Euler characteristic is the value of the Poincaré polynomial $P(t)$ at $t = -1$ and the "special values" of $P(t)$ at $t = -1$ are the higher Euler characteristics.

(b) Part (iii) of Theorem 1.4 provides a partial answer to **Q1** posed above. Namely, each factor of S^1 in M causes the vanishing of a higher Euler characteristic of M. Thus, the number of factors r of S^1 in M satisfies the inequality

$$r \leq \operatorname{ord}_{t=-1} P_M(t) = \operatorname{ord}_{u=0} Q_M(u)$$

with equality if and only if $\chi(N) \neq 0$. So $\chi_0(M) = 0, \cdots, \chi_{r-1}(M) = 0$.

(c) The higher Euler characteristics do not satisfy (1) in general; $P_{U \cup V} \neq P_U(t) + P_V(t) - P_{U \cap V}(t)$.

(d) There is a straightforward generalization of higher Euler characteristics of local systems (or sheaves) on nice topological spaces (or schemes).

Remark 1.9. Let $X \to B$ be a fibration with fiber F. Then the well known identity $\chi(X) = \chi(F)\chi(B)$ does not generalize to higher Euler characteristics. Lemma 1.1 does not generalize (from products) to fibrations. For instance, consider the Hopf fibration $S^3 \to S^2$ with fibers S^1. Lemma 1.1 (part (ii)) fails in this case: $\chi'(S^3) = 3 \neq 2 \times 1 + (-2) \times 0 = \chi(S^2)\chi'(S^1) + \chi'(S^2)\chi(S^1)$.

Proposition 1.10. *The higher Euler characteristics $\chi_j(Sym^n X)$ of any symmetric product $Sym^n(X)$ of X are determined by the Betti numbers of X.*

Proof. By I.G. Macdonald's formula [19], the Poincaré polynomial of $Sym^n(X)$ is determined by that of X. $\qquad \square$

Remark 1.11. Macdonald proved that $\chi(X)$ determines $\chi(Sym^n X)$; examples show that this does not generalize to the higher Euler characteristics. Namely, for any $j > 0$, one cannot calculate the integer $\chi_j(Sym^n X)$ for all $n > 1$ from just $\chi_j(X)$. There is no Macdonald formula for χ_j alone. $\square$

2. EXAMPLES OF SECONDARY EULER CHARACTERISTICS.

For any bounded complex $C^\bullet$ of finitely generated abelian groups

$$\cdots \to 0 \to C_0 \xrightarrow{d} C_1 \xrightarrow{d} \cdots C_n \to 0 \to \cdots ,$$

one defines $\chi(C^\bullet) = \sum_{i=0}^{i=n} (-1)^i \mathrm{rank}\, C_i$; it is elementary that $\chi(C^\bullet) = \sum_{i=0}^{i=n} (-1)^i \mathrm{rank}\, H_i(C^\bullet)$. We write $\chi'(C^\bullet) = \sum_{i=0}^{i=n} (-1)^{i-1} i (\mathrm{rank}\, C_i)$; this is of interest when $\chi(C^\bullet) = 0$.

Similarly, given any abelian category $\mathcal{A}$ and any bounded complex $C^\bullet$ of objects in $\mathcal{A}$, one defines

$$\chi(C^\bullet) = \sum_i (-1)^i [C_i] \text{ and } \chi'(C^\bullet) = \sum_i (-1)^{i-1} i [C_i],$$

which are elements of $K_0(\mathcal{A})$. Here $[X]$ denotes the class in $K_0(\mathcal{A})$ for any object X of $\mathcal{A}$.

Some of the well known occurences of secondary Euler characteristics include

- (Ray-Singer) [22] Let M be a compact oriented manifold without boundary of dimension N. The Franz-Reidemeister-Milner torsion (or simply R-torsion) $\tau(M, \rho) \in \mathbb{R}$ is defined for any acyclic orthogonal representation ρ of the fundamental group $\pi_1(M)$, Let K be a smooth triangulation of M and Δ_j be the combinatorial Laplacians associated with K and ρ. Then [22, Proposition 1.7]

$$\log \tau(M, \rho) = \frac{1}{2} \sum_{i=0}^{i=N} (-1)^{i+1} i \log \det (-\Delta_i).$$

  Ray-Singer conjectured (and J. Cheeger-W. Müller proved) that this is equal to analytic torsion (which they defined in terms of a Riemannian structure on M).

  (It is reasonable to introduce "the torsion Poincaré polynomial"

(6)
$$R(M, \rho)(t) = \sum_i \log \det (-\Delta_i) t^i \in \mathbb{R}[t];$$

  as its Taylor expansion $R(M, \rho) = \sum_j c_j(M, \rho)(t-1)^j$ at $t = -1$ contains $\log \tau(M, \rho)$ as $c_1(M, \rho)$, the other coefficients c_j can be considered as (logarithms of) *higher analytic torsion* [5] of M and ρ.)

- (Lichtenbaum) [18] For any smooth projective variety X over a finite field $\mathbb{F}_q$, the Weil-étale cohomology groups $H^i_W(X, \mathbb{Z})$ give a bounded complex $C^\bullet$ (of finitely generated abelian groups)

$$C^\bullet : \qquad \cdots H^i_W(X, \mathbb{Z}) \xrightarrow{\cup \theta} H^{i+1}_W(X, \mathbb{Z}) \cdots ;$$

  one has $\chi(C^\bullet) = 0$ and $\chi'(C^\bullet)$ is the order of vanishing of the zeta function $Z(X, t)$ at $t = 1$.

- (Grayson) [14, §3, p. 103] Let R be a commutative ring and N a finitely generated projective R-module. If $S^k CN$ is the k'th symmetric product of the mapping cone CN of the identity map on N, then Grayson's formula for the k'th Adams operation ψ^k reads

$$\psi^k[N] = \chi'(S^k CN).$$

(This raises the question: Is there a natural interpretation of $\chi_j(S^k CN)$ for $j > 1$?)

- (Fried) [11, Theorem 3] Let X be a closed oriented hyperbolic manifold of dimension $2n + 1 > 2$ and let ρ be an orthogonal representation of $\pi_1(X)$. Write V_ρ for the corresponding local system on X. The order of vanishing of the Ruelle zeta function $R_\rho(s)$ at $s = 0$ is given by

$$2 \sum_{i=0}^{i=n} (-1)^i (n + 1 - i) \dim H^i(X, V_\rho).$$

- (Bunke-Olbrich) [4] Given a locally symmetric space of rank one $Y = \Gamma \backslash G / K$ and a homogeneous vector bundle V (this depends on a pair σ, λ and the associated distribution globalization $V_{-\infty}$ (a complex representation of Γ), the order of vanishing of the Selberg zeta function $Z_S(s, \sigma)$ at $s = \lambda$ is given by $\chi'(\Gamma, V_\infty)$ (Patterson's conjecture)

$$\sum_i (-1)^{i+1} i \dim H^i(\Gamma, V_\infty).$$

3. K-THEORETIC VARIANTS

K-theory provides another general context to develop higher Euler characteristics.

As in [14], let $\mathcal{P}$ be an exact category with a suitable notion of tensor product (bi-exact), symmetric power and exterior power. Examples include the category $\mathcal{P}(X)$ of vector bundles over a scheme X, the category $\mathcal{P}(R)$ of finitely generated projective modules over a commutative ring R and for a fixed group Γ, the category $\mathcal{P}(\Gamma, R)$ of representations of Γ on finitely generated projective R-modules. For any object N in $\mathcal{P}$, let us write $[N]$ for the class of N in the Grothendieck ring $K_0(\mathcal{P})$. For any bounded complex M over $\mathcal{P}$, we write $\chi(M) = \sum_i (-1)^i [M_i] \in K_0(\mathcal{P})$.

Definition 3.1. (i) The higher Euler classes $\chi_j(M)$ of M are defined by

$$\chi_j(M) = \sum_i (-1)^{i-j} \binom{i}{j} [M_i] \in K_0(\mathcal{P}), \qquad j \geq 0.$$

(ii) The Poincaré function $P_M(t)$ is defined as

$$P_M(t) = \sum_i [M_i] t^i \in K_0(\mathcal{P})[t, t^{-1}].$$

Clearly, $\chi_0(M) = \chi(M)$ and $\chi_1(M) = \chi'(M)$. If M is concentrated in non-negative degrees, then $P_M(t)$ is the Poincaré polynomial of M. If $M[n]$ is the shifted complex (so that $M[n]_i = M_{i+n}$), then $t^n P_{M[n]}(t) = P_M(t)$. Defining $Q_M(u) \in K_0(\mathcal{P})[u, u^{-1}]$ by $Q_M(1 + t) = P_M(t)$, we have $(u - 1)^n Q_{M[n]}(u) = Q_M(u)$.

Theorem 3.2. *Let M and N be bounded complexes in $\mathcal{P}$ concentrated in non-negative degrees.*
(i) If $\chi_r(M)$ and $\chi_r(N)$ vanish for $0 \leq r < j$, then

$$\chi_k(M \otimes N) = 0, \quad \text{for } 0 \leq k < 2j$$
$$\chi_{2j}(M \otimes N) = \chi_j(M) . \chi_j(N),$$
$$\chi_{2j+1}(M \otimes N) = \chi_j(M) . \chi_{j+1}(N) + \chi_{j+1}(M) . \chi_j(N).$$

(ii) If $M = CN$ is the cone of a self-map $N \to N$, then $\chi_{j+1}(M) = \chi_j(N)$

(iii) Let $M = C^j(N) = C(\cdots C(N) \cdots)$ be an j-fold iterated cone on N. Then $\chi_j(M) = \chi(N)$ and $\chi_{k+j}(M) = \chi_k(N)$ for $k \geq 0$,

$$\chi_0(M) = 0, \cdots, \chi_{j-1}(M) = 0.$$

Proof. Defining $Q_M(u), Q_N(u) \in K_0(\mathcal{P})[u]$ by $Q_M(1+t) = P_M(t)$ and $Q_N(1+t) = P_N(t)$, the same arguments show that $Q_M(u) = \sum_j \chi_j(M)u^j$. This proves (i) as in Theorem 1.4.

(ii) As M is the total complex associated with CN, we have $M_0 = N_0$ and, for $i > 0$, that $M_i = N_i \oplus N_{i-1}$. Thus $P_M(t) = P_N(t)(1+t)$ which gives $Q_M(u) = uQ_N(u)$. This proves (ii). Part (iii) follows from (ii) by induction or one can observe that $Q_M(u) = u^j Q_N(u)$. □

Corollary 3.3. *Let M be a complex of $\mathcal{P}$ concentrated in non-negative degrees. If $\chi_0(M) \neq 0$, then $M \neq CN$. If $\chi_1(M) \neq 0$, then M is not the tensor product of two acyclic complexes.*

Remark 3.4. Theorem 3.2 is compatible with the intuition expressed in [14, p. 104]:

> ...we regard acyclic complexes as being infinitesimal in size when compared to arbitrary complexes. ....that we regard doubly acyclic complexes as being doubly infinitesimal in size when compared to arbitrary complexes. It also suggests that we regard the Adams operation ψ^k as being the differential of the functor $N \mapsto S^k N$ from the category of finitely generated projective modules to itself;....

Namely, for an acyclic complex C, one has $\chi_0(C) = 0$ but not always $\chi_1(C) = 0$; for the tensor product $C \times D$ of acyclic complexes, one has $\chi_0(C \otimes D) = 0 = \chi_1(C \otimes D)$ but not always $\chi_2(C \otimes D) = 0$. So C is like ϵ and $C \otimes D$ is like ϵ^2. The above text also suggests that for any functor $F : \mathcal{P} \to \mathcal{P}$, we regard $\chi_1(F(CN))$ as the differential of F and that $\chi_j(F(-))$ as a higher differential of F (when evaluated on acyclic complexes or their tensor products). Thus, the vanishing of $\chi_0(M), \chi_1(M), \cdots \chi_n(M)$ means M is like ϵ^n; the set of complexes which are infinitesimal of order n (like ϵ^n) gives a filtration on the acyclic complexes in $\mathcal{P}$. This provides an answer to **Q2** above. □

Homological Poincaré polynomials. Suppose for simplicity that $\mathcal{P}$ is an abelian category. For any short exact sequence of complexes

$$0 \to A \to B \to C \to 0$$

in $\mathcal{P}$, one has $P_B(t) = P_A(t) + P_C(t)$. This implies that

$$\chi_j(B) = \chi_j(A) + \chi_j(C).$$

However, a quasi-isomorphism $A \to B$ does not imply $P_A(t) = P_B(t)$. In order to get an invariant on the bounded derived category, it is convenient to define the homological Poincaré function of a bounded complex M as

$$P_M^h(t) = \sum_i [H^i(M)]t^i \in K_0(\mathcal{P})[t, t^{-1}].$$

Any quasi-isomorphism $A \to B$ implies $P_A^h(t) = P_B^h(t)$. However, for a short exact sequence as above, one does not have $P_B^h(t) = P_A^h(t) + P_C^h(t)$ in general. Using P^h, one can define $\chi_j^h(M) =$ homological higher Euler characteristics of M via the coefficients of $Q_M^h(t)$. By Euler's identity, $\chi_0(M) = \chi_0^h(M)$, but, in general, for $j > 0$, $\chi_j(M) \neq \chi_j^h(M)$ as $P_M(t)$ and $P_M^h(t)$ are not necessarily equal. For any short exact sequence as above, we have

$$\chi_1^h(B) = \chi_1^h(A) + \chi_1^h(C)$$

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only for acyclic complexes A, B and C. Theorems 1.4 and 3.2 remain valid with χ_j replaced with χ_j^h.

Remark 3.5. Suppose that the category $\mathcal{P}$ has a $\mathbb{Z}$-grading; an important example is the conjectural category of motives over a given field (the theory of weights give the $\mathbb{Z}$-grading).

For any object $M = \oplus_i M_i$, its component of weight $i \in \mathbb{Z}$ is M_i. We can define a Poincaré polynomial (in t, t^{-1}) as $P_M(t) \in K_0\mathcal{P}[t, t^{-1}]$ by $P_M(t) = \sum_i [M_i]t^i$. We can define the higher Euler characteristics $\chi_j(M)$ using $Q_M(u) \in K_0\mathcal{P}[u, u^{-1}]$ with $Q_M(1 + t) = P_M(t)$. Namely, $Q_M(u) = \sum_j \chi_j(M)u^j$. □

4. FINAL REMARKS

Motivic variants. Higher Euler characteristics can be defined in a motivic context.

(i) [2] For any subfield $k \hookrightarrow \mathbb{C}$, one has the category $\mathcal{M}_k^{AH}$ of absolute Hodge motives [6, p. 5] which has a natural $\mathbb{Z}$-grading coming from weights. Any smooth proper variety X over k defines an object $h(X)$ of $\mathcal{M}_k^{AH}$; using the weight decomposition of $h(X)$, one gets the motivic Poincaré polynomial of X and the motivic higher Euler characteristics $\chi_j^{mot}(X) \in K_0\mathcal{M}_k^{AH}$ of X The Betti realization gives a homomorphism $r : K_0\mathcal{M}_k^{AH}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}]$ of graded rings; the element $r(h(X))$ gives the usual Poincaré polynomial of $X(\mathbb{C})$ and thus the motivic higher Euler characteristics of X refine those of the topological space $X(\mathbb{C})$.

(ii) [21, 20] As motivic measures provide a refinement of the Euler characteristic in algebraic geometry, they provide many variants of higher Euler characteristics.

Consider the category Var_F of varieties (reduced separated schemes of finite type) over a field F. The Grothendieck ring $K_0\mathrm{Var}_F$ of varieties over F is defined as the quotient of the free abelian group on the set of isomorphism classes $[X]$ of varieties by the relations $[X] = [Y] + [X\backslash Y]$ where Y is a closed subvariety of X. The multiplication is induced by the product of varieties. When F is of positive characteristic, one needs also to impose the relation $[X] = [Y]$ for every surjective radical morphism $X \to Y$. A motivic measure is a ring homomorphism $K_0\mathrm{Var}_F \to R$ to a ring R. The Euler characteristic χ_c with compact support is the prototypical motivic measure: $\chi_c : K_0\mathrm{Var}_\mathbb{C} \to \mathbb{Z}$ is a ring homomorphism. Another motivic measure is the Poincaré characteristic $\mu_P : K_0\mathrm{Var}_\mathbb{C} \to \mathbb{Z}[u]$, determined by the following property: for any smooth proper variety X, one has $\mu_P(X) \in \mathbb{Z}[u]$ is the Poincaré polynomial of the topological space $X(\mathbb{C})$.

The higher Euler characteristics arise from the ring homomorphism $\mathbb{Z}[t] \to \mathbb{Z}$ sending t to -1, or the associated map of motivic measures $\mu_P \mapsto \chi_c$. Any pair of motivic measures related in a similar manner give rise to a notion of higher Euler characteristics. Given a motivic measure $\mu : K_0\mathrm{Var}_F \to A[t]$ and a homomorphism $A[t] \to A$ with associated motivic measure $\mu' : K_0\mathrm{Var}_F \to A[t] \to A$, we can consider the pair (μ, μ') as analogous to (μ_p, χ_c). Let us indicate an important example. For instance, the assignment $X \mapsto \mathrm{H}_X(u, v) := \sum_{p,q \geq 0} h^{p,q}(X)u^p v^q$, with X smooth projective, gives rise to the Hodge characteristic measure $\mu_\mathrm{H} : \mathrm{Var}(k) \to \mathbb{Z}[u, v]$. Since $\mathrm{H}_X(u, u)$ equals the Poincaré characteristic $\mu_P(X)$ of X, viewing $\mathbb{Z}[u, v] = \mathbb{Z}[u][v - u]$ provides an expansion of $\mu_\mathrm{H}(X) = \sum_j \chi_j^\mathrm{H}(X)(v - u)^j$. So we are dealing with the pair (μ_H, μ_P), related by the homomorphism $\mathbb{Z}[u, v] \to \mathbb{Z}[u]$ sending $u \mapsto u, v \mapsto u$. In this context, our higher Euler characteristics are the coefficients $\chi_j^\mathrm{H}(X) \in \mathbb{Z}[u]$. Note that $\chi_0^\mathrm{H}(X) \in \mathbb{Z}[u]$ is $\mu_P(X)$.

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[2]Motivic conjectures predict analogous results over arbitrary fields.

Remark 4.1. Suppose $\mathcal{P}$ is a $\mathbb{Z}$-graded neutral $\mathbb{Q}$-linear Tannakian category. For any object M of $\mathcal{P}$, the higher Euler characteristics $\chi_j(\mathrm{Sym}^n M) \in K_0\mathcal{P}$ are determined by the Poincaré polynomial $P_M(t)$: in this context, one has an abstract Macdonald formula [6, p. 8]. This generalizes Proposition 1.10.

Finite categories. C. Berger and T. Leinster [1, 17] have provided and studied various definitions of the Euler characteristic of a finite category $\mathcal{C}$. The series Euler characteristic [1, 2.3] $\chi(\mathcal{C})$ is defined to be value at $t = -1$ of a formal power series $f_{\mathcal{C}}(t) \in \mathbb{Q}(t)$. Define $g(u) \in \mathbb{Q}(u)$ by $g_{\mathcal{C}}(t + 1) = f_{\mathcal{C}}(t)$; so $g(0) = \chi(\mathcal{C})$. Then the higher Euler characteristics $\chi_j(C)$ of $\mathcal{C}$ are the coefficients of $g(u)$:

$$g(u) = \sum_j \chi_j(\mathcal{C})u^j.$$

Since g could have a pole at $u = 0$, this even gives a definition of lower Euler characteristics!

We end this paper with the

Question 4.2. (i) Given a ring homomorphism $f : A \to B$ between two commutative rings, consider the ideal J of $K_0(A)$ defined as

$$J = \mathrm{Ker}(f_* : K_0(A) \to K_0(B)).$$

Given a bounded complex X of finitely generated projective A-modules whose class lies in J, how to determine the integer r such that the class of X is in $J^r - J^{r+1}$?

(ii) Is there an analogue of Theorem 1.4 for higher analytic torsion [5]? Is the analytic Poincare polynomial (6) of a product $M \times N$ determined by that of M and N?

(iii) Is there an analogue of our results in the context of Kapranov's N-complexes [15]?

Island where all becomes clear.
Solid ground beneath your feet.
The only roads are those that offer access.
Bushes bend beneath the weight of proofs.
The Tree of Valid Supposition grows here
with branches disentangled since time immemorial.
The Tree of Understanding, dazzlingly straight and simple,
sprouts by the spring called Now I Get It.
The thicker the woods, the vaster the vista:
the Valley of Obviously.
If any doubts arise, the wind dispels them instantly.
Echoes stir unsummoned
and eagerly explain all the secrets of the worlds.
On the right a cave where Meaning lies.
On the left the Lake of Deep Conviction.
Truth breaks from the bottom and bobs to the surface.
Unshakable Confidence towers over the valley.

Its peak offers an excellent view of the Essence of Things.
For all its charms, the island is uninhabited,
and the faint footprints scattered on its beaches
turn without exception to the sea.
As if all you can do here is leave
and plunge, never to return, into the depths.
Into unfathomable life.

- W. Szymborska, *Utopia* (A large number, 1976)

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