Zero-one laws for multigraphs

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Motivation

Question

How does infinite structure arise as a "limit" of finite structure?

One answer: use model theoretic methods to "build" an infinite structure from larger and larger finite structures, for example, a Fraïssé limit.

Another answer: consider structure which arises "almost surely" from larger and larger finite structures.

Logical 0-1 laws

Fix a finite first-order language $\mathcal{L}$. For each n suppose $\mathbb{K}(n)$ is a set of $\mathcal{L}$-structures with universe $[n] = \{1, \ldots, n\}$. Set $\mathbb{K} = \bigcup_{n\in\mathbb{N}} \mathbb{K}(n)$.

Definition

We say $\mathbb{K}$ has a 0-1 *law* if for every $\mathcal{L}$-sentence ϕ,

$$\mu(\phi) = \lim_{n\to\infty} \frac{|\{G \in \mathbb{K}(n) : G \models \phi\}|}{|\mathbb{K}(n)|}.$$

is either 0 or 1.

The *almost sure theory* of $\mathbb{K}$ is $T_{as}(\mathbb{K}) = \{\phi \in \mathcal{L} : \mu(\phi) = 1\}$.

If $\mathbb{K}$ has a 0-1 law, then $T_{as}(\mathbb{K})$ is complete.

Example 1: graphs

Let $\mathcal{L} = \{R\}$ where R is a binary relation symbol.

Example (Glebskiĭ et. al. 1969, Fagin 1976)

Suppose $\mathbb{G}(n)$ is the set of all graphs with vertex set $[n]$ and $\mathbb{G} = \bigcup_{n \in \mathbb{N}} \mathbb{G}(n)$.

- $\mathbb{G}$ has a 0-1 law.
- $T_{as}(\mathbb{G})$ is the theory of the Fraïssé limit of the class of all finite graphs.
- This is called the theory of the “random graph.”

Two methods build the same theory.

Example 2: Triangle-free graphs

Let $\mathbb{F}(n)$ be the set of all triangle-free graphs with vertex set $[n]$ and $\mathbb{F} = \bigcup_{n \in \mathbb{N}} \mathbb{F}(n)$.

Let $\mathbb{B}(n)$ be the set of all bipartite graphs with vertex set $[n]$ and $\mathbb{B} = \bigcup_{n \in \mathbb{N}} \mathbb{B}(n)$.

Recall that for all n, $\mathbb{B}(n) \subseteq \mathbb{F}(n)$.

Theorem (Erdős, Kleitman, Rothschild 1976)

$$\lim_{n \to \infty} \frac{|\mathbb{B}(n)|}{|\mathbb{F}(n)|} = 1.$$

"almost all triangle-free graphs are bipartite."

Example 2: Triangle-free graphs

Theorem (Kolaitis, Prömel, Rothschild 1987)

The family $\mathbb{B}$ of bipartite graphs has a 0-1 *law.*

Corollary

The family $\mathbb{F}$ of triangle-free graphs has a 0-1 law and $T_{as}(\mathbb{F}) = T_{as}(\mathbb{B})$.

Example 2: Triangle-free graphs

The class of finite triangle free graphs has a Fraïssé limit called the Henson graph. Let T_{hg} denote its theory.

Let ϕ be the sentence saying "I contain a 5-cycle." Then

$$T_{hg} \models \phi.$$

Fact: a graph G is bipartite if and only if it omits all odd cycles. Therefore

$$T_{as}(\mathbb{F}) = T_{as}(\mathbb{B}) \models \neg\phi.$$

For triangle free graphs, the theory of the Fraïssé limit is not the almost sure theory.

Two methods give different answers.

Outline

Structure theorem for $\mathbb{F}$ + 0-1 law for $\mathbb{B} \Rightarrow$ 0-1 law for $\mathbb{F}$.

Goal: generalize to other first-order languages.

Recipe for new 0-1 laws:

Structure theorem

$$+$$

0-1 law for nice family

$$\Downarrow$$

0-1 law for more complicated family.

Weighted graphs

Given a set X, let $\binom{X}{2} = \{Y \subseteq X : |Y| = 2\}$.

Definition

A *weighted graph* is a pair (V, w) where V is a set of vertices and $w : \binom{V}{2} \to \mathbb{N}$ is a *weighting*.

Given a weighted graph (V, w) and $X \subseteq V$, set $S(X) = \sum_{xy \in \binom{X}{2}} w(x, y)$. Fix integers $k \geq 1$ and $r \geq 2$.

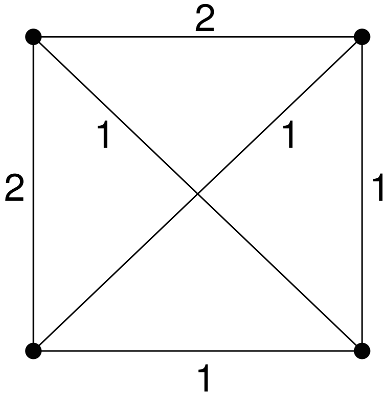
Definition

A (k, r)*-graph* is a weighted graph (V, w) with the property that every k-element set $X \subseteq V$, $S(X) \leq r$.

Definition

Given integers $k \geq 3$, $r \geq 2$ and $n \in \mathbb{N}$, let $\mathbb{F}_{k,r}(n)$ be the set of (k, r)-graphs with vertex set $[n]$. Let $\mathbb{F}_{k,r} = \bigcup_{n \in \mathbb{N}} \mathbb{F}_{k,r}(n)$.

$(k, r) = (3, 5)$



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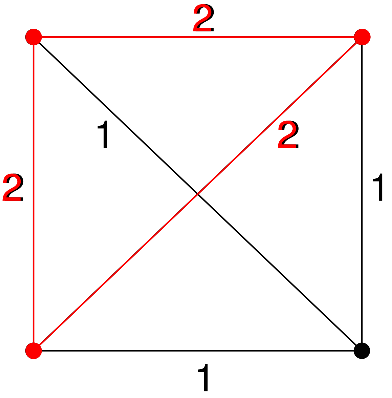

NO!

Questions

Note: in any (k, r)-graph there are no edges of weight greater than r.

Let $\mathcal{L}_r = \{R_0, \dots, R_r\}$ consist of k binary relation symbols.

Questions

Given fixed k and r and large n:

1. What does a typical element in $\mathbb{F}_{k,r}(n)$ look like?
2. Does $\mathbb{F}_{k,r}$ have a 0-1 law?
3. If so, how does it compare to other limit theories associated to this family?

Questions 1 and 2 may be extremely complicated depending on the values of k and r.

The two most tractable cases are:

1. $k = a\binom{r}{2}$ for some integer $a \geq 1$.
2. $k = a\binom{r}{2} - 1$ for some integer $a \geq 1$.

Nice Subfamilies

Definition

Given $a \geq 1$ and n, set

$$\mathbb{U}_a(n) = \left\{ G \in \mathbb{F}_{k,r}(n) : \forall xy \in \binom{[n]}{2}, w^G(x,y) \leq a \right\}.$$

Note $\mathbb{U}_a(n) \subseteq \mathbb{F}_{k,a\binom{k}{2}}(n)$ for all n.

Definition

Given $a \geq 1$ and n, set $\mathbb{T}_{a,k}(n)$ to be the elements $G \in \mathbb{F}_{k,r}$ such that there is a partition $P_1, \ldots, P_{k-1}$ of $[n]$ with the following properties.

- For all $1 \leq i \leq k-1$ and $xy \in \binom{P_i}{2}$, $w^G(x,y) \leq a-1$,
- For all $1 \leq i \neq j \leq k-1$ and $x \in P_i$, $y \in P_j$, $w^G(x,y) \leq a$.

Note $\mathbb{T}_{a,k}(n) \subseteq \mathbb{F}_{k,a\binom{k}{2}-1}(n)$ for all n.

Approximate Structure

Definition

Given $\delta > 0$ and weighted graphs G and G' with the same vertex set V, we say G and G' are *δ-close* if $\left|\left\{(x,y) \in V^2 : w^G(x,y) \neq w^{G'}(x,y)\right\}\right| \leq \delta n^2$.

Given $\delta > 0$, set

$$\mathbb{U}_a^\delta(n) = \{G \in \mathbb{F}_{k,r}(n) : G \text{ is } \delta\text{-close to an element of } \mathbb{U}_a(n)\}$$

and

$$\mathbb{T}_{a,k}^\delta(n) = \{G \in \mathbb{F}_{k,a\binom{k}{2}-1}(n) : G \text{ is } \delta\text{-close to an element of } \mathbb{T}_{a,k}(n)\}.$$

Approximate Structure

Theorem (Mubayi, T.)

For $\delta > 0$,

$$\lim_{n\to\infty} \frac{|\mathbb{U}_a^\delta(n)|}{|\mathbb{F}_{k,a\binom{k}{2}}(n)|} = 1$$

and

$$\lim_{n\to\infty} \frac{|\mathbb{T}_{a,k}^\delta(n)|}{|\mathbb{F}_{k,a\binom{k}{2}-1}(n)|} = 1.$$

Proof uses the hypergraph container method and stability theorems.

Case $r = a\binom{k}{2}$

Theorem (Mubayi, T.)

$$\lim_{n\to\infty} \frac{|\mathbb{U}_a(n)|}{|\mathbb{F}_{k,a\binom{k}{2}}(n)|} = 1.$$

Let $\mathbb{U}_a = \bigcup_{n\in\mathbb{N}} \mathbb{U}_a(n)$.

A standard argument shows that $\mathbb{U}_a$ has a 0-1 law.

Corollary

$\mathbb{F}_{k,a\binom{k}{2}}$ has a 0-1 law and $T_{as}(\mathbb{F}_{k,a\binom{k}{2}}) = T_{as}(\mathbb{U}_a)$.

Structure theorem + 0-1 law $\Rightarrow$ new 0-1 law.

Case $r = a\binom{k}{2}$ and $k = 3$.

Fact

When $k = 3$, the class of finite (k, r) graphs has a Fraïssé limit. Denote this theory $T_{fl}(\mathbb{F}_{3,3a})$.

Note for any $a \geq 1$,

$$T_{fl}(\mathbb{F}_{3,3a}) \models \text{“ there is an edge of weight } a+1\text{”.}$$

While

$$T_{as}(\mathbb{F}_{3,3a}) = T_{as}(\mathbb{U}_a) \models \text{“ there is no edge of weight } a+1\text{”.}$$

Therefore $T_{fl}(\mathbb{F}_{3,3a}) \neq T_{as}(\mathbb{F}_{3,3a})$.

Things are slightly more complicated when $k > 3$.

Case $r = a\binom{k}{2} - 1$

Conjecture (work in progress)

1. $\lim_{n\to\infty} \frac{|\mathbb{T}_{a,k}(n)|}{|\mathbb{F}_{k,a\binom{k}{2}-1}(n)|} = 1$.
2. $\mathbb{F}_{k,a\binom{k}{2}-1}$ has a 0-1 law.
3. The almost sure theory will disagree with any naturally corresponding Fraïssé limit.

Thank you for listening!