Discrete metric spaces: structure and enumeration

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$Forb_n(H)$: a history

Notation: Given $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$.
$K_n =$ the complete graph on n vertices.

Definition

Fix a finite graph H and integer n. Define $Forb_n(H)$ to be the set of graphs G with the following properties:

- $V(G) = [n]$ and
- G omits H as a (non-induced) subgraph.

Definition

Given $l \geq 2$, $Col_n(l)$ is the set of l-colorable graphs with vertex set $[n]$.

Recall for all $l \geq 2$ and n, $Col_n(l) \subseteq Forb_n(K_{l+1})$.

$Forb_n(H)$: a history

Case $H = K_3$:

Theorem (Erdős, Kleitman, Rothschild, 1976)

1. *Structure:*

$$\lim_{n\to\infty} \frac{|Col_n(2)|}{|Forb_n(K_3)|} = 1.$$

2. *Enumeration:*

$$|Forb_n(K_3)| = \left(1 + o\left(\frac{1}{n}\right)\right)|Col_2(n)| = 2^{\left(\frac{1}{2}\right)\frac{n^2}{2} + o(n^2)}.$$

$Forb_n(H)$: a history

There are many extensions and generalizations of this to other families of the form $Forb_n(H)$:

Theorem (Kolaitis, Prömel, Rothschild, 1987)

Fix $H = K_{l+1}$ for $l \geq 2$.

1. *Structure:*
$$\lim_{n \to \infty} \frac{|Col_n(l)|}{|Forb_n(K_{l+1})|} = 1.$$
2. *Enumeration: for any $p \geq 1$,*
$$|Forb_n(K_{l+1})| = \left(1 + o\left(\frac{1}{n^p}\right)\right)|Col_n(l)| = 2^{(1-\frac{1}{l})\frac{n^2}{2}+o(1)}.$$

$Forb_n(H)$: a history

Theorem (Prömel, Steger, 1992)

Suppose $l \geq 2$. Suppose H has $\chi(H) = l + 1$ and contains a color-critical edge.

1. *Structure:*
$$\lim_{n\to\infty} \frac{|Col_n(l)|}{|Forb_n(H)|} = 1.$$

2. *Enumeration: for any $p \geq 1$,*
$$|Forb_n(H)| = \left(1 + o\left(\frac{1}{n^p}\right)\right)|Col_n(l)| = 2^{(1-\frac{1}{l})\frac{n^2}{2}+o(1)}.$$

Metric spaces

Definition

Fix an integer $r \geq 2$. $M_r(n)$ is the set of metric spaces with underlying set $[n]$ and distances all in $[r]$.

Given a set X, $\binom{X}{2} = \{Y \subseteq X : |Y| = 2\}$.

Definition

Fix an integer $r \geq 2$. A *simple complete r-graph* is a pair (V, c) where V is a set of vertices and $c : \binom{V}{2} \to [r]$ is a function. c is called a *coloring*.

Elements $G \in M_r(n)$ are naturally simple complete r-graphs: just color edges xy with the distance $d(x, y)$.

Analogy

Definition

- A *violating triangle* is an r-graph $H = (V, c)$ with $V = \{x, y, z\}$ such that for some $i, j, k \in [r]$, with $i > j + k$,

$$c(x, y) = i,\ c(x, z) = j,\ \text{and } c(y, z) = k.$$

- Given two r-graphs G and H, G *omits* H, if for all injections $f : V(H) \to V(G)$, there is $xy \in \binom{V(H)}{2}$ such that $c^H(x, y) \neq c^G(f(x), f(y))$.

Observation:
$M_r(n)$ is the set of all simple and complete r-graphs G such that

- $V(G) = [n]$,
- G omits all violating triangles.

Questions

Questions

1. Given a fixed r, what is the asymptotic structure of elements of $M_r(n)$?
2. $|M_r(n)| = ???$

Metric sets

Definition

$A \subseteq [r]$ is a *metric set* if for all $a, b, c \in A$, $a \leq b + c$.

Notation: Given $s < r \in \mathbb{N}$, $[s, r] = \{s, s+1, \ldots, r\}$.

Lemma (Mubayi, T.)

- When $r \geq 2$ is even, $[\frac{r}{2}, r]$ is a unique largest metric subset of $[r]$.
- When $r \geq 3$ is odd, $[\frac{r-1}{2}, r-1]$, $[\frac{r+1}{2}, r]$ are the two largest metric subsets of $[r]$.

Example

If $r = 4$, $\{2, 3, 4\}$ is the unique largest metric subset of $[r]$.
If $r = 5$, $\{2, 3, 4\}$ and $\{3, 4, 5\}$ are the two largest metric subsets of $[r]$.

When r is odd, let $U_r = [\frac{r+1}{2}, r]$ and $L_r = [\frac{r-1}{2}, r-1]$.

$C_r(n)$

We now define a special subfamily $C_r(n) \subseteq M_r(n)$.
Idea: $C_r(n)$ contains only distances in the "top half" of $[r]$.

Definition

When r is even, $C_r(n) = \{G \in M_r(n) : \text{for all } a, b \in G, d(a,b) \in [\frac{r}{2}, r]\}$.

Example

When r is 4, this is the set of metric spaces on $[n]$ with all distances in $\{2, 3, 4\}$.

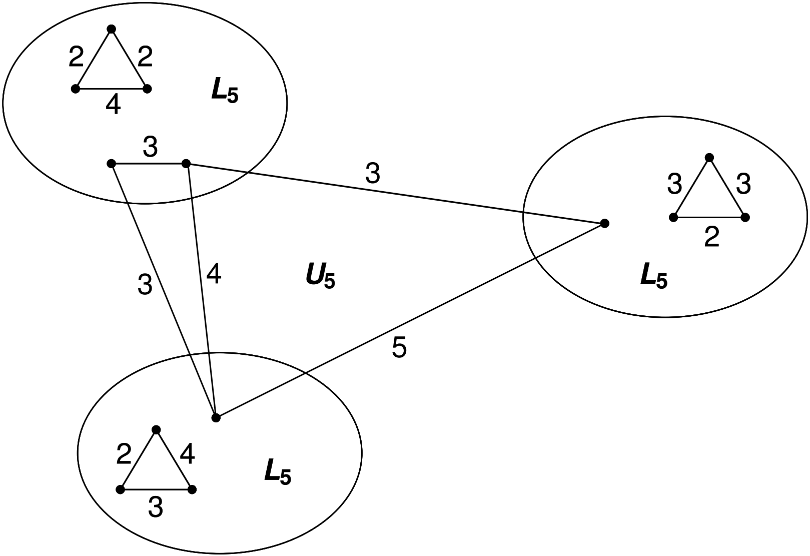
Definition

When r is odd, $C_r(n)$ is the set of all $G \in M_r(n)$ with the following property. There is a partition $P_1, \ldots, P_l$ of $[n]$ such that

- For all i and $ab \in \binom{P_i}{2}$, $d(a,b) \in L_r$.
- For all $i \neq j$, $(a,b) \in P_i \times P_j$, $d(a,b) \in U_r$.

$C_r(n)$

For example, when $r = 5$, an element of $C_r(n)$ could look like:



Counting $C_r(n)$

Observation:
When r is even $|[\frac{r}{2}, r]| = \lceil \frac{r+1}{2} \rceil$.
When r is odd, $|L_r| = |U_r| = \lceil \frac{r+1}{2} \rceil$.

When $r \geq 2$ is even,

$$|C_r(n)| = \left| \left[\frac{r}{2}, r \right] \right|^{\binom{n}{2}} = \left\lceil \frac{r+1}{2} \right\rceil^{\binom{n}{2}}.$$

When $r \geq 3$ is odd,

$$\left\lceil \frac{r+1}{2} \right\rceil^{\binom{n}{2}} \leq |C_r(n)| \leq n^n \left\lceil \frac{r+1}{2} \right\rceil^{\binom{n}{2}} = \left\lceil \frac{r+1}{2} \right\rceil^{\binom{n}{2}+o(n^2)}.$$

Questions

Questions

1. Given a fixed r, what is the asymptotic structure of elements of $M_r(n)$?
2. $|M_r(n)| =???$

Approximate structure

Definition

Given $\delta > 0$ and two elements $G, G' \in M_r(n)$, we say G and G' are δ-*close* if

$$\left|\left\{ab \in \binom{[n]}{2} : d^G(a,b) \neq d^{G'}(a,b)\right\}\right| \leq \delta n^2.$$

Theorem (Mubayi, T.)

For all $r \geq 2$ *and* $\delta > 0$*,*

$$\lim_{n\to\infty} \frac{\{G \in M_r(n) : G \textit{ is } \delta\textit{-close to an element of } C_r(n)\}|}{|M_r(n)|} = 1.$$

Proof uses multi-color version of Szemeredi's regularity lemma and a stability theorem.

Counting

Corollary (Mubayi, T.)

For all $r \geq 2$,

$$|M_r(n)| = \left\lceil \frac{r+1}{2} \right\rceil^{\binom{n}{2}+o(n^2)}.$$

The even case

Theorem (Mubayi, T.)

When r is even,

$$\lim_{n\to\infty} \frac{|C_r(n)|}{|M_r(n)|} = 1.$$

Corollary (Mubayi, T.)

When $r \geq 2$ is even

$$|M_r(n)| = (1 + o(1))|C_r(n)| = \left\lceil \frac{r+1}{2} \right\rceil^{\binom{n}{2}+o(1)}.$$

The odd case

Theorem (Mubayi, T.)

When $r \geq 3$ is odd,

$$\lim_{n\to\infty} \frac{|C_r(n)|}{|M_r(n)|} < 1.$$

Moreover,

$$|M_r(n)| \geq \left\lceil \frac{r+1}{2} \right\rceil^{\binom{n}{2}+\Omega(n\log n)}.$$

Open questions

When r is odd:

- What is the fine structure of $M_r(n)$?
- $|M_r(n)| = \lceil \frac{r+1}{2} \rceil^{\binom{n}{2} + ???}$.
- What is different about the even and odd cases?

Thank you for listening!