**Hmwk 1**: Due this FRIDAY by 12noon

**REVIEW**: Green's function $G(x,y)$

$G(x,y)$ helps express solution $u(x)$ of problem: $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$ ①

in terms of $f, g$.


**DEF.** $G(x,y) := \underbrace{\Phi(y-x)}_{\text{fund. soln; blows up at } y=x!} - \overbrace{\varphi^x(y)}^{\text{"corrector field"}}$ ; $x, y \in U$, $x \neq y$

$\begin{cases} \Delta \varphi^x(y) = 0 & \text{in } U \\ \varphi^x(y) = \Phi(y-x) & \text{on } \partial U \end{cases}$ (fixed $x \in U$)

**THM** If $u \in C^2(\bar{U})$ solves ①, then

$$u(x) = \int_U G(x,y)\, f(y)\, dy - \int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y), \quad x \in U.$$

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In principle, it is difficult to construct $G(x,y)$ explicitly.

**Digression:**
Another way to introduce the Green function is via the "Dirac measure with unit mass to the point $x$" $\delta_x(y) = \delta(y-x)$ ← "delta function"

For fixed $x$, $G(x,y)$ satisfies $\begin{cases} -\Delta_y G(x,y) = \delta_x(y) & \text{in } U \\ G = 0 & \text{on } \partial U. \end{cases}$

**THM 1** (Symmetry) $G(x,y) = G(y,x)$ $\forall x, y \in U$, $x \neq y$.

~ Proof: See pp. 35, 36 of Evans ~

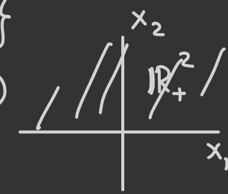
In what $\overset{\text{geometries}}{\text{cases}}$ is $G(x,y)$ $\overbrace{\text{"easily"}}^{G \text{ can be given in terms of } \Phi}$ constructed explicitly?

(i) Half space ; (ii) Ball

## (i) Half space

Consider the set $\quad \mathbb{R}^n_+ = \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0 \}$

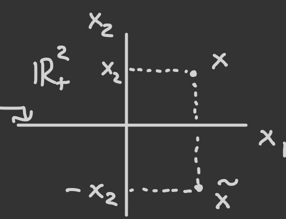
(n=2)




**DEF. 1** Green's function for $\mathbb{R}^n_+$ is

$$G(x,y) = \Phi(y-x) - \underbrace{\Phi(y-\tilde{x})}_{\varphi^x(y)}$$

$\partial U = \partial \mathbb{R}^2_+$
$(U = \mathbb{R}^2_+)$




sign reversal

where $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$ is the reflection in plane $\partial \mathbb{R}^n_+$ of point $x = (x_1, \dots, x_{n-1}, x_n)$.

$\boxed{\text{Omit THM 14 on p. 37 of Evans}}$ ~ Poisson's formula for half space

## (ii) Ball

Without loss of generality, let's consider the unit ball $B(0,1)$.

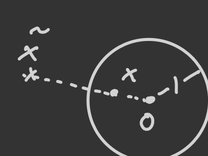
**DEF. 2** Green's function for the unit ball is

$$G(x,y) := \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \qquad \forall\, x,y \in B(0,1),\ x \neq y,$$

?

where $\tilde{x}$ is the point "dual" to $x$ with respect to $\partial B(0,1)$,

$$\tilde{x} = \frac{x}{|x|^2}, \qquad x \neq 0$$




— End of DEF. 2 —

**REMARK:** The mapping $x \mapsto \tilde{x}$ defines inversion through the unit sphere.

If $|x| < 1 \Rightarrow |\tilde{x}| > 1$

$|x| > 1 \Rightarrow |\tilde{x}| < 1$

$\boxed{\text{Read pp. 39-41 of Evans}}$

$\boxed{\text{OMIT THEOREM 15 on p. 41 of Evans}}$ Poisson's formula for ball.

## Energy Methods

Useful methodology for a wide class of PDEs.
It provides an indirect way for uniqueness of solutions via regularity assumptions.

**Example 1** Consider the boundary value problem (BVP)

$$(1)\ \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}, \quad u \in C^2(\bar{U}),\ f \in C(\bar{U}),\ g \in C(\partial U).$$

Note: Recall that we proved uniqueness of $u$ by max/min principles.
Now we are going to use an "energy method."

**THM 2** There exists <u>at most</u> one solution $u$ of (1), $u \in C^2(\bar{U})$.

**Proof**: Suppose both $u, \check{u}$ satisfy (1). Consider $w = u - \check{u}$.

$$\Rightarrow \begin{cases} -\Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}. \qquad \text{Suppose } w \neq 0$$


$$0 = \int_U -w\underbrace{\Delta w}_{\operatorname{div}(Dw)}\,dx = \int_U Dw\cdot Dw\,dx - \cancel{\int_{\partial U} \overset{0}{\overset{\shortparallel}{w}}\,\nu\cdot Dw\,dS} \to 0$$

$$\Rightarrow 0 = \int_U |Dw|^2\,dx \geq 0 \quad \Rightarrow \quad Dw \equiv 0 \text{ in } U$$

$$\Rightarrow \quad w \equiv 0 \text{ in } U \qquad (w = 0 \text{ on } \partial U,\ w \in C^2(\bar{U}))$$

$\square$

Ex. 2 (Modified Helmholtz eq.)

$$②\begin{cases} -\Delta u + k^2 u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}; \qquad (k^2>0) \quad u \in C^2(\bar{U})$$

Show uniqueness of solution

Proof: Let $w := u - \check{u}$ where $u, \check{u}$ satisfy ②.

$\llcorner$ Then $\begin{cases} -\Delta w + k^2 w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$

$$0 = \int_U -w\Delta w + k^2 w^2 \, dx = \int_U |Dw|^2 + k^2 w^2 \, dx \geq 0$$

$$\Rightarrow \{ Dw \equiv 0 \text{ and } w \equiv 0 \}$$

$\square$

Prob. 4, Hmwk 1: $\begin{cases} -\Delta u + k^2 u = 0 & \text{in } U \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$

Hint: Start w/ an integral that has $\frac{\partial w}{\partial \nu}$ inside the integral.

$$E = \int_U \tfrac{1}{2}|Dw|^2 + \tfrac{1}{2}k^2 w^2 \, dx + c\int_{\partial U} w \frac{\partial w}{\partial \nu} \, dS \; ;$$

suitable constant c must be found

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Go back to Poisson problem: $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$ ① $(u \in C^2(\bar{U}))$

DEF. 3 The energy functional for ① is

$$I[w] := \int_U \tfrac{1}{2}|Dw|^2 - wf \, dx$$

(w does NOT have to obey PDE in ①)

where $w \in A := \{ w \in C^2(\bar{U}) \; ; \; w = g \text{ on } \partial U \}$.

**THM 3** (Dirichlet Principle) The function $u \in C^2(\bar{U})$ solves ①

IF and ONLY IF

$$I[u] = \min_{w \in A} I[w].$$

**Proof** (a) If $u \in C^2(\bar{U})$ obeys $\left.\begin{matrix} -\Delta u = f \text{ in } U \\ u = g \text{ on } \partial U \end{matrix}\right\}$ ① THEN $I[u] \leq I[w]$ for every $w \in A$.

(b) If $I[u] = \min_{w \in A} I[w]$ THEN $u$ obeys ①.

**Prove (a) first.** arbitrary $w \in A = \{ w \in C^2(\bar{U}) : w = g \text{ on } \partial U \}$

$$0 = \int_U (-\Delta u - f)(u-w)\, dx = \int_U Du \cdot D(u-w)\, dx - \int_U f(u-w)\, dx$$

where are the bdry terms ?

$$-\int_{\partial U} \frac{\partial u}{\partial \nu} (u-w)\, dS$$

($u = g$, $w = g$)

$$\Rightarrow \int_U |Du|^2 - fu \; dx = \int_U \underbrace{Du \cdot Dw}_{\leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2} - fw \; dx \qquad (*)$$

Cauchy-Schwartz inequality: $Du \cdot Dw \leq \underbrace{|Du|\,|Dw|}_{\leq \frac{|Du|^2}{2} + \frac{|Dw|^2}{2}}$

Cauchy " :

Thus,

$$(*): \int_U |Du|^2 - fu \; dx \leq \frac{1}{2}\int_U |Du|^2\, dx + \int_U \frac{1}{2}|Dw|^2 - fw \; dx$$

$$\Rightarrow \underbrace{\int_U \frac{1}{2}|Du|^2 - fu \; dx}_{I[u]} \leq \underbrace{\int_U \frac{1}{2}|Dw|^2 - fw \; dx}_{I[w]} \qquad \forall w \in A$$

(b) Show that if $I[u] = \min_{w \in A} I[w]$ THEN $\begin{cases} -\Delta u = f \text{ in } U \\ u = g \text{ on } \partial U \end{cases}$