MATH / AMSC 673 LECTURE 8 09/24/2020

Hmwk 1 is DUE by 12:00 noon this FRIDAY

Reading: Evans 2.3

REVIEW: Energy methods

Consider the boundary value problem (BVP) $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$. ①

DEF. The energy functional for ① is

$$I[w] = \int_U \tfrac{1}{2}|Dw|^2 - wf \, dx \quad \text{where } w \in \mathcal{A} := \{ w \in C^2(\bar{U}) : w = g \text{ on } \partial U \}.$$

THM (Dirichlet Principle) $u \in C^2(\bar{U})$ solves ① IFF $I[u] = \min_{w \in \mathcal{A}} I[w]$.

Proof: (i) If u solves ① THEN $I[u] \le I[w] \quad \forall w \in \mathcal{A}$ ✓

(ii) If $I[u] = \min_{w \in \mathcal{A}} I[w]$ THEN $u \in C^2(\bar{U})$ solves ①.

Suppose $I[u] = \min_{w \in \mathcal{A}} I[w]$; $\mathcal{A} = \{ w \in C^2(\bar{U}) : w = g \text{ on } \partial U \}$

We want to consider a small "change" of u: $w = u + \tau v$, $\tau \in \mathbb{R}$

$v \in C_c^\infty(U)$ (v: "test function")

└ of compact support

Notice that $w = u + \tau v \in \mathcal{A}$.

Consider $\varphi(\tau) := I[u + \tau v]$; $\varphi : \mathbb{R} \to \mathbb{R}$ $(\tau \in \mathbb{R})$

By hypothesis, $\varphi(\tau)$ must have a min at $\tau = 0$.

$$\varphi(\tau) = \int_U \tfrac{1}{2} \overbrace{|Du + \tau Dv|^2}^{D \cdot D} - (u + \tau v) f \, dx$$

$$= \int_U \tfrac{1}{2}\left(|Du|^2 + \tau^2 |Dv|^2 + 2\tau \, Du \cdot Dv \right) - (u + \tau v) f \, dx$$

quadratic function of τ

Must have: $0 = \varphi'(0) = \int_U \overbrace{Du \cdot Dv}^{\text{integration by parts}} - vf \, dx = \int_U (-\Delta u - f) v \, dx \quad \forall v \in C_c^\infty(U)$

$\Rightarrow -\Delta u = f$ in U $\square$

Heat or Diffusion Eq.

$u_t - \Delta u = 0$ in $U\times(0,\infty)$ or $u_t - \Delta u = f$ in $U\times(0,\infty)$; $u=u(x,t)$,

where $u: \bar{U}\times(0,\infty) \to \mathbb{R}$, $f: U\times(0,\infty)\to\mathbb{R}$.

First, we will explore "classical solutions" for $u(x,t)$.

In particular, consider the initial value problem (IVP)

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U\times(0,\infty) \\ u = g & \text{on } U\times\{t=0\} \end{cases}$$

$\boxed{\text{Consider } U=\mathbb{R}^n}$

Let us consider the "fundamental solution" (for $U=\mathbb{R}^n$).

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty) \qquad (2)$$

DEF. 1 (Invariance of solution under dilation scaling)

Define the dilation scaling $u \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t)$, $\forall \lambda>0$, $\alpha,\beta\in\mathbb{R}$.

Suppose that $u(x,t)$ obeys PDE (2). Then, u is called invariant under dilation scaling if

"Self-similar solution"

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t) \quad \forall \lambda>0 \,;\; (x,t)\in\mathbb{R}^n\times(0,\infty)$$

—·— (End of DEF. 1)

Set $\lambda = t^{-1}$: $\quad u(x,t) = t^{-\alpha} u(t^{-\beta}x, 1) =: t^{-\alpha} v\left(\frac{x}{t^\beta}\right)$; $y = \frac{x}{t^\beta}$.

Goal: What can α,β and $v(y)$ be $\left(y=\frac{x}{t^\beta}\right)$ so that this $u(x,t)$ satisfies the heat eqn. (2) in $\mathbb{R}^n\times(0,\infty)$? $(u_t - \Delta u = 0)$

- $u_t = -\alpha t^{-\alpha-1} v(y) - \beta x\, t^{-\alpha-\beta-1}\cdot Dv(y) = -\alpha t^{-\alpha-1} v(y) - \beta t^{-\alpha-1} y\cdot Dv(y)$
- $\Delta_x u = t^{-\alpha-2\beta}\Delta_y v(y)$

$$u_t = \Delta_x u \iff \underline{-\alpha t^{-\alpha-1} v(y) - \beta t^{-\alpha-1} y\cdot Dv(y)} = \underline{t^{-\alpha-\overset{1}{2\beta}}\Delta_y v(y)} \qquad (*)$$

It would be necessary to end up with an eqn in y only!

Can we pick α or β so that the factors of $t^{(\cdot)}$ cancel out?

Pick $\quad 2\beta = 1 \quad \Rightarrow \quad \boxed{\beta = \frac{1}{2}}$

⊛ gives: $\quad \alpha v(y) + \frac{1}{2}\, y \cdot Dv + \Delta v = 0 \qquad$ (eqn in y only!)

Look for radial solutions: $\quad v(y) =: w(\underbrace{|y|}_{r}) \;;\quad y = \frac{x}{t^\beta} = \frac{x}{\sqrt{t}}$.

$\Delta v = w''(r) + \frac{n-1}{r} w'(r) \qquad$ (Exercise) $\qquad \left(\Delta v = \sum_{i=1}^{n} v_{y_i y_i}\right)$

Equation for v becomes: $\qquad$ (PDE $\Rightarrow$ ODE)

$$\alpha w(r) + \frac{1}{2} r w'(r) + w''(r) + \frac{n-1}{r} w'(r) = 0$$

Multiply both sides by $\quad r^{n-1}$:

$$\underbrace{\overset{n/2}{\alpha}\, r^{n-1} w(r) + \frac{1}{2} r^{n} w'(r)}_{\frac{1}{2}(r^n w)'} + \underbrace{r^{n-1} w''(r) + (n-1) r^{n-2} w'(r)}_{(r^{n-1} w')'} = 0$$

Pick $\boxed{\alpha = \frac{n}{2}} \Rightarrow$

Thus, we solve: $\quad \frac{d}{dr}\left\{ \frac{1}{2} r^n w + r^{n-1} w' \right\} = 0$

$$\Rightarrow \quad \frac{1}{2} r^n w(r) + r^{n-1} w'(r) = C = \text{const.}$$

Impose: $\; r^n w(r), r^{n-1} w'(r) \to 0 \;$ as $\; r \to \infty$. $\;$ Thus, $\; \underline{C = 0}$

Hence, solve: $\quad \frac{1}{2} r^n w(r) + r^{n-1} w'(r) = 0$

$$\Leftrightarrow \quad w'(r) + \frac{r}{2} w(r) = 0 \quad \Leftrightarrow \quad w(r) = B e^{-r^2/4}$$

$r = |y| = \left|\frac{x}{t^\beta}\right|$.

Finally: $\quad u(x,t) = t^{-\alpha} v\left(\frac{x}{t^\beta}\right) = t^{-\alpha} w\left(\frac{|x|}{t^\beta}\right) \;;\quad \alpha = \frac{n}{2},\; \beta = \frac{1}{2}$

$$\Rightarrow \quad u(x,t) = c\, t^{-n/2} e^{-|x|^2/4t}, \quad t > 0 \quad (x \in \mathbb{R}^n)$$

└ Define the "fundamental solution" accordingly!

DEF. 2 The fundamental solution for the heat eqn in $\mathbb{R}^n$ is

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} & , \ t>0, \ x \in \mathbb{R}^n \Leftarrow \text{SINGULAR at } (x,t)=(0,0) \\ 0 & , \ t=0 . \end{cases}$$

Note: c was chosen to be $\frac{1}{(4\pi)^{n/2}}$. Why ?

LEMMA 1: With the above choice of the const. prefactor c, we have

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1 \quad , \quad \forall t>0. \qquad (\text{"normalization to unity"})$$

— · —

Proof: Integrate directly by using $\int_{\mathbb{R}} e^{-x_1^2}\, dx_1 = \sqrt{\pi}$. □

— · —

What is the role of this $\Phi(x,t)$ in $\mathbb{R}^n$?

We want to use $\Phi(x,t)$ in order to construct solutions of the initial value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Define the spatial convolution

$$u(x,t) = \Phi * g = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy, \quad \underline{t>0}. \qquad (3)$$

THM 1 Consider $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and u defined by (3).

THEN:

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$;

(ii) $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$;

(iii) $\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0) \quad \forall x^0 \in \mathbb{R}^n.$

Proof (i) Because $\Phi(x-y,t)$ is C^∞ in $\mathbb{R}^n \times [\delta,\infty)$, $\delta>0$, with uniformly bounded derivatives, then $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$.

(ii) $$u_t - \Delta u = \int_{\mathbb{R}^n} \underbrace{\{\Phi_t - \Delta_x \Phi\}}_{0 \text{ by construction of } \Phi!}(x-y,t)\, g(y)\, dy \qquad (t>0)$$

$\Rightarrow u_t - \Delta u = 0$ for $t>0$.

(iii) Show that $u(x,t) = \Phi * g$ satisfies the initial condition.

(To be continued in LECTURE 9!)