AMSC/CMSC 460: Final Exam

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May 16, 2019

Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 2 hours.
- Good luck!

Additional instructions:

- You should solve only 6 out of the 7 problems. Each problem = 10 points.
- No extra credit will be given for solving more than 6 problems.
- If you solve more than 6 problems, you should clearly indicate which problems you would like to be graded - otherwise, the first 6 problems in each part will be graded.

Solve 6 problems out of the following 7 problems

1. Find the most accurate approximation to the second derivative, $f''(x)$, using $f(x-2h), f(x), f(x+4h)$. What is the order of accuracy of this approximation?

2. Let $D(h)$ be a first-order approximation to $f'(x)$ such that

$$f'(x) = D(h) + C_1 h + C_2 h^2 + \dots$$

   (a) Use Richardson's extrapolation to find a second-order approximation of $f'(x)$.

   (b) What is the result of part (a) if

$$D(h) = \frac{f(x+h) - f(x-3h)}{4h}.$$

3. Find a linear polynomial, $P_1^*(x)$, that minimizes

$$\int_{-1}^{1} \frac{(x^2 - Q_1(x))^2}{\sqrt{1-x^2}} dx,$$

   among all polynomials $Q_1(x)$ of degree ≤ 1.

4. (a) Find a quadrature of the form

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx = A_0 f(x_0) + A_1 f(x_1),$$

   that is exact for all polynomials of degree ≤ 3.

   (b) Use the result of part (a) to approximate $\int_{-\infty}^{\infty} x^6 e^{-x^2} dx$.

5. Consider the ODE $y'(t) = f(t, y(t))$ together with the initial condition $y(0) = y_0$.

   (a) Write an equivalent integral formulation to the ODE and show how to obtain Euler's method using a rectangular quadrature.

   (b) Let $f(t, y(t)) = t^2 y(t)$. Compute two iterations of Euler's method for the ODE $y' = f(t, y(t))$, starting from $y(0) = 1$. Assume that the time step is $h = 0.1$.

6. Let $f(x) = x^4$ in $[-1, 1]$.

   (a) Write the Lagrange form of the interpolating polynomial $P_2(x)$, of degree ≤ 2, that interpolates the values of $f(x)$ at 3 Chebyshev points.

   (b) Explain the advantages of interpolating at Chebyshev points.

7. Let $f(x) = e^{-x} - x$.

   (a) Prove that $f(x)$ must have an least one root in the interval $[0, 10]$.

   (b) Explain why $f(x)$ has only one root in the interval $[0, 10]$.

   (c) Write Newton's method for approximating a root of $f(x)$, and compute two iterations of the method, starting from $x_0 = 1$.

- Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) = 0, \ \forall n \geq 1.$$

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0, \quad m \neq n.$$

$$\int_{-1}^{1} \frac{(T_n(x))^2}{\sqrt{1-x^2}} dx = \begin{cases} \pi, & n = 0, \\ \frac{\pi}{2}, & n = 1, 2, \ldots \end{cases}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C, \qquad \int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2}(\arcsin x - x\sqrt{1-x^2}) + C.$$

- Hermite polynomials

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \ \forall n \geq 1$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) = \delta_{nm} 2^n n! \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \Gamma\left(\frac{m+1}{2}\right), \quad \text{for even } m$$

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}.$$