AMSC/CMSC 460: Midterm 2

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems: (Each problem = 10 points)

1. (a) Assume $h > 0$. Find the most accurate approximation of $f''(x)$ using $f(x-h)$, $f(x+h)$, and $f(x+2h)$.

Solution:

Using the method of undetermined coefficients, we approximation

$$f''(x) \approx Af(x-h) + Bf(x+h) + Cf(x+2h).$$

We write the Taylor expansions:

$$\begin{aligned} f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \dots \\ f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \dots \\ f(x+2h) &= f(x) + 2hf'(x) + \frac{(2h)^2}{2}f''(x) + \frac{(2h)^3}{6}f'''(x) + \dots \end{aligned}$$

This leads to the following system of equations:

$$\begin{aligned} A + B + C &= 0, \\ -A + B + 2C &= 0, \\ A + B + 4C &= \frac{2}{h^2}. \end{aligned}$$

The solution of this system is: $A = \frac{1}{3h^2}$, $B = -\frac{1}{h^2}$, $C = \frac{2}{3h^2}$. Hence, the approximation is

$$f''(x) \approx \frac{f(x-h) - 3f(x+h) + 2f(x+2h)}{3h^2}.$$

(b) What is the order of accuracy of this approximation?

Solution:

The coefficient of the next term in the expansion ($f'''(x)$) is not zero. Hence this is the error term, which means that the method is $O(h)$, i.e., a first-order approximation.

2. (a) Let $w(x) = \sin(x)$. Find two polynomials, $P_0(x)$ (of degree 0) and $P_1(x)$ (of degree 1) that are orthogonal with respect to $w(x)$ on $[0, \pi]$.

Solution:

The first polynomial is $P_0(x) = 1$. For $P_1(x)$ we set $P_1(x) = x - cP_0 = x - c$. We find c such that P_0 and P_1 are orthogonal to each other:

$$0 = \langle P_0, P_1 \rangle_w = \int_0^\pi (x - c)\sin(x)dx = \ldots = -2c + \pi.$$

Hence $c = \frac{\pi}{2}$ and $P_1(x) = x - \frac{\pi}{2}$.

(b) Normalize the polynomials you found in part (a).
You may use: $\int x\sin(x)dx = \sin(x) - x\cos(x)$ and $\int x^2\sin(x)dx = 2x\sin(x) + (2 - x^2)\cos(x)$.

Solution:

Let $\tilde{P}_0(x) = cP_0(x)$. Hence

$$1 = \left\langle \tilde{P}_0, \tilde{P}_0 \right\rangle_w = c^2 \int_0^\pi \sin(x)dx = \ldots = 2c^2.$$

Hence, $c = \frac{1}{\sqrt{2}}$ which means that $\tilde{P}_0(x) = \frac{1}{\sqrt{2}}$.
For the second polynomial, we let $\tilde{P}_1(x) = cP_1(x)$. Hence

$$1 = \left\langle \tilde{P}_1, P_1 \right\rangle_w = c^2 \int_0^\pi \left(x - \frac{\pi}{2}\right)^2 \sin(x)dx = \ldots = c^2 \left(\frac{\pi^2}{2} - 4\right).$$

This means that $c = \frac{1}{\sqrt{\frac{\pi^2}{2} - 4}}$, and

$$\tilde{P}_1(x) = \frac{x - \frac{\pi}{2}}{\sqrt{\frac{\pi^2}{2} - 4}}.$$

3. Let $f(x) = x^2 + 1$. Find the weighted linear least squares approximation to $f(x)$ with respect to $w(x) = 2$ on $[-1, 1]$.

Solution:

The first polynomial is $P_0(x) = 1$. For $P_1(x)$ we set $P_1(x) = x - cP_0(x) = x - c$ and find c such that P_0 and P_1 are orthogonal to each other:

$$0 = \langle P_0, P_1 \rangle_w = \int_{-1}^1 (x - c)2dx = \ldots = -4c,$$

which means that $c = 0$ and $P_1(x) = x$.

Now, the linear least squares approximation can be written as

$$Q_1(x) = c_0 P_0(x) + c_1 P_1(x).$$

Here

$$c_0 = \frac{\langle f, P_0\rangle_w}{\|P_0\|_w} = \frac{\int_{-1}^{1}(x^2+1)2dx}{\int_{-1}^{1} 1\cdot 1\cdot 2dx} = \ldots = \frac{4}{3},$$

and

$$c_1 = \frac{\langle f, P_1\rangle_w}{\|P_1\|_w} = \frac{\int_{-1}^{1}(x^2+1)x2dx}{\int_{-1}^{1} x^2 2dx} = 0.$$

Therefore

$$Q_1(x) = c_0 P_0(x) = \frac{4}{3}.$$

4. Find a cubic spline, $s(x)$, that interpolates

x	-1	0	1
y	1	0	1

on $[-1, 1]$ given that $s''(-1) = s''(1) = 0$. Use the interpolation points as the spline nodes.

Note: Unfortunately you cannot solve this problem by guessing the answer. Solving it does require some calculations.

Solution:

Let

$$s(x) = \begin{cases} s_0(x), & -1 \le x \le 0, \\ s_1(x), & 0 \le x \le 1, \end{cases} = \begin{cases} a_0x^3 + b_0x^2 + c_0x + d_0, & -1 \le x \le 0, \\ a_1x^3 + b_1x^2 + c_1x + d_1, & 0 \le x \le 1. \end{cases}$$

From $s(0) = 0$ we have $d_0 = d_1 = 0$. The other interpolation conditions are: $s(-1) = 1$, i.e., $-a_0 + b_0 - c_0 = 1$, and $s(1) = 1$, from which $a_1 + b_1 + c_1 = 1$. From $s_0'(0) = s_1'(0)$ we have $c_0 = c_1$, and from $s_0''(0) = s_1''(0)$ we have $b_0 = b_1$.

Then $s''(-1) = -6a_0 + 2b_0 = 0$, and $s''(1) = 6a_1 + 2b_1 = 0$. Solving the system, we have $a_0 = 1/2$, $b_0 = 3/2$, $c_0 = 0$, $d_0 = 0$, and $a_1 = -1/2$, $b_1 = 3/2$, $c_1 = 0$, $d_1 = 0$.

This means that the spline we are seeking for is:

$$s(x) = \begin{cases} \frac{1}{2}x^3 + \frac{3}{2}x^2, & -1 \le x \le 0, \\ -\frac{1}{2}x^3 + \frac{3}{2}x^2, & 0 \le x \le 1. \end{cases}$$