

AMSC/CMSC 460
Final Exam - Solutions
Spring 2017

1) a) $P_1(x) = f(1)l_0(x) + f(1+h)l_1(x)$

$$l_0(x) = \frac{x - (1+h)}{-h}$$

$$l_1(x) = \frac{x-1}{h}$$

b) $\int_{1-h}^{1+h} f(x) dx \approx \int_{1-h}^{1+h} P_1(x) dx = f(1) \int_{1-h}^{1+h} l_0(x) dx + f(1+h) \int_{1-h}^{1+h} l_1(x) dx.$

$$\int_{1-h}^{1+h} l_0(x) dx = \dots = 2h.$$

$$\int_{1-h}^{1+h} l_1(x) dx = \dots = 0.$$

$$\Rightarrow \int_{1-h}^{1+h} f(x) dx \approx 2hf(1).$$

$$2. a) P_0 = 1$$

$$P_1 = x - c$$

$$0 = \langle P_0, P_1 \rangle = \int_0^1 1 \cdot (x - c) = \frac{x^2}{2} - cx \Big|_0^1 = \frac{1}{2} - c \Rightarrow c = \frac{1}{2}$$

To normalize:

$$\|P_0\|^2 = \int_0^1 P_0^2 dx = 1$$

$$\|P_1\|^2 = \int_0^1 (x - \frac{1}{2})^2 dx = \int_0^1 (x^2 - x + \frac{1}{4}) dx = \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4} \Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

\Rightarrow The normalized polynomials are

$$\tilde{P}_0 = 1, \quad \tilde{P}_1(x) = \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

$$b) Q_1(x) = c_0 P_0(x) + c_1 P_1(x).$$

$$c_0 = \frac{\langle f, P_0 \rangle}{\|P_0\|^2} = \frac{\int_0^1 e^x dx}{1} = \frac{e-1}{1} = e-1.$$

$$c_1 = \frac{\langle f, P_1 \rangle}{\|P_1\|^2} = \frac{\int_0^1 e^x (x - \frac{1}{2}) dx}{\int_0^1 (x - \frac{1}{2})^2 dx} = \frac{1 - \frac{1}{2}(e-1)}{\frac{1}{12}} = 6(3-e)$$

$$\Rightarrow Q_1(x) = e-1 + 6(3-e)(x - \frac{1}{2}).$$

c) $x_0 = \frac{1}{2}$ as the root of $P_1(x)$. $A_0 = 1$ since $1 = \int_0^1 dx = A_0$.
(We get for free that the quadrature is exact for x).

$$3) a) \begin{cases} y'(t) = f(t, y(t)) \\ y(a) = y_0. \end{cases}$$

The integral form of the ODE:

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s)) ds.$$

Replace the integral by a midpoint quadrature:

$$\int_t^{t+h} f(s, y(s)) ds \approx h f\left(t + \frac{h}{2}, y\left(t + \frac{h}{2}\right)\right).$$

We obtain $y(t + \frac{h}{2})$ by predicting it using the Euler's method:

$$y\left(t + \frac{h}{2}\right) \approx y(t) + \frac{h}{2} f\left(t, y(t)\right).$$

$$\Rightarrow \begin{cases} \omega_{j+1/2} = \omega_j + \frac{h}{2} f(t_j, \omega_j) \\ \omega_{j+1} = \omega_j + h f\left(t_{j+1/2}, \omega_{j+1/2}\right). \end{cases}$$

$$b) I = O(h) + c_1 h^2 + c_2 h^4 + O(h^6)$$

$$\Rightarrow I = D(2h) + c_1 (2h)^2 + c_2 (2h)^4 + O(h^6)$$

$$\Rightarrow I = \frac{4D(h) - D(2h)}{3} + O(h^4).$$

$$\text{with } D(h) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

$$\text{and } D(2h) = \frac{f(x-2h) - 2f(x) + f(x+2h)}{4h^2}$$

$$4. a) \quad P_3(x) = \underset{1}{f(0)} + \underset{1}{f'(0)}x + f[0,0,1]x^2 + f[0,0,1,1]x^2(x-1)$$

$$f[0,0,1] = \frac{f[0,1] - f[0,0]}{1-0} = \frac{f'(1) - f(0)}{1} = -1$$

$$f[0,1,1] = \frac{f'(1) - f[0,1]}{1} = 1$$

$$f[0,0,1,1] = \frac{f[0,1,1] - f[0,0,1]}{1-0} = \frac{1 - (-1)}{1} = 2.$$

$$\Rightarrow P_3(x) = 1 + x - x^2 + 2x^2(x-1) = 1 + x - 3x^2 + 2x^3$$

$$b) \quad A = LL^T \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$