# AMSC 460 - Midterm #2 - Solutions.

1. a) The table of divided differences corresponding to the data is:

| $x$ | $f(x)$ | $f[\cdot,\cdot]$ | $f[\cdot,\cdot,\cdot]$ | $f[\cdot,\cdot,\cdot,\cdot]$ |
|---|---|---|---|---|
| $x_0 = 0$ | $1$ | $-1$ | $\frac{1}{2}$ | $\frac{-\frac{1}{2}}{3} = -\frac{1}{6}$ |
| $x_1 = 1$ | $0$ | $0$ | $0$ | |
| $x_2 = 2$ | $0$ | $0$ | | |
| $x_3 = 3$ | $0$ | | | |

The Newton form of the interpolating polynomial is

$$P_3(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

$$= 1 - x + \tfrac{1}{2}x(x-1) - \tfrac{1}{6}x(x-1)(x-2).$$

b) Since there are 4 interpolation conditions, we are looking for a polynomial of degree $\leq 3$.

$$P_3(x) = f(0) + f[0,0]x + f[0,0,1]x^2 + f[0,0,1,1]x^2(x-1).$$

$f(0) = 1$

$f[0,0] = f'(0) = 0$

$$f[0,0,1] = \frac{f[0,1] - f[0,0]}{1-0} = \underbrace{\frac{f(1)-f(0)}{1-0}} - f'(0) = \frac{0-1}{1} - 0 = -1$$

$$f[0,1,1] = \frac{f[1,1] - f[0,1]}{1-0} = f'(1) - (-1) = 1.$$

$$f[0,0,1,1] = \frac{f[0,1,1] - f[0,0,1]}{1-0} = \frac{1-(-1)}{1} = 2$$

$$\Rightarrow P_3(x) = 1 - x^2 + 2x^2(x-1)$$

2) a) Since in every subinterval the function $f(x)$ is a cubic polynomial, all that we need to do is to verify that it is continuous as are its first and second derivatives.

Continuity of $f(x)$ at $x=0$ : $-b=-d \Rightarrow \underline{b=d}$.

" " " " $x=1$ $\underline{c=e}$

$$f'(x) = \begin{cases} a+3b(x-1)^2, & x\in(-\infty,0], \\ c+3d(x-1)^2, & x\in[0,1], \\ e+3f(x-1)^2, & x\in[1,\infty). \end{cases}$$

$f'(0)$ : $a+3b=c+3d$ but since $b=d$, this implies $\underline{a=c}$

$f'(1)$ : $\underline{c=e}$ (which we already know).

$$f''(x) = \begin{cases} 6b(x-1) & x\in(-\infty,0] \\ 6d(x-1) & x\in[0,1] \\ 6f(x-1) & x\in[1,\infty). \end{cases}$$

$f''(0) \Rightarrow -6b=-6d \Rightarrow b=d$ which we know.

$f''(1) \Rightarrow 0=0$. Nothing new.

Overall, $f(x)$ is a cubic spline if $\underline{a=c=e}$ and $\underline{b=d}$.

b) $f(1)=c=2 \Rightarrow a=c=e=2$.

$-2=f(0)=-b \Rightarrow b=2\ (=d)$

$f(2)=7=2e+f=4+f \Rightarrow f=3$

$$\Rightarrow f(x) = \begin{cases} 2x+2(x-1)^3, & x\in(-\infty,0] \\ 2x+2(x-1)^3, & x\in[0,1] \\ 2x+3(x-1)^3, & x\in[1,\infty) \end{cases}$$

3) a) Set $P_0 = 1$.

Let $P_1 = x - cP_0 = x - c$.

$$\langle P_1, P_0 \rangle_w = \int_0^1 (x-c)e^x\,dx = e^x(x-1) - ce^x \Big|_0^1 = c(1-e)+1 = 0$$

$$\Rightarrow c = \frac{1}{e-1} \Rightarrow \underline{P_1 = x - \frac{1}{e-1}}.$$

b) Let $\tilde{P}_0 = cP_0$

Then $1 = \|\tilde{P}_0\|_w^2 = \langle \tilde{P}_0, \tilde{P}_0 \rangle_w = \langle cP_0, cP_0 \rangle_w = c^2 \langle P_0, P_0 \rangle_w$

$$= c^2 \int_0^1 e^x \cdot 1^2\,dx = c^2 e^x \Big|_0^1 = c^2(e-1) \Rightarrow \underline{c = \frac{1}{\sqrt{e-1}}}$$

(i.e. $\tilde{P}_0 = \frac{1}{\sqrt{e-1}}$).

c) Let $Q_1(x) = c_0 P_0(x) + c_1 P_1(x)$.

Then

$$c_0 = \frac{\langle f, P_0 \rangle_w}{\langle P_0, P_0 \rangle_w} = \frac{\int_0^1 e^{-x} \cdot 1 \cdot e^x\,dx}{\int_0^1 1^2 e^x\,dx} = \frac{1}{e-1}.$$

$$c_1 = \frac{\langle f, P_1 \rangle_w}{\langle P_1, P_1 \rangle_w} = \frac{\int_0^1 e^{-x}\left(x - \frac{1}{e-1}\right)e^x\,dx}{\int_0^1 \left(x - \frac{1}{e-1}\right)^2 e^x\,dx} = \frac{\frac{1}{2} - \frac{1}{e-1}}{A}$$

$$A = \int_0^1 \left(x - \frac{1}{e-1}\right)^2 e^x\,dx = \int_0^1 x^2 e^x\,dx - \frac{2}{e-1}\int_0^1 x e^x\,dx + \frac{1}{(e-1)^2}\int_0^1 e^x\,dx$$

$$= e^x(x^2 - 2x + 2)\Big|_0^1 - \frac{2}{e-1}\left(e^x(x-1)\right)\Big|_0^1 + \frac{1}{(e-1)^2} e^x\Big|_0^1$$

$$= e(1-2+2) - 2 - \frac{2}{e-1}(0-(-1)) + \frac{1}{(e-1)^2}(e-1) = e - 2 - \frac{1}{e-1}.$$