AMSC/CMSC 460: Midterm 2

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems: (Each problem = 10 points)

1. (a) Explain the advantages of interpolating at Chebyshev points.

   (b) Compute the unique interpolating polynomial of degree ≤ 2 that interpolates data sampled from $f(x) = x^2$ at an appropriate number of Chebyshev points on the interval $[-1, 1]$.

   (c) Repeat part (b) with $f(x) = x^4$.

   Note: Chebyshev polynomials are given by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) = 0, \forall n \geq 1.$$

2. Find a spline of degree 2, $S(x)$, on the interval $[0, 2]$, for which $S(0) = 0$, $S(1) = 2$, $S(2) = 0$, and $S'(0) = 0$. Use the points $0, 1, 2$ as the knots.

3. Use the Gram-Schmidt process to find orthogonal polynomials of degrees 0, 1, 2, on the interval $[0, 1]$, with respect to the weight $w(x) = 1 + x$.

   Note: you do not need to normalize the polynomials. For the quadratic polynomial, $P_2(x)$, write the coefficients but do not explicitly calculate the integrals.

4. Let $f(x) = x^2$. Find the quadratic polynomial $Q_2^*(x)$ that minimizes

$$\int_{-\infty}^{\infty} e^{-x^2}(f(x) - Q_2(x))^2 dx,$$

   among all quadratic polynomials $Q_2(x)$.

   Note: You may use:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \forall n \geq 1$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x) = \delta_{nm}2^n n!\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \Gamma\left(\frac{m+1}{2}\right), \quad \text{for even } m$$

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}.$$