Solving systems of linear equations

(Following Kincaid & Ch.).

Topics

1. The LU and Cholesky factorizations.
2. Pivoting
3. Norms and the analysis of errors.

1. The LU & Cholesky factorizations

A system of n equations in n unknowns: $Ax=b$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Examples of easy to solve systems:

1) Diagonal

$$\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\Rightarrow$ $\underset{\text{decoupled}}{n \text{ equations}} \Longrightarrow$ $x = \begin{pmatrix} b_1/a_{11} \\ \vdots \\ b_n/a_{nn} \end{pmatrix}$ assuming $a_{ii} \neq 0$.

2) A is lower triangular

$$A = \begin{pmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

assume $a_{ii} \neq 0$.

<u>Solution</u>: Solve for x_1 from eq. 1.
Solve for x_2 " " 2 given x_1
etc.

$\Rightarrow$ We can obtain the x_i's one by one.

This is called Solution by <u>forward substitution</u>.

$$\left(x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j}{a_{ii}} \right)$$

$i = 2, \dots, n$

3) <u>A is upper triangular</u>

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} \qquad a_{ii} \neq 0.$$

Solution for $x_n \to x_{n-1} \to \dots \to x_1$.

<u>Back substitution</u>

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

4) <u>Permuting the equations</u> to get an upper or lower triangular matrix.

Example

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} a_{31} & & \\ a_{11} & a_{12} & \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_3 \\ b_1 \\ b_2 \end{pmatrix}$$

LU factorization

Suppose A can be factored into a product:

$$A = \underset{\text{lower triangular}}{L}\,\overset{\text{upper triangular.}}{U}$$

Solving $Ax=b$ then means $LUx=b$

So the solution can be obtained in 2 simple steps:

Solve for z: $Lz=b$.

Solve for x: $Ux=z$.

- A form $A=LU$ is called <u>LU-decomposition</u>

- Not always possible
- When exists: not unique.

- Deriving an <u>algorithm</u> for the LU decomposition:

- Let
$$L = \begin{pmatrix} l_{11} & & \bigcirc \\ l_{21} & l_{22} & \\ \vdots & \ddots & \\ l_{n1} & \cdots & l_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ & u_{22} & \cdots u_{2n} \\ \bigcirc & & \vdots \\ & & u_{nn} \end{pmatrix}$$

- if $A = LU$

then $$a_{ij} = \sum_{s=1}^{n} l_{is} u_{sj} = \sum_{s=1}^{\min(i,j)} l_{is} u_{sj} \ . \quad \textcircled{*}$$

$$(l_{is} = 0 \text{ for } s > i \text{ and } u_{sj} = 0 \text{ for } s > j).$$

- In each step in this process:
  determine one new row of U
  & one new column in L.

At step k: assume that rows $1, \dots, k-1$ have been computed in U & columns $1, \dots, k-1$ are computed in L.

Set $i = j = k$ in $\textcircled{*}$:

$$\underbrace{a_{kk}}_{\text{known}} = \sum_{s=1}^{k-1} \underbrace{l_{ks} u_{sk}}_{\text{known}} + \underbrace{l_{kk} u_{kk}}_{\text{unknown}} \quad \textcircled{**}$$

- If u_{kk} or l_{kk} has been specified, use (***) to determine the other.

- We now write the k^{th} row $(i=k)$ and k^{th}-column: $(j=k)$

$$\overset{\text{known}}{a_{kj}} = \overbrace{\sum_{s=1}^{k-1} l_{ks}u_{sj}}^{\text{known}} + \overset{\text{now known}}{l_{kk}}\,\underset{\text{unknown}}{u_{kj}} \,, \quad k+1 \le j \le n .$$

$$\underset{\text{known}}{a_{ik}} = \sum_{s=1}^{k-1} l_{is}u_{sk} + \underset{\text{unknown}}{l_{ik}}\,\underset{\text{now known}}{u_{kk}} \quad k+1 \le i \le n .$$

As long as l_{kk} & u_{kk} are $\neq 0$, we can find the unknown elements.

Terminology:

(1) Doolittle's factorization : L = unit lower triangular
obtain by specifying $l_{ii}=1$ $\forall i$ $\longrightarrow$ $(l_{ii}=1 \ \forall i)$.

(2) Crout's factorization: U = unit upper triangular
obtain by specifying $u_{ii}=1 \ \forall i$. $(u_{ii}=1 \ \forall i)$

(3) Cholesky's factorization: $U = L^T$
and then $l_{ii}=u_{ii}$ $\forall i$.

Example:

$$A=\begin{pmatrix} 60 & 30 & 20 \\ 30 & 20 & 15 \\ 20 & 15 & 12 \end{pmatrix}$$

$$L=\begin{pmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{pmatrix}, \quad U=\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{pmatrix}$$

(1) **Doolittle** Set $l_{11}=1 \Rightarrow u_{11}=60$

$$a_{12} = l_{11} u_{12}$$
$$30 = 1 \cdot u_{12} \Rightarrow u_{12}=30.$$

$$a_{13} = l_{11} u_{13}$$
$$20 = 1 \cdot u_{13} \Rightarrow u_{13}=20.$$

$$a_{21} = l_{21} u_{11}$$
$$30 = l_{21} \cdot 60 \Rightarrow l_{21}=\tfrac{1}{2}.$$

$$a_{31} = l_{31} u_{11}$$
$$20 = l_{31} \cdot 60 \Rightarrow l_{31}=\tfrac{1}{3}.$$

Set $l_{22}=1$

$$a_{22} = l_{21}\cdot u_{12} + l_{22}\cdot u_{22}$$
$$20 = \frac{1}{2}\cdot 30 + 1\cdot u_{22} \quad \Rightarrow u_{22}=5$$

$$a_{23} = l_{21}\cancel{u_{13}} + l_{22}u_{23} \quad \Rightarrow u_{23}=5.$$
$$15 = \frac{1}{2}\cdot 20 + 1\cdot u_{23}$$

$$a_{32} = l_{31}u_{12} + l_{32}u_{22} \quad \Rightarrow l_{32}=1.$$
$$15 = \frac{1}{3}\cdot 30 + l_{32}\cdot 5$$

Set $l_{33}=1$

$$a_{33} = l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \quad \Rightarrow u_{33}=\frac{1}{3}$$
$$12 = \frac{1}{3}\cdot 20 + 1\cdot 5 + 1\cdot u_{33}$$

$$\Downarrow$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 60 & 30 & 20 \\ 0 & 5 & 5 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}}_{U}$$

Crout factorization

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix}\begin{pmatrix} 60 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}}_{\text{Combine}}\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad A = \underbrace{\begin{pmatrix} 60 & 0 & 0 \\ 30 & 5 & 0 \\ 20 & 5 & \frac{1}{3} \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{U}.$$

Cholesky factorization

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{60} & & \\ & \sqrt{5} & \\ & & \sqrt{\frac{1}{3}} \end{pmatrix}}\underbrace{\begin{pmatrix} \sqrt{60} & & \\ & \sqrt{5} & \\ & & \sqrt{\frac{1}{3}} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \underbrace{\begin{pmatrix} \sqrt{60} & 0 & 0 \\ \frac{1}{2}\sqrt{60} & \sqrt{5} & 0 \\ \frac{1}{3}\sqrt{60} & \sqrt{5} & \frac{1}{3}\sqrt{3} \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} \sqrt{60} & \frac{1}{2}\sqrt{60} & \frac{1}{3}\sqrt{60} \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & \frac{1}{3}\sqrt{3} \end{pmatrix}}_{L^T}$$

In this case,
A is
- real
- symmetric
- positive definite.

This is also a unique factorization.

2. Pivoting

- Gaussian elimination:

$$\begin{pmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 34 \\ 27 \end{pmatrix}$$

Subtract 2 × R1 from R2
Subtract $\frac{1}{2}$ × R1 from R2.

$2, \frac{1}{2}$ = multipliers

Pivot element. The 1st row: pivot row

After the 1st step:

$$\begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & -12 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 21 \end{pmatrix}$$

2nd row pivot row

The pivot element

Subtract 3 × R2 from R3

$$\begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \end{pmatrix}$$

An equivalent system to the original one.

upper triangular.

Solving by <u>backward substitution</u> $\Rightarrow X = (\cdots)$

The multipliers we arrange in a unit lower triangular matrix

$$L = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ \frac{1}{2} & 3 & 1 \end{pmatrix}$$

Together with: $U = \begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

We got

$$\begin{pmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 9 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ \frac{1}{2} & 3 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\boxed{A = LU}$$

<u>Comment</u>: It is clear why this is correct by going back in how U was obtained. For example

$$R_2 \to R_2^{orig} - 2R_1$$

So $R_2^{orig} = R_2 + 2R_1$ which explains the 2nd row of the matrix L $(2\ 1\ 0)$.

etc.

Comment: Clearly, for everything to work, the pivot elements must be nonzero.

Problems:

1) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The simple version of the algorithm fails. (Can't add a multiple of the 1st row to R2 to zero the coefficient of x_1).

2) $\begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Small ε.

Gaussian elimination:

$$\begin{pmatrix} \varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 2-\frac{1}{\varepsilon} \end{pmatrix}$$

$$\Rightarrow \quad x_2=\frac{2-\frac{1}{\varepsilon}}{1-\frac{1}{\varepsilon}}$$

$$x_1=(1-x_2)\cdot\frac{1}{\varepsilon}$$

- What is the real solution?

$$X_2 = \cancel{\frac{2-\varepsilon}{2\varepsilon-1}}\ \frac{2\varepsilon-1}{\varepsilon-1} \approx 1.$$

$$X_1 = \frac{1}{\varepsilon}\left(1-\frac{2\varepsilon-1}{\varepsilon-1}\right) = \frac{1}{\varepsilon}\ \frac{\varepsilon-1-2\varepsilon+1}{\varepsilon-1} = \frac{1}{1-\varepsilon} \approx 1.$$

But on a computer:

$$\left.\begin{array}{l} 2-\frac{1}{\varepsilon} \approx -\frac{1}{\varepsilon}. \\ 1-\frac{1}{\varepsilon} \approx -\frac{1}{\varepsilon} \end{array}\right. \Rightarrow \quad X_2 \approx 1$$

and

$$X_1 = (1-X_2)\frac{1}{\varepsilon} \approx 0.$$

The problem: the smallness of a_{11} *relative* to the other elements in the row

B) $$\begin{pmatrix} 1 & \frac{1}{\varepsilon} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} \\ 2 \end{pmatrix}$$

Gaussian elimination:

$$\begin{pmatrix} 1 & \frac{1}{\varepsilon} \\ 0 & 1-\frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} \\ 2-\frac{1}{\varepsilon}. \end{pmatrix}$$

The solution is:

$$x_2 = \frac{2-\frac{1}{\varepsilon}}{1-\frac{1}{\varepsilon}} \approx 1$$

$$x_1 = \frac{1}{\varepsilon} - \frac{1}{\varepsilon} x_2$$

on a computer ≈ 0

real: $\frac{1}{\varepsilon}\left[1-\frac{2\varepsilon-1}{\varepsilon-1}\right] = \frac{1}{1-\varepsilon} \approx 1$.

② If we change the order of the equations, the problem disappears:

$$\begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1-2\varepsilon \end{pmatrix}$$

Solution: $x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \approx 1$

$$x_1 = 2 - x_2 \approx 1.$$

Conclusion: The algorithm must allow for interchanging rows when necessary.

Gaussian elimination with scaled row pivoting

Key: Select the order.

Step I: Compute the "scale" of each row:

$$s_i = \max_{1\le j\le n} |a_{ij}| = \max\{|a_{i1}|, \dots, |a_{in}|\}, \quad 1\le i\le n.$$

Example: $A = \begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ 3 & -2 & 1 \end{pmatrix}, \quad s = (6, 8, 3).$

Step II: Look at the ratios $\left\{ \frac{|a_{11}|}{s_1}, \frac{|a_{21}|}{s_2}, \frac{|a_{31}|}{s_3} \right\}$

pick the row with the largest scale (ratio) as the pivot row.

In this case $\left(\frac{2}{6}, \frac{1}{8}, \frac{3}{3} \right)$ the largest $\Downarrow$ R3 is the pivot

$\Rightarrow$ Exchange R3 with R1.

Now elimination: ~~$\begin{pmatrix} 3 & -2 & 1 \\ \frac{1}{3} & -\frac{16}{3} & \frac{23}{3} \\ \frac{2}{3} & & \end{pmatrix}$~~

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & -6+\frac{2}{3} & 8-\frac{1}{3} \\ 0 & 3+\frac{4}{3} & -6-\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & -\frac{16}{3} & \frac{23}{3} \\ 0 & \frac{13}{3} & \frac{-20}{3} \end{pmatrix}$$

Now we compare $\left\{ \frac{|a_{22}|}{s_2}, \frac{|a_{32}|}{s_3} \right\}$ ← the original value of the permutatio

For this case $\left\{ \frac{\frac{16}{3}}{8}, \frac{\frac{13}{3}}{6} \right\}$

↑ the largest.

So we exchange $R_2 \Leftrightarrow R_3$.

etc.

Comment:

The scale we divide by is always the original calculated number.

Q: When can Gaussian elimination be used safetly w/o pivoting?

Example: If the matrix is diagonally dominant:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad , \quad 1\le i\le n .$$

THM: Gaussian elimination preserves the d.d. of a matrix (w/o pivoting).

Thm: Every diagonally dominant matrix is nonsingular and has an LU-decomposition.

Comments

Scaled Gaussian elimination with pivoting when applied to a d.d. matrix doesn't change the order of the rows.

Norms and the analysis of errors

Norm - vectors

Def: On a vector space V a norm is a function $\|\cdot\|: V \to \mathbb{R}^+$ st.

(i) $\|x\| > 0$ if $x \neq 0$, $x \in V$

(ii) $\|\lambda x\| = |\lambda| \|x\|$ $\lambda \in \mathbb{R}$, $x \in V$

(iii) $\|x+y\| \leq \|x\| + \|y\|$ $\forall x, y \in V$.

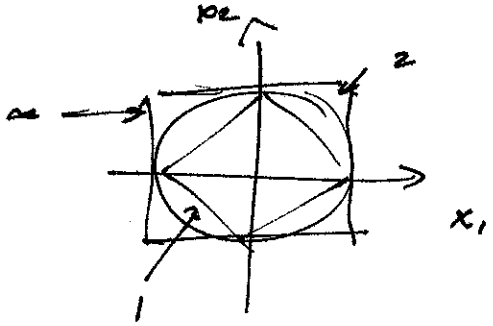
Examples: ① in $\mathbb{R}^n$

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

② The unit ball in $\mathbb{R}^2$



Matrix norm

Given a vector norm $\|\cdot\|$, the associated matrix norm is (for $A_{n\times n}$)

$$\circledast \quad \|A\| = \sup\{\|Ax\| : x\in\mathbb{R}^n, \|x\|=1\}.$$

Thm: if $\|\cdot\|$ is a vector norm, then $\|A\|$ defined by $\circledast$ is a norm on the space of $n\times n$ matrices.

Proof:

(i) $\|\lambda A\| = \sup\{\|\lambda Ax\|, \|x\|=1\} = |\lambda| \sup\{\|Ax\|, \|x\|=1\} = |\lambda|\,\|A\|.$

(ii) $\|A+B\| = \sup\{\|(A+B)x\|, \|x\|=1\}$

$\leq \sup\{\|Ax\| + \|Bx\|, \|x\|=1\}$

$\leq \sup\{\|Ax\|, \|x\|=1\} + \sup\{\|Bx\|, \|x\|=1\}$

$= \|A\| + \|B\|.$

(iii) If $A\neq 0$, it has at least one nonzero element, say in column j. Let $u=(0,\dots,0,\overset{j}{\overset{\downarrow}{1}},0,\dots,0)$. Set $x=\frac{u}{\|u\|}$. Then $\|x\|=1$.

$\Rightarrow \|A\| \geq \|Ax\| = \frac{\|Au\|}{\|u\|} = \frac{\|A^j\|}{\|u\|} > 0.$

Properties of matrix norms

1. $\|Ax\| \leq \|A\| \|x\|$

2. $\|I\| = 1$

3. $\|AB\| \leq \|A\| \cdot \|B\|.$

Example:

If $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

Then $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$

Condition

Assume that A is invertible.

Then

$$\boxed{K(A) = \|A\| \cdot \|A^{-1}\|.}$$

The condition number of A.

any norm.

Question: Assume that $Ax=b$. $A_{n\times n}$ invertible.

Perturb $A^{-1} \longrightarrow B$

Then the solution $x = A^{-1}b$ becomes $\tilde{x} = Bb$.

How large is the perturbation in the solution?

Answer: any vector norm.

$$\|x-\tilde{x}\| = \|x - Bb\| = \|x - BAx\| = \|(I-BA)x\|$$

$$\le \|I-BA\|\,\|x\|.$$

The relative perturbation

$$\Rightarrow \quad \frac{\|x-\tilde{x}\|}{\|x\|} \le \|I-BA\|$$

Question: Perturb $b \longrightarrow \tilde{b}$.
If $Ax=b$ and $A\tilde{x}=\tilde{b}$, how close are x and $\tilde{x}$?

Answer: $\|x-\tilde{x}\| = \|A^{-1}b - A^{-1}\tilde{b}\| = \|A^{-1}(b-\tilde{b})\| \le \|A^{-1}\|\,\|b-\tilde{b}\|.$

To obtain relative estimate $\longrightarrow$

$$= \|A^{-1}\|\,\|Ax\|\,\frac{\|b-\tilde{b}\|}{\|b\|} \le \|A^{-1}\|\,\|A\|\,\|x\|\,\frac{\|b-\tilde{b}\|}{\|b\|}$$

$$\Rightarrow \quad \frac{\|x-\tilde{x}\|}{\|x\|} \le \underbrace{\|A\|\,\|A^{-1}\|}_{\kappa(A)}\,\frac{\|b-\tilde{b}\|}{\|b\|}.$$

* The condition number depends on the norm.

* If the c.n. is small then small perturbation in b will lead to small perturbations in x.

* $K(A) \geq 1$.

Example: $A = \begin{pmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{pmatrix}$ $A^{-1} = \frac{1}{\varepsilon^2}\begin{pmatrix} 1 & -1-\varepsilon \\ -1+\varepsilon & 1 \end{pmatrix}$.

In the ∞ norm: $\|A\|_\infty = 2+\varepsilon$

$\|A^{-1}\|_\infty = \frac{1}{\varepsilon^2}(2+\varepsilon)$

$\Rightarrow K(A) = \left(\frac{2+\varepsilon}{\varepsilon}\right)^2 > \frac{4}{\varepsilon^2}$.

If $\varepsilon < 0.01$ then $K(A) > 40{,}000$.

Hence a small perturbation in b may induce a relative perturbation $40{,}000$ times greater in the solution of the system $Ax=b$.