NON-COLLISION SINGULARITIES IN THE PLANAR TWO-CENTER-TWO-BODY PROBLEM

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Contents

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1. Introduction

1.1. **Statement of the main result.** We study a two-center two-body problem. Consider two fixed centers Q_1 and Q_2 of masses $m_1 = m_2 = 1$ located at distance χ from each other and two small particles Q_3 and Q_4 of masses $m_3 = m_4 = \mu \ll 1$. Q_is interact with each other via Newtonian potential. If we choose coordinates so that Q_2 is at $(0,0)$ and Q_1 is at $(-\chi, 0)$ then the Hamiltonian of this system can be written as

$$H = \frac{|P_3|^2}{2\mu} + \frac{|P_4|^2}{2\mu} - \frac{\mu}{|Q_3|} - \frac{\mu}{|Q_3 - (-\chi, 0)|} - \frac{\mu}{|Q_4|} - \frac{\mu}{|Q_4 - (-\chi, 0)|} - \frac{\mu^2}{|Q_3 - Q_4|}. \tag{1.1}$$

We assume that the total energy of the system is zero.

We want to study singular solutions of this system, that is, the solutions which can not be continued for all positive times. We will exhibit a rich variety of singular solutions. Fix $\varepsilon_0 < \chi$. Let $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^{\infty}$ be a sequence of 3s and 4s.

Definition 1.1. *We say that* $(Q_3(t), Q_4(t))$ *is a* **singular solution with symbolic sequence** $\boldsymbol{\omega}$ *if there exists a positive increasing sequence* $\{t_j\}_{j=0}^{\infty}$ *such that*

- $t^* = \lim_{j\to\infty} t_j < \infty$.
- $|Q_3(t_j) - Q_2| \leq \varepsilon_0$, $|Q_4(t_j) - Q_2| \leq \varepsilon_0$.
- *If* $\omega_j = 4$ *then for* $t \in [t_{j-1}, t_j]$, $|Q_3(t) - Q_2| \leq \varepsilon_0$ *and* $\{Q_4(t)\}_{t\in[t_{j-1},t_j]}$ *winds around* Q_1 *exactly once.*
  If $\omega_j = 3$ *then for* $t \in [t_{j-1}, t_j]$, $|Q_4(t) - Q_2| \leq \varepsilon_0$ *and* $\{Q_3(t)\}_{t\in[t_{j-1},t_j]}$ *winds around* Q_1 *exactly once.*
- $|\dot{Q}_i(t)| \to \infty$ *as* $t \to t^*$.

During the time interval $[t_{j-1}, t_j]$ we refer to Q_{ω_j} as the traveling particle and to $Q_{7-\omega_j}$ as the captured particle. Thus ω_j prescribes which particle is the traveler during the j trip.

We denote by $\Sigma_{\boldsymbol{\omega}}$ the set of initial conditions of singular orbits with symbolic sequence $\boldsymbol{\omega}$. Note that if $\boldsymbol{\omega}$ contains only finitely many 3s then there is a collision of Q_3 and Q_2 at time t^*. If $\boldsymbol{\omega}$ contains only finitely many 4s then there is a collision of Q_4 and Q_2 at time t^*. Otherwise at we have a **collisionless singularity** at t^*.

Theorem 1. *There exists* $\mu_* \ll 1$ *such that for* $\mu < \mu_*$ *the set* $\Sigma_{\boldsymbol{\omega}} \neq \emptyset$.

Moreover there is an open set U *on the zero energy level and a foliation of* U *by two-dimensional surfaces such that for any leaf* S *of our foliation* $\Sigma_{\boldsymbol{\omega}} \cap S$ *is a Cantor set.*

Remark 1.1. *By rescaling space and time variables we can assume that* $\chi \gg 1$. *In the proof we shall make this assumption and set* $\varepsilon_0 = 2$.

Remark 1.2. *It follows from the proof that the Cantor set described in Theorem 1 can be chosen to depend continuously on* S. *In other words* $\Sigma_{\boldsymbol{\omega}}$ *contains a set which is local a product of a five dimensional disc and a Cantor set. The fact that on* *each surface we have a Cantor set follows from the fact that we have a freedom of choosing how many rotations the captured particle makes during j-th trip.*

Remark 1.3. *The construction presented in this paper also works for small nonzero energies. Namely, it is sufficient that the total energy is much smaller than the kinetic energies of the individual particles. The assumption that the total energy is zero is made to simplify notation since then the energies of Q_3 and Q_4 have the same absolute values.*

Remark 1.4. *One can ask if Theorem 1 holds for other choices of masses. The fact that the masses of the fixed centers Q_1 and Q_2 are the same is not essential and is made only for convenience. The assumption that Q_3 and Q_4 are light is important since it allows us to treat their interaction as a perturbation except during the close encounters of Q_3 and Q_4. The fact that the masses of Q_3 and Q_4 are equal allows us to use an explicit periodic solution of a certain limiting map (*Gerver map*) which is found in* [G2]*. It seems likely that the conclusion of Theorem 1 is valid if $m_3 = \mu, m_4 = c\mu$ where c is a fixed constant close to 1 and μ is sufficiently small but we do not have a proof of that.*

1.2. **Motivations.**

1.2.1. *Non-collision singularity in N-body problem.* Our work is motivated by the following fundamental problem in celestial mechanics. *Describe the set of initial conditions of the Newtonian N-body problem leading to global solutions.* The compliment to this set splits into the initial conditions leading to the collision and non-collision singularities.

It is clear that the set of initial conditions leading to collisions is non-empty for all $N > 1$ and it is shown in [Sa1] that it has zero measure. Much less is known about the non-collision singularities. The main motivation for our work is provided by following basic problems.

Conjecture 1. *The set of non-collision singularities is non-empty for all $N > 3$.*

Conjecture 2. *The set of non-collision singularities has zero measure for all $N > 3$.*

Conjecture 1 probably goes back to Poincaré who was motivated by King Oscar II prize problem about analytic representation of collisionless solutions of the N-body problem. It was explicitly mentioned in Painlevé's lectures [Pa] where the author proved that for $N = 3$ there are no non-collision singularities. Soon after Painlevé, von Zeipel showed that if the system of N bodies has a non-collision singularity then some particle should fly off to infinity in finite time. Thus non-collision singularities seem quite counterintuitive. However in [MM] Mather and McGehee constructed a system of four bodies on the line where the particles go to infinity in finite time after an infinite number of binary collisions (it was known since the work of Sundman [Su] that binary collisions can be regularized so that the solutions can be extended beyond the collisions). Since Mather-McGehee example had collisions it did not solve Conjecture 1 but it made it plausible. Conjecture 1 was proved independently by Xia [X] for the spacial five-body problem and by Gerver [G1] for a planar $3N$ body problem where N is sufficiently large. The problem still remained open for $N = 4$ and for small N in the planar case. However in [G2] (see also [G3]) Gerver sketched a scenario which may lead to a non-collision singularity in the planar four-body problem. Gerver has not published the details of his construction due to a large amount of computations involved (it suffices to mention that even technically simpler large N case took 68 pages in [G1]). The goal of this paper is to realize Gerver's scenario in the simplified setting of two-center-two-body problem. Some of the estimates obtained here are used in the companion paper [Xu] which proves Conjecture 1 for the planar four body problem.

Conjecture 2 is mentioned by several authors, see e.g. [Sim, Sa3, K]. It is known that the set of initial conditions leading to the collisions has zero measure [Sa1] and that the same is true for non-collisions singularities if $N = 4$. To obtain the complete solution of this conjecture one needs to understand better of the structure of the non-collision singularities and our paper is one step in this direction.

1.2.2. *Well-posedness in other systems.* Recently the question of global well-posedness in PDE attracted a lot of attention motivated in part by the Clay Prize problem about well-posedness of the Navier-Stokes equation. One approach to constructing a blowup solutions for PDEs is to find a fixed point of a suitable renormalization scheme and to prove the convergence towards this fixed point (see e.g. [LS]). The same scheme is also used to analyze two-center-two-body problem and so we hope that the techniques developed in this paper can be useful in constructing singular solutions in more complicated systems.

1.2.3. *Poincaré's second species solution.* In his book [Po], Poincaré claimed the existence of the so-called second species solution in three-body problem, which are periodic orbits converging to collision chains as $\mu \to 0$. The concept of second species solution was generalized to the non-periodic case. In recent years significant progress was made in understanding second species solutions in both restricted [BM, FNS] and full [BN] three-body problem. However the understanding of general second species solutions generated by infinite aperiodic collision chains is still incomplete. Our result can be considered as a generalized version of second species solution. All masses are positive and there are infinitely many close encounters. Therefore the techniques developed in this paper can be useful in the study of the second species solutions.

1.3. **Extension to the 4-body problem.** Consider the same setting as in our main result but suppose that Q_1 and Q_2 are also free (not fixed). Then we can expect that during each encounter light particle transfers a fixed proportion of their energy and momentum to the heavy particle. The exponential growth of energy and momentum would cause Q_1 and Q_2 to go to infinity in finite time leading to a non-collision singularity.

Unfortunately a proof of this involves a significant amount of additional computations due to higher dimensionality of the full four-body problem. A good news is that similarly to the problem at hand, the Poincaré map of the full four-body problem will have only two strongly expanding directions whose origin could be understood by looking at our two-center-two-body problem. The other directions will be dominated by the most expanding ones. This allows our strategy to extend to the full four-body problem leading to the complete solution of the Painlevé conjecture. However, due to the length of the arguments, the details are presented in a separate paper [Xu].

1.4. **Plan of the paper.** The paper is organized as follows. Section 2 and 3 constitute the framework of the proof. In Section 2 we give a proof of the main Theorem 1 based on a careful study of the hyperbolicity of the properties of the Poincaré map. In Section 3, we summarize all later calculations and we prove the hyperbolicity results of Section 3. All the later sections provide calculations needed in Section 3. We define the local map to study the local interaction between Q_3 and Q_4 and global map to cover the time interval when Q_4 is traveling between Q_1 and Q_2. Sections 4, 6, 7 and 8 are devoted to the global map, while Sections 9,10, and 12 study local map. Relatively short Sections 5 and 11 contain some technical results pertaining to both local and global maps. Finally, we have two appendices. In Appendix A, we include an introduction to the Delaunay coordinates for Kepler motion, which is used extensively in our calculation. In Appendix B, we summarize the information about Gerver's model in [G2].

2. Proof of the main theorem

2.1. **Idea of the proof.** The proof of the Theorem 1 is based on studying the hyperbolicity of the Poincaré map. Our system has four degrees of freedom. We pick the zero energy surface and then consider a Poincaré section. The resulting Poincaré map is six dimensional. In turns out that for orbits of interest (that is, the orbits where the captured particle rotates around Q_2 and the traveler moves back and forth between Q_1 and Q_2) there is an invariant cone family which consists of vectors close to a certain two dimensional subspace such that all vectors in the cone are strongly expanding. This expansion comes from the combination of shearing (there are long stretches when the motion of the light particles is well approximated by the Kepler motion and so the derivatives are almost upper triangular) and twisting caused by the close encounters between Q_4 and Q_3 and between Q_4 and Q_1. We restrict our attention to a two dimensional surface whose tangent space belong to the invariant cone and construct on such a surface a Cantor set of singular orbits as follows. The two parameters coming from the two dimensionality of the surface will be used to control the phase of the close encounter between the particles and their relative distance. The strong expansion will be used to ensure that the choices made at the next step will have a little effect on the parameters at the previous steps. This Cantor set construction based on the instability of near colliding orbits is also among the key ingredients of the singular orbit constructions in [MM] and [X].

2.2. **Main ingredients.** In this section we present the main steps in proving Theorem 1. In Subsection 2.3 we describe a simplified model for constructing singular solutions given by Gerver [G2]. This model is based on the following simplifying assumptions:

- $\mu = 0,\ \chi = \infty$ so that Q_3(resp. Q_4) moves on a standard ellipse (resp. hyperbola).
- The particles Q_3, Q_4 do not interact except during a close encounter.
- Velocity exchange during close encounters can be modeled by an elastic collision.
- The action of Q_1 on light particles can be ignored except that during the close encounters of the traveler particle with Q_1 the angular momentum of the traveler with respect to Q_2 can be changed arbitrarily.

The main conclusion of [G2] is that the energy of the captured particle can be increased by a fixed factor while keeping the shape of its orbit unchanged. Gerver designs a procedure with two steps of collisions having the following properties:

- The incoming and outgoing asymptotes of the traveler are horizontal.
- The major axis of the captured particle remains vertical.
- After two steps of collisions, the elliptic orbit of the captured particle has the same eccentricity but smaller semimajor axis compared with the elliptic orbit before the first collision (see Fig 1 and 2).

For quantitative information, see Appendix B.

Since the shape is unchanged after the two trips described above the procedure can be repeated. Then the kinetic energies of the particles grow exponentially and so the time needed for j-th trip is exponentially small. Thus the particles can make infinitely many trips in finite time leading to a singularity. Our goal therefore is to get rid of the above mentioned simplifying assumptions.

In Subsection 2.4 we study near collision of the light particles. This assumption that velocity exchange can be modeled by elastic collision is not very restrictive since both energy and momentum are conserved during the exchange and any change of velocities conserving energy and momentum amounts to rotating the relative velocity by some angle and so it can be effected by an elastic collision. In Subsection 2.5 we state a result saying that away from the close encounters we can disregard interaction between the light particles and the action of Q_1 to the particle which is captured by Q_2 can indeed be disregarded. In Subsection 2.6 we study the Poincaré map corresponding to one trip of one light particle around Q_1. After some technical preparations we present the main result of that section Lemma 2.7 which says that

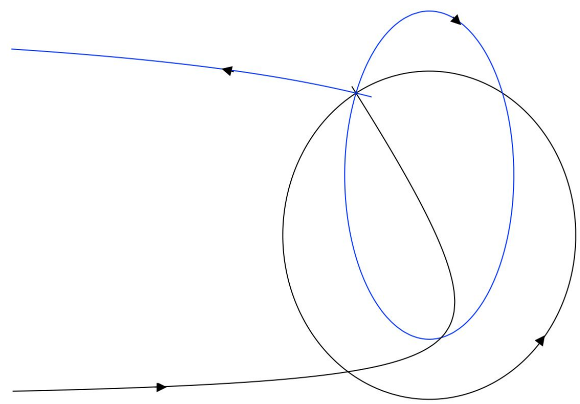

FIGURE 1. Angular momentum transfer

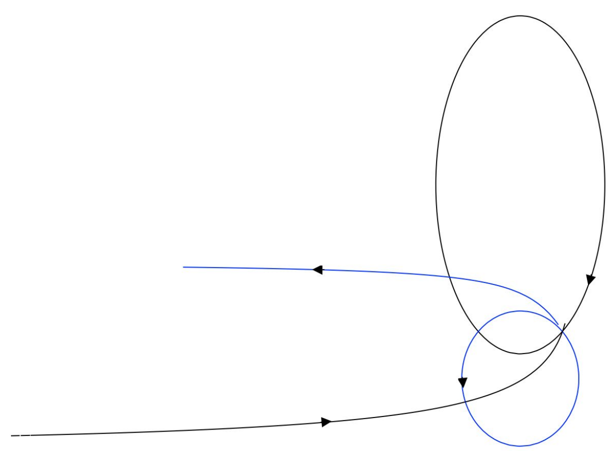

FIGURE 2. Energy transfer

after this trip the angular momentum of the traveler particle indeed can change in an arbitrary way. Finally in Subsection 2.7 we show how to combine the above ingredients to construct a Cantor set of singular orbits.

2.3. **Gerver map.** Following [G2], we discuss in this section the limit case $\mu = 0, \chi = \infty$. We assume that Q_3 has elliptic motion and Q_4 has hyperbolic motion with respect to the focus Q_2. Since $\mu = 0$, Q_3 and Q_4 do not interact unless they have exact collision. Since we assume that Q_4 just comes from the interaction from Q_1 located at $(-\infty, 0)$ and the new traveler particle is going to interact with Q_1 in the future, the slope of incoming asymptote θ_4^- of Q_4 and that of the outgoing asymptote $\bar{\theta}^+$ of the traveler particle should satisfy $\theta^- = 0,\ \bar{\theta}^+ = \pi$.

The Kepler motions of Q_3 and Q_4 has three first integrals E_i, G_i and g_i where E_i denotes the energy, G_i denotes the angular momentum and g_i denotes the argument of periapsis. Since the total energy of the system is zero we have $E_4 = -E_3$. It turns out convenient to use eccentricities $e_i = \sqrt{1 + 2G_i^2 E_i}$ instead of G_i since the proof of Theorem 1 involves a renormalization transformation and e_i are scaling invariant. The Gerver map describes the parameters of the elliptic orbit change during the interaction of Q_3 and Q_4. The orbits of Q_3 and Q_4 intersect in two points. We pick one of them. We use a discrete parameter $j \in \{1, 2\}$ to describe the first or the second collision in Gerver's construction.

Since Q_3 and Q_4 only interact when they are at the same point the only effect of the interaction is to change their velocities. Any such change which satisfies energy and momentum conservation can be described by an elastic collision. That is, velocities before and after the collision are related by

$$v_3^+ = \frac{v_3^- + v_4^-}{2} + \left|\frac{v_3^- - v_4^-}{2}\right| n(\alpha), \quad v_4^+ = \frac{v_3^- + v_4^-}{2} - \left|\frac{v_3^- - v_4^-}{2}\right| n(\alpha), \tag{2.1}$$

where $n(\alpha)$ is a unit vector making angle α with $v_3^- - v_4^-$.

With this in mind we proceed to define the Gerver map $\mathbf{G}_{e_4,j,\omega}(E_3, e_3, g_3)$. This map depends on two discrete parameters $j \in \{1, 2\}$ and $\omega \in \{3, 4\}$. The role of j has been explained above, and ω will tell us which particle will be the traveler after the collision.

To define $\mathbf{G}$ we assume that Q_4 moves along the hyperbolic orbit with parameters $(-E_3, e_4, g_4)$ where g_4 is fixed by requiring that the incoming asymptote of Q_4 is horizontal. We assume that Q_3 and Q_4 arrive to the j-th intersection point of their orbit simultaneously. At this point their velocities are changed by (2.1). After that the particle proceed to move independently. Thus Q_3 moves on an orbit with parameters $(\bar{E}_3, \bar{e}_3, \bar{g}_3)$, and Q_4 moves on an orbit with parameters $(\bar{E}_4, \bar{e}_4, \bar{g}_4)$.

If $\omega = 4$, we choose α so that after the exchange Q_4 moves on hyperbolic orbit and $\bar{\theta}_4^+ = \pi$ and let

$$\mathbf{G}_{e_4,j,4}(E_3, e_3, g_3) = (\bar{E}_3, \bar{e}_3, \bar{g}_3).$$

If $\omega = 3$ we choose α so that after the exchange Q_3 moves on hyperbolic orbit and $\bar{\theta}_3^+ = \pi$ and let

$$\mathbf{G}_{e_4,j,3}(E_3, e_3, g_3) = (\bar{E}_4, \bar{e}_4, \bar{g}_4).$$

In the following, to fix our notation, we always call the captured particle Q_3 and the traveler Q_4.

We will denote the ideal orbit parameters in Gerver's paper [G2] of Q_3 and Q_4 before the first (respectively second) collision with * (respectively **). Thus, for example, G_4^{**} will denote the angular momentum of Q_4 before the second collision. Moreover, the actual values after the first (respectively, after the second) collisions are denoted with a *bar* or *double bar*.

Note $\mathbf{G}$ has a skew product form

$$\bar{e}_3 = f_e(e_3, g_3, e_4), \quad \bar{g}_3 = f_g(e_3, g_3, e_4), \quad \bar{E}_3 = E_3 f_E(e_3, g_3, e_4).$$

This skew product structure will be crucial in the proof of Theorem 1 since it will allow us to iterate $\mathbf{G}$ so that E_3 grows exponentially while e_3 and g_3 remains almost unchanged.

The following fact plays a key role in constructing singular solutions.

Lemma 2.1 ([G2])**.** *There exist (e_3^*, g_3^*), such that for sufficiently small $\bar{\delta} > 0$ given $\omega', \omega'' \in \{3, 4\}$, there exist $\lambda_0 > 1$ and functions $e_4'(e_3, g_3)$, $e_4''(e_3, g_3)$, defined in a small (depending on $\bar{\delta}$) neighborhood of (e_3^*, g_3^*), such that*

(a) *for e_4^*, e_4^{**} given by $e_4'(e_3^*, g_3^*) = e_4^*$ and $e_4''(e_3^*, g_3^*) = e_4^{**}$, we have*

$$(e_3, g_3, E_3)^{**} = \mathbf{G}_{e_4^*, 1, \omega'}(e_3, g_3, E_3)^*, \quad (e_3, -g_3, \lambda_0 E_3)^* = \mathbf{G}_{e_4^{**}, 2, \omega''}(e_3, g_3, E_3)^{**},$$

(b) *If (e_3, g_3) lie in a $\bar{\delta}$ neighborhood of (e_3^*, g_3^*), we have*

$$(\bar{e}_3, \bar{g}_3, \bar{E}_3) = \mathbf{G}_{e_4'(e_3, g_3), 1, \omega'}(e_3, g_3, E_3), \quad (\bar{\bar{e}}_3, -\bar{\bar{g}}_3, \bar{\bar{E}}_3) = \mathbf{G}_{e_4''(e_3, g_3), 2, \omega''}\left(\bar{e}_3, \bar{g}_3, \bar{E}_3\right),$$

and

$$\bar{\bar{e}}_3 = e_3^*, \quad \bar{\bar{g}}_3 = g_3^*, \quad \bar{\bar{E}}_3 = \lambda(e_3, g_3) E_3, \text{ where } \lambda_0 - \bar{\delta} < \lambda < \lambda_0 + \bar{\delta}.$$

Part (a) is the main result of the above lemma. It allows us to increase energy after two collisions without changing the shape of the orbit in the limit case $\mu = 0, \chi = \infty$. Part (b) is of a more technical nature. It allows us to fight the perturbation coming from the fact that $\mu > 0$ and $\chi < \infty$.

Lemma 2.1 is a slight restatement of the main result of [G2]. Namely part (a) is proven in Sections 3 and 4 of [G2] and part (b) is stated in Section 5 of [G2] (see equations (5-10)–(5-13)). The proof of part (b) proceeds by a routine numerical computation. For the reader's convenience we review the proof of Lemma 2.1 in Appendix B explaining how the numerics is done.

Remark 2.1. *In fact Gerver produces a one parameter family of the periodic solution. Namely one can take e_3^* to be any number $(0, \frac{\sqrt{2}}{2})$ and $g_3^* = 0$. In the course of the proof of Theorem 1 we need to check several non-degeneracy conditions. This will be done numerically for $e_3^* = \dfrac{1}{2}$.*

Remark 2.2. *We try to minimize the use of numerics in our work. The use of numerics is always preceded by mathematical derivations. Readers can see that the numerics in this paper can also be done without using computer. We prefer to use the computer since computers are more reliable than humans when doing routine computations.*

2.4. **Asymptotic analysis, local map.** We assume that the two centers are at distance $\chi \gg 1$ and that Q_3, Q_4 have positive masses $0 < \mu \ll 1$. We also assume that Q_3 and Q_4 have initial orbit parameters $(E_3, \ell_3, e_3, g_3, e_4, g_4) \in \mathbb{R}^4 \times \mathbb{T}^2$ in the section $\{x_4(0) = -2,\ \dot{x}_4(0) > 0\}$ (the choice of this section in justified by Lemma 2.3 below). Here ℓ_3 stands for the mean anomaly of Q_3, see Appendix A. We let particles move until one of the particles reach the surface $\{x_4 = -2,\ \dot{x}_4 < 0\}$ moving on hyperbolic orbit. We measure the final orbit parameters $(\bar{E}_3, \bar{l}_3, \bar{e}_3, \bar{g}_3, \bar{e}_4, \bar{g}_4)$. We call the mapping moving initial positions of the particles to their final positions the **local map** $\mathbb{L}$. In Fig. 3 of Section 3.2 the local map is to the right of the section $\{x = -2\}$.

Lemma 2.2. *Suppose that the initial orbit parameters $(E_3, \ell_3, e_3, g_3, e_4, g_4)$ are such that the traveler particle(s) satisfy $\theta^- = O(\mu)$ and $\bar{\theta}^+ = \pi + O(\mu)$ then the following asymptotics holds uniformly*

$$(\bar{E}_3, \bar{e}_3, \bar{g}_3) = \mathbf{G}_{e_4}(E_3, e_3, g_3) + o(1), \text{ as } \mu \to 0, \chi \to \infty.$$

The lemma tells us Gerver map is a good approximation of the local map $\mathbb{L}$ for the real case $0 < \mu \ll 1 \ll \chi < \infty$ for the orbits of interest. Lemma 2.2 will be proven in Section 10 where we also present some additional information about the local map (see Lemma 10.2).

We also need the following fact. Fix a small number $\tilde{\theta}$.

Lemma 2.3. *Suppose the initial orbit parameters $(E_3, \ell_3, e_3, g_3, e_4, g_4)$ for the local map are such that $E_3 = -\dfrac{1}{2} + O(\mu)$, and the incoming and outgoing asymptotes of Q_4 satisfy $\theta^- = O(\mu)$ and $|\bar{\theta}^+ - \pi| \le \tilde{\theta} \ll 1$. Then for μ sufficiently small and χ sufficiently large we have $|Q_3| \le 2 - \delta$ where $\delta > 0$ is a constant independent of μ and χ.*

The proof of this lemma is also given in to Section 10.

2.5. **Asymptotic analysis, global map.** As before we assume that the two centers are at distance $\chi \gg 1$. We assume that initially Q_3 moves on an elliptic orbit, Q_4 moves on hyperbolic orbit and $\{x_4(0) = -2, \dot{x}_4(0) < 0\}$. We assume that $|y_4(0)| < C$ and after moving around Q_1 it hits the surface $\{x_4 = -2,\ \dot{x}_4 > 0\}$ so that $|y_4| < C$. We call the mapping moving initial positions of the particles to their final positions the (pre) **global map** $\mathbb{G}$. In Section 2.6 we will slightly modify the definition of the global map but it will not change the essential features discussed here. In Fig 3 from Section 3.2, the global map is to the left of the section $\{x = -2\}$. We let $(E_3, \ell_3, e_3, g_3, e_4, g_4)$ denote the initial orbit parameters measured in the section $\{x_4 = -2,\ \dot{x}_4 < 0\}$ and $(\bar{E}_3, \bar{l}_3, \bar{e}_3, \bar{g}_3, \bar{e}_4, \bar{g}_4)$ denote the final orbit parameters measured in the section $\{x_4 = -2,\ \dot{x}_4 > 0\}$. Fix a large constant C.

Lemma 2.4. *Assume that $|y_4| < C$ holds both at initial and final moments. Then uniformly in χ, μ we have the following estimates*

$$(a)\ \bar{E}_3 - E_3 = O(\mu), \quad \bar{G}_3 - G_3 = O(\mu), \quad \bar{g}_3 - g_3 = O(\mu).$$
$$(b)\ \theta_4^+ = \pi + O(\mu), \quad \bar{\theta}_4^- = O(\mu).$$

The proof of this lemma is given in Section 4.

2.6. **Admissible surfaces.** Given a sequence $\boldsymbol{\omega}$ we need to construct orbits having singularity with symbolic sequence $\boldsymbol{\omega}$.

We will study the Poincaré map $\mathcal{P} = \mathbb{G} \circ \mathbb{L}$ to the surface $\{x_4 = -2,\ \dot{x}_4 > 0\}$. It is a composition of the local and global maps defined in the previous sections.

Given δ consider open sets in the phase space defined by

$$U_1(\delta) = \left\{ \left| E_3 - \left(-\frac{1}{2}\right) \right|,\ |e_3 - e_3^*|,\ |g_3 - g_3^*|,\ |\theta_4^-| < \delta, \quad |e_4 - e_4^*| < \sqrt{\delta} \right\},$$

$$U_2(\delta) = \left\{ |E_3 - E_3^{**}|,\ |e_3 - e_3^{**}|,\ |g_3 - g_3^{**}|,\ |\theta_4^-| < \delta, \quad |e_4 - e_4^{**}| < \sqrt{\delta} \right\}.$$

We will also need the renormalization map $\mathcal{R}$ defined as follows. Partition our section $\{x_4 = -2,\ \dot{x}_4 > 0\}$ into cubes of size $1/\sqrt{\chi}$ and on each cube we rescale the space and time so that

- in the center of the cube Q_3 has elliptic orbit with energy $-\dfrac{1}{2}$.
- the potential of the fixed centers is still $1/|Q_i - Q_j|$, $i = 1, 2,\ j = 3, 4$.

In addition we reflect the coordinates with respect to x axis. We define $\lambda = |E_3| > 1$ as the dilation rate where E_3 is the energy of Q_3 at the center of each cube. We push forward each cube to the section $\{x_4 = -2/\lambda,\ \dot{x}_4 > 0\}$. *We include the piece of orbits from the section $\{x_4 = -2,\ \dot{x}_4 > 0\}$ to $\{x_4 = -2/\lambda,\ \dot{x}_4 > 0\}$ to the global map and apply the $\mathcal{R}$ to the section* $\{x_4 = -2/\lambda,\ \dot{x}_4 > 0\}$. So the locally constant map $\mathcal{R}$ amounts to zooming in the configuration by multiplying by λ and slowing down the velocity by dividing $\sqrt{\lambda}$. This is then followed by a reflection. We have $\mathcal{R}(\{x_4 = -2/\lambda,\ \dot{x}_4 > 0\}) = \{x_4 = -2,\ \dot{x}_4 > 0\}$, and

$$\mathcal{R}(E_3, \ell_3, e_3, g_3, e_4, g_4) = (E_3/\lambda, \ell_3, e_3, -g_3, e_4, -g_4).$$

Note that the rescaling changes (for the orbits of interest, increases) the distance between the fixed centers by sending χ to $\lambda\chi$. Observe that at each step we have the freedom of choosing the centers of the cubes. We describe how this choice is made in the next section. In the following we give a proof of the main theorem based on the three lemmas, whose proofs are in the next section.

Lemma 2.5. *There are cone families $\mathcal{K}_1$ on $T_{U_1}(\mathbb{R}^4 \times \mathbb{T}^2)$ and $\mathcal{K}_2$ on $T_{U_2}(\mathbb{R}^4 \times \mathbb{T}^2)$, each of which contains a two dimensional plane and a constant c such that for all $\boldsymbol{x} \in U_1(\delta)$ satisfying $\mathcal{P}(\boldsymbol{x}) \in U_2(\delta)$, and for all $\boldsymbol{x} \in U_2(\delta)$ satisfying $\mathcal{R}\circ\mathcal{P}(\boldsymbol{x}) \in U_1(\delta)$,*

(a) *$d\mathcal{P}(\mathcal{K}_1) \subset \mathcal{K}_2$, $d(\mathcal{R} \circ \mathcal{P})(\mathcal{K}_2) \subset \mathcal{K}_1$.*
(b) *If $v \in \mathcal{K}_1$, then $\|d\mathcal{P}(v)\| \geq c\chi\|v\|$.*
If $v \in \mathcal{K}_2$, then $\|d(\mathcal{R} \circ \mathcal{P})(v)\| \geq c\chi\|v\|$.

We call a C^1 surface $S_1 \subset U_1(\delta)$ (respectively $S_2 \subset U_2(\delta)$) **admissible** if $TS_1 \subset \mathcal{K}_1$ (respectively $TS_2 \subset \mathcal{K}_2$).

Lemma 2.6. (a) *The vector $\tilde{w} = \dfrac{\partial}{\partial \ell_3}$ is in $\mathcal{K}_i$.*
(b) *Any plane Π in $\mathcal{K}_i$ the map projection map $\pi_{e_4,\ell_3} = (de_4, d\ell_3) : \Pi \to \mathbb{R}^2$ is one-to-one. In other words (e_4, ℓ_3) can be used as coordinates on admissible surfaces.*

We call an admissible surface **essential** if π_{e_4,ℓ_3} is an $I \times \mathbb{T}^1$ for some interval I. In other words given $e_4 \in I$ we can prescribe ℓ_3 arbitrarily.

Lemma 2.7. (a) *Given an essential admissible surface $S_1 \in U_1(\delta)$ and $\tilde{e}_4 \in I(S_1)$ there exists $\tilde{\ell}_3$ such that $\mathcal{P}((\tilde{e}_4, \tilde{\ell}_3)) \in U_2(K\delta)$. Moreover if dist$(\tilde{e}_4, \partial I) > 1/\chi$ then there is a neighborhood $V(\tilde{e}_4)$ of $(\tilde{e}_4, \tilde{\ell}_3)$ such that $\pi_{e_4,\ell_3} \circ \mathcal{P}$ maps V surjectively to*
$$\{|e_4 - e_4^*| < K\delta\} \times \mathbb{T}^1.$$
(b) *Given an essential admissible surface $S_2 \subset U_2(\delta)$ and $\tilde{e}_4 \in I(S_2)$ there exists $\tilde{\ell}_3$ such that $\mathcal{R} \circ \mathcal{P}((\tilde{e}_4, \tilde{\ell}_3)) \in U_1(K\delta)$. Moreover if dist$(\tilde{e}_4, \partial I) > 1/\chi$ then there is a neighborhood $V(\tilde{e}_4)$ of $(\tilde{e}_4, \tilde{\ell}_3)$ such that $\pi_{e_4,\ell_3} \circ \mathcal{R} \circ \mathcal{P}$ maps V surjectively to*
$$\{|e_4 - e_4^{**}| < K\delta\} \times \mathbb{T}^1.$$
(c) *For points in $V(\tilde{e}_4)$ from parts (a) and (b), the particles avoid collisions before the next return and the minimal distance between the particles satisfies $\mu\delta \leq d \leq \dfrac{\mu}{\delta}$.*

Note that by Lemma 2.5 the diameter of $V(\tilde{e}_4)$ is $O(\delta/\chi)$.

2.7. **Construction of the singular orbit.** Fix a number ε which is small but is much larger than both μ and $1/\chi$. Let S_0 be an admissible surface such that the diameter of S_0 is much larger than $1/\chi$ and such that on S_0 we have
$$|e_3 - \hat{e}_3| < \varepsilon, \quad |g_3 - \hat{g}_3| < \varepsilon.$$
where $(\hat{e}_3, \hat{g}_3)$ is close to (e_3^*, g_3^*). For example, we can pick a point $\boldsymbol{x} \in U_1(\delta)$ and let $\hat{w}$ be a vector in $\mathcal{K}_1(\boldsymbol{x})$ such that $\dfrac{\partial}{\partial \ell_3}(\hat{w}) = 0$. Then let
$$S_0 = \{(E_3, \ell_3, e_3, g_3, e_4, g_4)(\boldsymbol{x}) + a\hat{w} + (0, b, 0, 0, 0, 0)\}_{a \leq \varepsilon/\bar{K}}$$

where $\bar{K}$ is a large constant.

We wish to construct a singular orbit in S_0. We define S_j inductively so that S_j is component of $\mathcal{P}(S_{j-1}) \cap U_2(\delta)$ if j is odd and S_j is component of $(\mathcal{R} \circ \mathcal{P})(S_{j-1}) \cap U_1(\delta)$ if j is even (we shall show below that such components exist). Let $\boldsymbol{x} = \lim_{j\to\infty}(\mathcal{R}\mathcal{P}^2)^{-j} S_{2j}$. We claim that $\boldsymbol{x}$ has singular orbit. Indeed by Lemma 2.1 the unscaled energy of Q_4 satisfies $E(j) \geq (\lambda_0 - \tilde{\delta})^{j/2}$ where $\tilde{\delta} \to 0$ as $\delta \to 0$. Accordingly the velocity of Q_4 during the trip j is bounded from below by $c\sqrt{E(j)} \geq c(\lambda_0 - \tilde{\delta})^{j/4}$. Therefore $t_{j+1} - t_j = O((\lambda_0 - \tilde{\delta})^{-j/4})$ and so $t_* = \lim_{j\to\infty} t_j < \infty$ as needed.

It remains to show that we can find a component of $\mathcal{P}(S_{2j})$ inside $U_2(\delta)$ and a component of $(\mathcal{R} \circ \mathcal{P}(S_{2j+1}))$ inside $U_1(\delta)$. Note that Lemma 2.7 allows to choose such components inside larger sets $U_2(K\delta)$ and $U_1(K\delta)$.

First note that by Lemma 2.4 on $\mathcal{P}(S_{2j}) \bigcap U_1(K\delta)$ and on $(\mathcal{R} \circ \mathcal{P}^2)(S_{2j}) \bigcap U_2(K\delta)$ we have $\theta_4^- = O(\mu)$. Also by Lemma 2.7 e_4 can be prescribed arbitrarily. In other words we have a good control on the orbit of Q_4.

In order to control the orbit of Q_3 note that by Lemma 2.5(b) the preimage of S_{2j} has size $O(1/\chi)$ and so by Lemmas 2.2, 2.4 and 2.6 given ε we have that e_3 and g_3 have oscillation less than ε on S_{2j} if μ is small enough. Namely part (b) of Lemma 2.6 shows that e_3 and g_3 have oscillation $O(1/\chi)$ on the preimage of S_{2j} while Lemmas 2.2 and 2.4 show that the oscillations do not increase much after application of local and global map. Thus there exist $(\hat{e}_3, \hat{g}_3)$ such that on S_{2j} we have

$$|e_3 - \hat{e}_3| < \varepsilon, \quad |g_3 - \hat{g}_3| < \varepsilon.$$

Also due to rescaling defined in Section 2.6 and Lemma 2.4, we have

$$\left| E_3 - \left(-\frac{1}{2}\right) \right| = O\left(\frac{1}{\sqrt{\chi}} + \mu\right).$$

Set

$$\tilde{S}_{2j+1} = \mathcal{P}V(e'(\hat{e}_3, \hat{g}_3)), \quad \tilde{S}_{2j+2} = (\mathcal{R} \circ \mathcal{P})V(e''(\hat{e}_3, \hat{g}_3)). \tag{2.2}$$

Then on $\tilde{S}_{2j+1}$ we shall have

$$|e_3 - e_3^{**}| < K\varepsilon, \quad |g_3 - g_3^{**}| < K\varepsilon \text{ and } |E_3 - E_3^{**}| < K\varepsilon$$

while on $\tilde{S}_{2j+2}$ we shall have

$$|e_3 - e_3^{*}| < K^2\varepsilon, \quad |g_3 - g_3^{*}| < K^2\varepsilon \text{ and } \left| E_3 + \frac{1}{2} \right| < K(1/\sqrt{\chi} + \mu).$$

Denote

$$S_{2j+1} = \tilde{S}_{2j+1} \cap \{|e_4 - e''(e_3^*, g_3^*)| < \sqrt{\delta}\}, \quad S_{2j+2} = \tilde{S}_{2j+2} \cap \{|e_4 - e'(e_3^*, g_3^*)| < \sqrt{\delta}\}.$$

Taking ε so small that $K^2\varepsilon < \delta$ we get that $S_{2j+1} \in U_2(\delta)$, $S_{2j+2} \in U_1(\delta)$ as needed.

Finally we use the freedom to choose the appropriate partition in the definition of $\mathcal{R}$ to ensure that $\mathcal{R}$ is continuous on the preimage of $V(e'(\hat{e}_3, \hat{g}_3))$ so that $V(e'(\hat{e}_3, \hat{g}_3))$ is a smooth surface.

Remark 2.3. *In fact we do not need to use exactly* $e'(\hat{e}_3, \hat{g}_3)$ *and* $e''(\hat{e}_3, \hat{g}_3)$ *in* (2.2)*. Namely any* $V(e_4^\dagger)$ *and* $V(e_4^\ddagger)$ *would do provided that*

$$\left|e_4^\dagger - e_4'(\hat{e}_3, \hat{g}_3)\right| < \varepsilon, \quad \left|e_4^\ddagger - e_4''(\hat{e}_3, \hat{g}_3)\right| < \varepsilon.$$

Different choices of $e_4^\dagger$ *and* $e_4^\ddagger$ *allow us obtain different orbits. Since such freedom exists at each step of our construction we have a Cantor set of singular orbits with a given symbolic sequence* $\boldsymbol{\omega}$*.*

3. Hyperbolicity of the Poincaré map

3.1. **Construction of invariant cones.** Here we derive Lemma 2.5, 2.6 and 2.7 from the asymptotics of the derivative of local and global maps.

Lemma 3.1. *There exist continuous functions* $\mathbf{u}_j(\boldsymbol{x})$, $\mathbf{l}_j(\boldsymbol{x})$ *and* $B_j(\boldsymbol{x})$ *such that if* $\boldsymbol{x} \in U_j(\delta),\ j = 1, 2$ *is such that* $\mathbb{L}(\boldsymbol{x})$ *satisfies* $\theta_4^- = O(\mu), |\bar{\theta}_4^+ - \pi| \le \tilde{\theta} \ll 1$ *where* $\tilde{\theta}$ *is independent of* μ, χ*, then we have*

$$d\mathbb{L}(\boldsymbol{x}) = \frac{1}{\mu}\mathbf{u}_j(\boldsymbol{x}) \otimes \mathbf{l}_j(\boldsymbol{x}) + B_j(\boldsymbol{x}) + o(1).$$

Moreover there exist a linear functional $\hat{\mathbf{l}}_j$ *and a vector* $\hat{\mathbf{u}}_j$ *such that*

$$\mathbf{l}_j = \hat{\mathbf{l}}_j + o(1), \quad \mathbf{u}_j = \hat{\mathbf{u}}_j + o(1), \quad B_j = \hat{B}_j + o(1), \ \text{as } \delta,\ \mu,\ 1/\chi \to 0.$$

This lemma is proven in Section 12.

Lemma 3.2. *Let* $\boldsymbol{x}$ *and* $\boldsymbol{y} = \mathbb{G}(\boldsymbol{x}) \in U_{3-j}(\delta)$*, be such that* $|y(\boldsymbol{x})| \le C$, $|y(\boldsymbol{y})| \le C$ *Then there exist linear functionals* $\bar{\mathbf{l}}_j(\boldsymbol{x})$ *and* $\bar{\bar{\mathbf{l}}}_j(\boldsymbol{x})$ *and vectorfields* $\bar{\mathbf{u}}_j(\boldsymbol{y})$ *and* $\bar{\bar{\mathbf{u}}}_j(\boldsymbol{y}),\ j = 1, 2,$ *such that*

$$d\mathbb{G}(\boldsymbol{x}) = \chi^2 \bar{\mathbf{u}}_j(\boldsymbol{y}) \otimes \bar{\mathbf{l}}_j(\boldsymbol{x}) + \chi \bar{\bar{\mathbf{u}}}_j(\boldsymbol{y}) \otimes \bar{\bar{\mathbf{l}}}_j(\boldsymbol{x}) + O(\mu^2\chi).$$

Moreover there exist vector w_j *and linear functionals* $\hat{\bar{\mathbf{l}}}_j, \hat{\bar{\bar{\mathbf{l}}}}_j$ *such that if* $\delta, \mu, \frac{1}{\chi} \to 0$ *then*

$$\bar{\mathbf{l}}_j(\boldsymbol{x}) \to \hat{\bar{\mathbf{l}}}_j, \quad \bar{\bar{\mathbf{l}}}_j(\boldsymbol{x}) \to \hat{\bar{\bar{\mathbf{l}}}}_j.$$

and

$$span(\bar{\mathbf{u}}_j(\boldsymbol{y}), \bar{\bar{\mathbf{u}}}_j(\boldsymbol{y})) \to span(w_j, \tilde{w}).$$

where $\tilde{w} = \dfrac{\partial}{\partial \ell_3}$.

This lemma is proven in Section 3.2.

Lemma 3.3. *The following non degeneracy conditions are satisfied.*

(a1) *span*$(\hat{\mathbf{u}}_1, B(\hat{\mathbf{l}}_1(\tilde{w})d\mathcal{R}w_2 - \hat{\mathbf{l}}_1(d\mathcal{R}w_2)\tilde{w}))$ *is transversal to* $Ker(\hat{\bar{\mathbf{l}}}_1) \cap Ker(\hat{\bar{\bar{\mathbf{l}}}}_1)$.
(a2) $de_4(span(d\mathcal{R}w_2, d\mathcal{R}\tilde{w})) \neq 0$.
(b1) *span*$(\hat{\mathbf{u}}_2, B(\hat{\mathbf{l}}_2(\tilde{w})w_1 - \hat{\mathbf{l}}_2(w_1)\tilde{w}))$ *is transversal to* $Ker(\hat{\bar{\mathbf{l}}}_2) \cap Ker(\hat{\bar{\bar{\mathbf{l}}}}_2)$.
(b2) $de_4(w_1) \neq 0$.

This lemma is proven in Section 3.3.

Definition 3.1. *We now take* $\mathcal{K}_1$ *to be the set of vectors which make an angle less than a small constant* η *with span*$(d\mathcal{R}w_2, \tilde{w}_2)$, *and* $\mathcal{K}_2$ *to be the set of vectors which make an angle less than a small constant* η *with span*$(w_1, \tilde{w}_1)$.

Proof of Lemma 2.5. Consider for example the case where $\boldsymbol{x} \in U_2(\delta)$. We claim that if δ, μ are small enough then $d\mathbb{L}(\text{span}(w_1, \tilde{w}))$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\bar{\mathbf{l}}}_2$. Indeed take Γ such that $\mathbf{l}(\Gamma) = 0$. If $\Gamma = aw_1 + \tilde{a}\tilde{w}$ then $a\mathbf{l}_2(w_1) + \tilde{a}\mathbf{l}_2(\tilde{w}) = 0$. It follows that the direction of Γ is close to the direction of $\hat{\Gamma} = \hat{\mathbf{l}}_2(\tilde{w})w_1 - \hat{\mathbf{l}}_2(w_1)\tilde{w}$. Next take $\tilde{\Gamma} = bw + \tilde{b}\tilde{w}$ where $b\mathbf{l}_2(w_1) + \tilde{b}\mathbf{l}_2(\tilde{w}) \neq 0$. Then the direction of $d\mathbb{L}\tilde{\Gamma}$ is close to $\hat{\mathbf{u}}_2$ and the direction of $d\mathbb{L}(\Gamma)$ is close to $B(\hat{\Gamma})$ so our claim follows.

Thus for any plane Π close to $\text{span}(w_1, \tilde{w})$ we have that $d\mathbb{L}(\Pi)$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\bar{\mathbf{l}}}_2$. Take any $Y \in \mathcal{K}_2$. Then either Y and w_1 are linearly independent or Y and $\tilde{w}$ are linearly independent. Hence $d\mathbb{L}(\text{span}(Y, w_1))$ or $d\mathbb{L}(\text{span}(Y, \tilde{w}))$ is transversal to $\text{Ker}\bar{\mathbf{l}}_2 \cap \text{Ker}\bar{\bar{\mathbf{l}}}_2$. Accordingly either $\bar{\mathbf{l}}_2(d\mathbb{L}(Y)) \neq 0$ or $\bar{\bar{\mathbf{l}}}_2(d\mathbb{L}(Y)) \neq 0$. If $\bar{\mathbf{l}}_2(d\mathbb{L}(Y)) \neq 0$ then the direction of $d(\mathbb{G} \circ \mathbb{L})(Y)$ is close to $\bar{\mathbf{u}}$. If $\bar{\mathbf{l}}_2(d\mathbb{L}(Y)) = 0$ then the direction of $d(\mathbb{G} \circ \mathbb{L})(Y)$ is close to $\bar{\bar{\mathbf{u}}}$. In either case $d(\mathcal{R}\mathbb{G} \circ \mathbb{L})(Y) \in \mathcal{K}_1$ and $\|d(\mathbb{G} \circ \mathbb{L})(Y)\| \geq c\chi\|Y\|$. This completes the proof in the case $\boldsymbol{x} \in U_2(\delta)$. The case where $\boldsymbol{x} \in U_1(\delta)$ is similar. □

Proof of Lemma 2.6. Part (a) follows from the definition of $\mathcal{K}_i$. Also by part (b) of Lemma 3.3 the map $\pi : \text{span}(w, \tilde{w}) \to \mathbb{R}^2$ given by $\pi(\Gamma) = (d\ell_3(\Gamma), de_4(\Gamma))$ is invertible. Namely if $\Gamma = aw + \tilde{a}\tilde{w}$ then

$$a = \frac{de_4(\Gamma)}{de_4(w)}, \quad \tilde{a} = d\ell_3(\Gamma) - ad\ell_3(w).$$

Accordingly π is invertible on planes close to $\text{span}(w, \tilde{w})$ proving our claim. □

To prove Lemma 2.7 we need two auxiliary results.

Sublemma 3.4. *Given* $\tilde{e}_4$ *there exists* $\tilde{\ell}_3$ *such that* $\mathcal{P}(\tilde{e}_4, \tilde{\ell}_3) \in U_2(\delta)$.

The proof of this sublemma is postponed to Section 11.2.

Sublemma 3.5. *Let* $\mathcal{F}$ *be a map on* $\mathbb{R}^2$ *which fixes the origin and such that if* $|\mathcal{F}(z)| < R$ *then* $\|d\mathcal{F}(X)\| \geq \bar{\chi}\|X\|$. *Then for each* a *such that* $|a| < R$ *there exists* z *such that* $|z| < R/\bar{\chi}$ *and* $\mathcal{F}(z) = a$.

Proof. Without the loss of generality we may assume that $a = (r, 0)$. Let $V(z)$ be the direction field defined by the condition that the direction of $d\mathcal{F}(V(z))$ is parallel to $(1, 0)$. Let $\gamma(t)$ be the integral curve of V passing through the origin and parameterized by the arclength. Then $\mathcal{F}(\gamma(t))$ has form $(\sigma(t), 0)$ where $\sigma(0) = 0$ and $|\dot\sigma(t)| \geq \bar\chi$ as long as $|\sigma| < R$. Now the statement follows easily. □

Proof of Lemma 2.7. (a) We claim that it suffices to show that for each $(\bar e_4, \bar\ell_3)$ such that $|\bar e_4 - e_4^{**}| < \sqrt{\delta}$ there exist $(\hat e_4, \hat\ell_3)$ such that

$$\mathcal{P}(\hat e_4, \hat\ell_3) = (\bar e_4, \bar\ell_3). \tag{3.1}$$

Indeed in that case Sublemma 4.9 from Section 4.3 says that the outgoing asymptote is almost horizontal. Therefore by Lemma 2.2 our orbit has (E_3, e_3, g_3) close to $\mathbf{G}_{\tilde e_4,2,4}(E_3(\hat e_4, \hat\ell_3), e_3(\hat e_4, \hat\ell_3), g_3(\hat e_4, \hat\ell_3))$. Next Lemma 2.4 shows that after the application of $\mathbb{G}$, (E_3, e_3, g_3) change little and θ_4^- becomes $O(\mu)$ so that $\mathcal{P}(\hat e_4, \hat\ell_3) \in U_2(K\delta)$.

We will now prove (3.1). Our coordinates allow us to treat $\mathcal{P}$ as a map $\mathbb{R}\times\mathbb{T} \to \mathbb{R}\times\mathbb{T}$. Due to Lemma 2.5 we can apply Sublemma 3.5 to the covering map $\tilde{\mathcal{P}} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\bar\chi = c\chi$ obtaining (3.1). Part (b) of the lemma is similarly proven.

Part (c) follows from Lemma 10.2 proven in Section 10. □

3.2. **Expanding directions of the global map.** Estimating the derivative of the global map is the longest part of the paper. It occupies Sections 5–8.

It will be convenient to use the Delaunay coordinates (L_3, ℓ_3, G_3, g_3) for Q_3 and (G_4, g_4) for Q_4. Delaunay coordinates are action-angle coordinates for the Kepler problem. We collect some facts about the Delaunay coordinates in Appendix A.

We divide the plane into several pieces by lines $x_4 = -2$ and $x_4 = -\dfrac{\chi}{2}$. Those lines cut the orbit of Q_4 into 4 pieces:

- $\{x_4 = -2,\ \dot x_4 < 0\} \to \left\{x_4 = -\dfrac{\chi}{2},\ \dot x_4 < 0\right\}$. We call this piece (I).
- $\left\{x_4 = -\dfrac{\chi}{2},\ \dot x_4 < 0\right\} \to \left\{x_4 = -\dfrac{\chi}{2},\ \dot x_4 > 0\right\}$ turning around Q_1. We call it (III).
- $\left\{x_4 = -\dfrac{\chi}{2},\ \dot x_4 > 0\right\} \to \{x_4 = -2,\ \dot x_4 > 0\}$. We call it (V)
- $\{x_4 = -2,\ \dot x_4 > 0\} \to \{x_4 = -2,\ \dot x_4 < 0\}$ turning around Q_2.

We composition of the first three pieces constitutes the global map. The last piece defines the local map. See Fig 3. Notice that when we define $\mathcal{R}$ in Section 2.6, after the second collision in Gerver's construction, the global map sends $\{x_4 = -2,\ \dot x_4 < 0\}$ to $\{x_4 = -2/\lambda,\ \dot x_4 > 0\}$. Then $\mathcal{R}$ sends $\{x_4 = -2/\lambda,\ \dot x_4 > 0\}$ to $\{x_4 = -2,\ \dot x_4 > 0\}$ before applying local map. So without leading to confusion, when we are talking about sections after the second collision, we always talk about $\mathcal{R} \circ \mathbb{G}$ so that the section $\{x_4 = -2, \dot x_4 < 0\}$ is sent to $\{x_4 = -2, \dot x_4 > 0\}$.

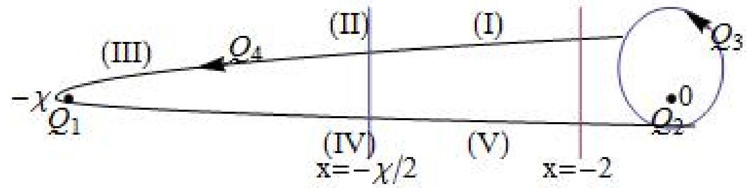

FIGURE 3. Poincaré sections

The line $x_4 = -\dfrac{\chi}{2}$ is convenient because if Q_4 is moving to the right of the line $x_4 = -\dfrac{\chi}{2}$, its motion can be treated as a hyperbolic motion focused at Q_2 with perturbation caused by Q_1 and Q_3. If Q_4 is moving to the left of this line, its motion can be treated as a hyperbolic motion focused at Q_1 perturbed by Q_2 and Q_3.

Since we use different guiding centers to the left and right of the line of $x_4 = -\dfrac{\chi}{2}$ we will need to change variables when Q_4 hits this line. This will give rise to two more matrices for the derivative of the global map: (II) will correspond to the change of coordinates from right to left and (IV) will correspond for the change of coordinates from left to right. Thus $d\mathbb{G} = (V)(IV)(III)(II)(I)$. In turn, each of the matrices (II) and (IV) will be products of three matrices corresponding to changing one variable at a time. Thus we will have $(II) = [(iii)(ii)](i)$ and $(IV) = (iii')[(ii')(i')]$.

The asymptotics of the above mentioned matrices is presented in the two propositions below.

To refer to a certain subblock of a matrix $(\sharp)$, we use the following convention:

$$(\sharp) = \left[\begin{array}{c|c} (\sharp)_{33} & (\sharp)_{34} \\ \hline (\sharp)_{43} & (\sharp)_{44} \end{array}\right].$$

Thus $(\sharp)_{33}$ is a 4×4 matrix and $(\sharp)_{44}$ is a 2×2 matrix. To refer to the $(i,j)-th$ entry of a matrix $(\sharp)$ (in the Delaunay coordinates mentioned above) we use $(\sharp)(i,j)$. For example, $(I)(1,3)$ means the derivative of L_3 with respect to G_3 when the orbit moves between sections $\{x_4 = -2\}$ and $\left\{x_4 = -\dfrac{\chi}{2}\right\}$.

Proposition 3.6. *Under the assumptions of Lemma 3.2 the matrices introduced above satisfy the following estimates.*

$$(I) = \left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) \\ O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\mu) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) \\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right],$$

$$(i) = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{\tilde{G}_{4R}/k_R\tilde{L}_3}{k_R^2\tilde{L}_3^2+\tilde{G}_{4R}^2}+O(\frac{1}{\chi}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & \frac{1}{k_R^2\tilde{L}_3^2+\tilde{G}_{4R}^2}+O(\frac{1}{\chi}) & -\frac{1}{k_R\tilde{L}_3}+O(\frac{1}{\chi}) \end{array}\right],$$

$$[(iii)(ii)] = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & 1 & -\chi \\ O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & -\frac{1}{\tilde{L}_3}+O(1/\chi) & \frac{\chi}{\tilde{L}_3}+O(1) \end{array}\right],$$

$$(III) = \left[\begin{array}{cccc|cc} 1+O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) \\ O(\chi) & O(1) & O(1) & O(1) & O(1) & O(1) \\ O(1/\chi) & O(1/\chi) & 1+O(1/\chi) & O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) \\ O(1/\chi) & O(1/\chi) & O(1/\chi) & 1+O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) \\ \hline O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(1) & O(1) \\ O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(1) & O(1) \end{array}\right],$$

$$[(ii')(i')] =$$

$$\left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline O(1) & O(1/\chi^2) & O(1/\chi^2) & O(1/\chi^2) & \frac{\chi}{\hat{L}_3^2}+O(1) & -\frac{\chi}{\hat{L}_3}+O(1) \\ O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & \frac{1}{\hat{L}_3^2}+O(1/\chi) & -\frac{1}{\hat{L}_3}+O(1/\chi) \end{array}\right],$$

$$(iii') =$$

$$\left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{\hat{G}_{4R}/(k_R)}{(k_R^2\hat{L}_3^2+G_{4R}^2)}+O(\frac{1}{\chi}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & -\frac{k_R\hat{L}_3}{k_R^2\hat{L}_3^2+\hat{G}_{4R}^2}+O(\frac{1}{\chi}) & k_R\hat{L}_3+O(\frac{1}{\chi}) \end{array}\right],$$

$$(V) = \left[\begin{array}{cccc|cc} O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\chi) & 1+O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(\mu^2\chi) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\mu^2\chi) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) \\ \hline O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right].$$

where $k_R = 1+\mu$, $\tilde L_3, \tilde G_4$ are the initial values of $\mathbb{G}$ of L_3, G_4 and $\hat L_3, \hat G_4$ are the initial values of $\mathbb{G}$ of L_3, G_4. Moreover, the matrix of the renormalization map $\mathcal{R}$ has the form $\mathrm{diag}\{\sqrt{\lambda}, 1, -\sqrt{\lambda}, -1, -\sqrt{\lambda}, -1\}$, where the constant λ is the dilation rate defined in Section 2.6 and the "$-$" appears due to the reflection.

Proposition 3.7. *The $O(1)$ blocks in Proposition 3.6 can be written as a continuous function of $\boldsymbol{x}$ and $\boldsymbol{y}$ plus an error which vanishes in the limit $\mu \to 0, \chi \to \infty$. Moreover the $O(1)$ blocks have the following limits for orbits of interest.*

$$(I)_{44} = \begin{bmatrix} 1+\dfrac{\tilde L_4^2}{2(\tilde L_4^2+\tilde G_4^2)} & -\dfrac{\tilde L_4}{2} \\ \dfrac{\tilde L_4^3}{2(\tilde L_4^2+\tilde G_4^2)^2} & 1-\dfrac{\tilde L_4^2}{2(\tilde L_4^2+\tilde G_4^2)} \end{bmatrix}, \quad (III)_{44} = \begin{bmatrix} \dfrac{1}{2} & -\dfrac{L_4}{2} \\ \dfrac{3}{2L_4} & \dfrac{1}{2} \end{bmatrix}$$

$$(V)_{44} = \begin{bmatrix} 1-\dfrac{1/2\hat L_4^2}{\hat L_4^2+\hat G_4^2} & -1/2\hat L_4 \\ \dfrac{1/2\hat L_4^3}{(\hat L_4^2+\hat G_4^2)^2} & 1+\dfrac{1/2\hat L_4^2}{\hat L_4^2+\hat G_4^2} \end{bmatrix}.$$

In addition for map (I) we have

$$((I)(5,1),(I)(6,1))^T = \left(-\frac{\tilde G_4\tilde L_4}{2(\tilde L_4^2+\tilde G_4^2)}, -\frac{\tilde G_4\tilde L_4^2}{2(\tilde L_4^2+\tilde G_4^2)^2}\right)^T.$$

where tilde, hat have the same meaning as the previous Proposition.

Here and below the phrase *after the first collision* means that the initial orbit has parameters $(\frac{1}{2}, e_3^{**}, g_3^{**})+o(1)$ for Q_3, G_4 satisfies $G_4+G_3^{**} = G_3^*+G_4^*+o(1)$ and that at the final moment the angular momentum of Q_4 is close to G_4^{**}. The phrase *after the second collision* means that the initial orbit has parameters $(\frac{1}{2}, e_3^*, g_3^*)+o(1)$ for Q_3, G_4 satisfies $G_4+G_3^* = G_3^{**}+G_4^{**}+o(1)$ and that at the final moment the angular momentum of Q_4 is close to G_4^*.

The estimates of $(I),(III),(V)$ from Proposition 3.6 are proven in Sections 4–7. The estimates of $(II),(IV)$ are given in Section 8. Proposition 3.7 is proven in Section 6.2. Now we prove Lemma 3.2 based on the Proposition 3.7.

Proof of Lemma 3.2. $d\mathbb{G}$ is a product of several matrices. We will divide the product into three groups. The following estimates are obtained from Proposition 3.6 by direct computation.

$$(i)(I) = \left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) \\ O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\mu) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) \\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right],$$

$$\begin{aligned} \boldsymbol{M} &= [(ii')(i')](III)[(iii)(ii)] \\ &= \left[\begin{array}{cccc|cc} 1+O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi) \\ O(\chi) & O(1) & O(1) & O(1) & O(1) & O(\chi) \\ O(1/\chi) & O(1/\chi) & 1+O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi) \\ O(1/\chi) & O(1/\chi) & O(1/\chi) & 1+O(1/\chi) & O(1/\chi^2) & O(1/\chi) \\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(\chi) & O(\chi^2) \\ O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(1) & O(\chi) \end{array}\right], \end{aligned}$$

$$(V)(iii') = \left[\begin{array}{cccc|cc} O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\chi) & 1+O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(\mu^2\chi) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\mu^2\chi) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) \\ \hline O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right].$$

We decompose $(i)(I)$ and $(V)(iii')$ as

$$\begin{aligned} (i)(I) = &\left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) & 0 & 0 \\ O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & 0 & 0 \\ O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & 0 & 0 \\ O(\mu) & O(\mu) & O(\mu) & 1+O(\mu) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \times \\ &\left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & O(\mu) & O(\mu) \\ 0 & 1 & 0 & 0 & O(\mu) & O(\mu) \\ 0 & 0 & 1 & 0 & O(\mu) & O(\mu) \\ 0 & 0 & 0 & 1 & O(\mu) & O(\mu) \\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right] := [b][a] \end{aligned} \tag{3.2}$$

$$(V)(iii') = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & O(\mu) & O(\mu) \\ 0 & 1 & 0 & 0 & O(1) & O(1) \\ 0 & 0 & 1 & 0 & O(\mu) & O(\mu) \\ 0 & 0 & 0 & 1 & O(\mu) & O(\mu) \\ \hline O(1) & O(\mu^2) & O(\mu) & O(\mu) & O(1) & O(1) \\ O(1) & O(\mu^2) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right] \times$$

$$\left[\begin{array}{cccc|cc} O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & 0 & 0 \\ O(\chi) & 1+O(\mu) & O(\mu) & O(\mu) & 0 & 0 \\ O(\mu^2\chi) & O(\mu) & 1+O(\mu) & O(\mu) & 0 & 0 \\ O(\mu^2\chi) & O(\mu) & O(\mu) & 1+O(\mu) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] := [d][c]$$

Note that $[d]$ and $[a]$ are bounded so they do not change the order of magnitude of the derivative growth. On the other hand, denoting $\boldsymbol{D} = [c]\boldsymbol{M}[b]$ we obtain

$$\boldsymbol{D} = \left[\begin{array}{cccc|cc} O(\mu\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu) & O(\mu\chi) \\ O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) & O(1) & O(\chi) \\ O(\mu\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu) & O(\mu\chi) \\ O(\mu\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu) & O(\mu\chi) \\ \hline O(\mu\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\mu^2\chi) & O(\chi) & O(\chi^2) \\ O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(1) & O(\chi) \end{array}\right].$$

Note that $\boldsymbol{D}_{44} = \boldsymbol{M}_{44}$. In particular

$$\frac{\boldsymbol{D}(5,6)}{\chi^2} = \left(\frac{k_R}{L_3^2}, \frac{k_R}{L_3}\right)(III)_{44}\begin{pmatrix} 1 \\ 1 \\ \frac{1}{L_3}\end{pmatrix} + o(1).$$

It follows that if χ is large and μ is small then $\dfrac{\boldsymbol{D}(5,6)}{\chi^2}$ is uniformly bounded from above and below. Hence $\boldsymbol{D}$ can be represented as

$$\boldsymbol{D} = \chi^2 \bar{\mathbf{u}}' \otimes \bar{\mathbf{l}}' + \chi \bar{\bar{\mathbf{u}}}' \otimes \bar{\bar{\mathbf{l}}}' + O(\mu^2\chi),$$

where

$$\bar{\mathbf{u}}' = (O(\mu/\chi), O(1/\chi), O(\mu/\chi), O(\mu/\chi), 1, O(1/\chi))^T, \quad \bar{\mathbf{l}}' = \left(0,0,0,0,\frac{\boldsymbol{D}(5,5)}{\boldsymbol{D}(5,6)}, 1\right),$$

$$\bar{\bar{\mathbf{u}}}' = (O(\mu), 1, O(\mu), O(\mu), O(\mu), 0)^T, \quad \bar{\bar{\mathbf{l}}}' = (1, O(\mu), O(\mu), O(\mu), 0, 0)$$

and we have used the fact that $\dfrac{\boldsymbol{D}(5,5)}{\boldsymbol{D}(5,6)} = O\left(\dfrac{1}{\chi}\right)$. In the limit $\mu \to 0, \chi \to \infty$, we have

$$\bar{\mathbf{u}}' = (0,0,0,0,1,0)^T, \quad \bar{\mathbf{l}}' = (0,0,0,0,0,1),$$
$$\bar{\bar{\mathbf{u}}}' = (0,1,0,0,0,0)^T, \quad \bar{\bar{\mathbf{l}}}' = (1,0,0,0,0,0).$$

This allows us to compute the limiting values of $\bar{\mathbf{l}}$ and $\bar{\bar{\mathbf{l}}}$. Since $d\mathbb{G}$ is obtained from $\boldsymbol{D}$ by multiplying from the right and the left by bounded matrices we get

$$d\mathbb{G} = \chi^2 \bar{\mathbf{u}} \otimes \bar{\mathbf{l}} + \chi \bar{\bar{\mathbf{u}}} \otimes \bar{\bar{\mathbf{l}}} + O(\mu^2\chi),$$

where

$$\bar{\mathbf{u}} = [d]\bar{\mathbf{u}}',\ \bar{\bar{\mathbf{u}}} = [d]\bar{\bar{\mathbf{u}}}',\ \bar{\mathbf{l}} = \bar{\mathbf{l}}'[a],\ \bar{\bar{\mathbf{l}}} = \bar{\bar{\mathbf{l}}}'[a].$$

Similarly Proposition 3.7 shows that as $\chi \to \infty$, $\mu \to 0$ $\bar{\bar{\mathbf{u}}} \to (0,1,0,0,0,0)^T$ and it allows us to compute the limiting components of $\bar{\mathbf{u}}$ except that we do not have the exact expression for $d\ell_3(\bar{\mathbf{u}})$. However we do not need to know this component because we only interested in the span of $\bar{\mathbf{u}}$ and $\bar{\bar{\mathbf{u}}}$ and $d\ell_3(\bar{\mathbf{u}})$ can be suppressed by subtracting a suitable multiple of $\bar{\bar{\mathbf{u}}}$. It turns out that $\bar{\bar{\mathbf{l}}}$ has the same asymptotics as $\bar{\bar{\mathbf{l}}}'$, and $\tilde{w}$ the same as $\bar{\bar{\mathbf{u}}}'$. The functional $\bar{\mathbf{l}}$ is the limit of the sixth row of $[a]$, which is also the sixth row of $(i)(I)$. The vector $\bar{\mathbf{u}}$ is the fifth column of $[d]$, which is also the fifth column of $(V)(iii')$. Thus the asymptotic parameters of $d\mathbb{G}$ can be summarized as follows:

$$\begin{aligned} &\hat{\bar{\mathbf{l}}} = \left(-\frac{\tilde{G}_4/\tilde{L}_4}{\tilde{L}_4^2+\tilde{G}_4^2}, 0, 0, 0, \frac{1}{\tilde{L}_4^2+\tilde{G}_4^2}, -\frac{1}{\tilde{L}_4}\right), \quad \hat{\bar{\bar{\mathbf{l}}}} = (1,0,0,0,0,0), \\ &w = \left(0,0,0,0,1,-\frac{\hat{L}_4}{\hat{L}_4^2+\hat{G}_4^2}\right)^T, \qquad \tilde{w} = (0,1,0,0,0,0)^T. \end{aligned} \tag{3.3}$$

□

3.3. **Checking transversality.**

We study the local map numerically. The $O(1/\mu)$ part of $d\mathbb{L}$ in Lemma 3.1 is

Lemma 3.8. *The vectors $\mathbf{l}, \mathbf{u}$ in the $O(1/\mu)$ part of the matrix $d\mathbb{L}$ satisfy the following:*

(a) *As $\mu \to 0$, we have*

$$\hat{\mathbf{l}}_j \cdot \tilde{w} \neq 0, \quad \hat{\mathbf{l}}_j \cdot w_{3-j} \neq 0, \quad \hat{\bar{\mathbf{l}}}_j \cdot \hat{\mathbf{u}}_j \neq 0,$$

$j = 1, 2$ meaning the first or the second collision.

(b) *If Q_3 and Q_4 switch roles after the collisions, the vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ get a "$-$" sign.*

The computation is done using the choice of $E_3^* = -\frac{1}{2}$ and $e_3^* = \frac{1}{2}$, at Gerver's collision points.

To check the nondegeneracy condition, it is enough to know the following.

Lemma 3.9. *Let $\boldsymbol{x} \in U_j(\delta)$ where δ is small enough. If we take the directional derivative at $\boldsymbol{x}$ of the local map along a direction $\Gamma \in span\{\bar{\mathbf{u}}_{3-j}, \bar{\bar{\mathbf{u}}}_{3-j}\}$, such that*

$$\bar{\mathbf{l}}_j \cdot (d\mathbb{L}\Gamma) = 0, \;\; j = 1, 2$$

then

$$\lim_{\mu \to 0, \chi \to \infty} \frac{\partial E_3^+}{\partial \Gamma} \neq 0,$$

where E_3^+ is the energy of Q_3 after the close encounter with Q_4. These derivatives are computed in Gerver's case starting with $E_3^* = -1/2, e_3^* = 1/2$ and evaluated at Gerver's collision points. See the Appendix B.2 for concrete values.

The proofs of the two lemmas are postponed to Section 12.

Now we can check the nondegeneracy condition.

Proof of Lemma 3.3. We prove (b1) and (b2). The proofs of (a1) and (a2) are similar and are left to the reader.

To check (b2), de_4 we differentiate $e_4 = \sqrt{1 + (G_4/L_4)^2}$ to get

$$de_4 = \frac{1}{e_4}\left(\frac{G_4}{L_4^2} dG_4 - \frac{G_4^2}{L_4^3} dL_4\right).$$

Thus (3.3) gives $de_4 w = \dfrac{G_4}{L_4^2} \neq 0$ as claimed.

Next we check (b1) which is equivalent to the following condition

$$\det \begin{pmatrix} \hat{\bar{\mathbf{l}}}_2(\hat{\mathbf{u}}_2) & \hat{\bar{\mathbf{l}}}_2(\hat{B}_2\Gamma')) \\ \hat{\bar{\bar{\mathbf{l}}}}_2(\hat{\mathbf{u}}_2) & \hat{\bar{\bar{\mathbf{l}}}}_2(\hat{B}_2\Gamma') \end{pmatrix} \neq 0. \tag{3.4}$$

where $\Gamma' = \hat{\mathbf{l}}_2(\tilde{w})w_1 - \hat{\mathbf{l}}_2(w_1)\tilde{w}$. The vector $\Gamma' \neq 0$ due to part (a) of Lemma 3.8.

Let Γ be a vector satisfying $\hat{\bar{\mathbf{l}}}_2 \cdot (d\mathbb{L}\Gamma) = 0$ chosen as follows. $d\mathbb{L}\Gamma$ is a vector in $span\{\hat{\mathbf{u}}_i, \hat{B}_i\Gamma_i'\}$, so it can be represented as $d\mathbb{L}\Gamma_i = b\hat{\mathbf{u}}_2 + b'\hat{B}_2\Gamma'$. Thus we can take $b = -\hat{\bar{\mathbf{l}}}_2 \cdot \hat{B}_2\Gamma'$ and $b' = \hat{\bar{\mathbf{l}}}_2 \cdot \hat{\mathbf{u}}_2$ to ensure that $d\mathbb{L}\Gamma_i \in Ker\hat{\bar{\mathbf{l}}}_2$. Note that we have $b' \neq 0$ by part (a) of Lemma 3.8. Hence

$$\det \begin{pmatrix} \hat{\bar{\mathbf{l}}}_2(\hat{\mathbf{u}}_2) & \hat{\bar{\mathbf{l}}}_2(\hat{B}_2\Gamma') \\ \hat{\bar{\bar{\mathbf{l}}}}_2(\mathbf{u}_2) & \hat{\bar{\bar{\mathbf{l}}}}_2(\hat{B}_2\Gamma') \end{pmatrix} = \frac{1}{b'} \det \begin{pmatrix} \hat{\bar{\mathbf{l}}}_2(\hat{\mathbf{u}}_2) & \hat{\bar{\mathbf{l}}}_2(d\mathbb{L}\Gamma) \\ \hat{\bar{\bar{\mathbf{l}}}}_2(\hat{\mathbf{u}}_2) & \hat{\bar{\bar{\mathbf{l}}}}_2(d\mathbb{L}\Gamma) \end{pmatrix} = \hat{\bar{\bar{\mathbf{l}}}}_2(d\mathbb{L}\Gamma)$$

where the last equality holds since $\hat{\bar{\mathbf{l}}}_2(d\mathbb{L}\Gamma) = 0$. By Lemma 3.8 $\hat{\bar{\bar{\mathbf{l}}}}_i = (1, 0, 0, 0, 0, 0)$. Therefore $\hat{\bar{\bar{\mathbf{l}}}}_2(d\mathbb{L}\Gamma) = \dfrac{\partial E_3^+}{\partial \Gamma}$ and so (b2) follows from Lemma 3.9. □

Remark 3.1. *Let us describe the physical and geometrical meanings of the vectors $\bar{\mathbf{l}}, \bar{\bar{\mathbf{l}}}, \bar{\mathbf{u}}, \bar{\bar{\mathbf{u}}}, \mathbf{l}, \mathbf{u}$ and the results in this section.*

(1) *The structure of* $d\mathbb{L}$ *shows that a significant change of the behavior of the outgoing orbit parameters occurs when we vary the orbit parameters in the direction of* $\mathbf{l}$*, which is actually varying the closest distance (called impact parameter) between* Q_3 *and* Q_4 *(see Section 12, especially Corollary 12.1). The vector* w *in* $d\mathbb{G}$ *means that after the global map, the variable* G_4 *gets significant change as asserted by Lemma 2.7. So* $\hat{\mathbf{l}}_i \cdot w_{3-i} \neq 0$ *in Lemma 3.8 means that by changing* G_4 *after the global map, we can change the impact parameter and hence change the outgoing orbit parameters after the local map significantly. Similarly we see* $\hat{\mathbf{l}}_i \cdot \tilde{w} \neq 0$ *means the same outcome by varying* ℓ_3 *instead of* G_4*.*

(2) *The result* $\hat{\hat{\mathbf{l}}}_i \cdot \hat{\mathbf{u}}_i \neq 0$ *in Lemma 3.8 means that by changing the outgoing orbit parameter of the local map in* $\hat{\mathbf{u}}$ *direction, which is in turn changed significantly by changing the impact parameter in the local map, we can change the final orbit parameter of the global map in the* $\bar{\mathbf{u}}$ *direction significantly. The vector* $\hat{\hat{\mathbf{l}}}$ *has clear physical meaning. If we differentiate the outgoing asymptote* $\theta_4^+ = g_4^+ - \arctan \dfrac{G_4^+}{L_4^+}$*, where + means after close encounter of* Q_3 *and* Q_4*, we get* $d\theta_4^+ = L_4^+ \hat{\hat{\mathbf{l}}}$*.*

(3) *Lemma 3.9 means that if we vary the incoming orbit parameter of the local map in the direction* Γ *such that there is no significant change of the outgoing parameters of the local map in certain direction, then the energy (and, hence, semimajor axis) of the ellipse after* Q_3, Q_4 *interaction will change accordingly. One may think this as varying the incoming orbit parameter while holding the outgoing asymptotes unchanged. The change of energy means the change of periods of the ellipses according to Kepler's law. Ellipses with different periods will accumulate huge phase difference during one return time* $O(\chi)$ *of* Q_4*. This is the mechanism that we use to fine tune the phase of* Q_3 *such that* Q_3 *comes to the correct phase to interact with* Q_4*. Since the phase is defined up to* 2π*, we get a Cantor set as initial condition of singular orbits.*

4. C^0 estimates for global map

4.1. **Equations of motion in Delaunay coordinates.** We use Delaunay variables to describe the motion of Q_3 and Q_4 (for reader's convenience we collect the basic properties of Delaunay variables in Appendix A). We have eight variables (L_3, ℓ_3, G_3, g_3) and (L_4, ℓ_4, G_4, g_4). We eliminate L_4 using the energy conservation and ℓ_4 will play the role of independent variable.

After setting $v_{3,4} = P_{3,4}/\mu$ and dividing (1.1) by μ the Hamiltonian takes the form

$$H = \frac{v_3^2}{2} + \frac{v_4^2}{2} - \frac{1}{|Q_3|} - \frac{1}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{1}{|Q_4 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}. \tag{4.1}$$

When Q_4 is moving to the left of the section $\{x_4 = -\chi/2\}$, we consider the motion of Q_3 as elliptic motion with focus at Q_2, and Q_4 as hyperbolic motion with focus at Q_1, perturbed by other interactions. We can write the Hamiltonian in terms of Delaunay variables as

$$H_L = -\frac{1}{2L_3^2} + \frac{1}{2L_4^2} - \frac{1}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}.$$

When Q_4 is moving to the right of the section $\{x_4 = -\chi/2\}$, we consider the motion of Q_3 as an elliptic motion with focus at Q_2, and that of Q_4 as a hyperbolic motion with focus at Q_2 attracted by the pair Q_2, Q_3 which has mass $1 + \mu$ plus a perturbation. For $|Q_4| \geq 2$ we have the following Taylor expansion

$$\frac{\mu}{|Q_3 - Q_4|} = \frac{\mu}{|Q_4|} + \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right).$$

Hence the Hamiltonian takes form

$$H = \frac{v_3^2}{2} + \frac{v_4^2}{2} - \frac{1}{|Q_3|} - \frac{1+\mu}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{1}{|Q_4 - (-\chi, 0)|} - \frac{\mu Q_3 \cdot Q_4}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right).$$

In terms of the Delaunay variables we have
(4.2)
$$H_R = -\frac{1}{2L_3^2} + \frac{(1+\mu)^2}{2L_4^2} - \frac{1}{|Q_3 + (\chi, 0)|} - \frac{1}{|Q_4 + (\chi, 0)|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right).$$

We shall use the following notation. The coefficients of $\frac{1}{2L_4^2}$ in the Hamiltonian will be called $k_L = 1$ and $k_R = 1 + \mu$. The terms in the Hamiltonian containing Q_4 will be denoted by

$$(4.3)\quad V_R = -\frac{1}{|Q_4 + (\chi, 0)|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right), \text{ and } V_L = -\frac{1}{|Q_4|} - \frac{\mu}{|Q_3 - Q_4|}.$$

Here subscripts L and R mean that the corresponding expressions are used when Q_4 is to the left (respectively to the right) of the line $Q = -\frac{\chi}{2}$. Likewise for the terms containing Q_3 we define
(4.4)
$$U_R = -\frac{1}{|Q_3 + (\chi, 0)|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right), \ U_L = -\frac{1}{|Q_3 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}.$$

The use of subscripts R, L here is the same as above. Let us write down the full Hamiltonian equations with the subscripts R and L suppressed.

$$
\left\{
\begin{array}{ll}
\dot{L}_3 = -\dfrac{\partial Q_3}{\partial \ell_3}\cdot\dfrac{\partial U}{\partial Q_3}, & \dot{\ell}_3 = \dfrac{1}{L_3^3} + \dfrac{\partial Q_3}{\partial L_3}\cdot\dfrac{\partial U}{\partial Q_3},\\
\dot{G}_3 = -\dfrac{\partial Q_3}{\partial g_3}\cdot\dfrac{\partial U}{\partial Q_3}, & \dot{g}_3 = \dfrac{\partial Q_3}{\partial G_3}\cdot\dfrac{\partial U}{\partial Q_3},\\
\dot{L}_4 = -\dfrac{\partial Q_4}{\partial \ell_4}\cdot\dfrac{\partial V}{\partial Q_4}, & \dot{\ell}_4 = -\dfrac{k^2}{L_4^3} + \dfrac{\partial Q_4}{\partial L_4}\cdot\dfrac{\partial V}{\partial Q_4},\\
\dot{G}_4 = -\dfrac{\partial Q_4}{\partial g_4}\cdot\dfrac{\partial V}{\partial Q_4}, & \dot{g}_4 = \dfrac{\partial Q_4}{\partial G_4}\cdot\dfrac{\partial V}{\partial Q_4}.
\end{array}
\right.
\tag{4.5}
$$

Next we use the energy conservation to eliminate L_4. We have

$$
\begin{aligned}
\frac{L_4^3}{k_R^2} &= k_R L_3^3\cdot\left(1-3L_3^2\left(\frac{1}{|Q_3+(\chi,0)|}+\frac{1}{|Q_4+(\chi,0)|}\right.\right.\\
&\quad\left.\left.+\ \frac{\mu Q_4\cdot Q_3}{|Q_4|^3}+O\left(\frac{\mu}{|Q_4|^3}\right)+O(1/\chi^2)\right)\right) := k_R L_3^3 + W_R,\\
\frac{L_4^3}{k_L^2} &= k_L L_3^3\left(1-3L_3^2\left(\frac{1}{|Q_3+(\chi,0)|}+\frac{1}{|Q_4|}-\frac{\mu}{|Q_4-Q_3|}+O(1/\chi^2)\right)\right)\\
&:= k_L L_3^3 + W_L.
\end{aligned}
\tag{4.6}
$$

We use ℓ_4 as the independent variable. Dividing (4.5) by $\dot{\ell}_4$ and using (4.6) to eliminate L_4 we obtain

$$
\left\{
\begin{aligned}
\frac{dL_3}{d\ell_4} &= (kL_3^3+W)\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\\
\frac{d\ell_3}{d\ell_4} &= -(kL_3^3+W)\left(\frac{1}{L_3^3}+\frac{\partial Q_3}{\partial L_3}\cdot\frac{\partial U}{\partial Q_3}\right)\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\\
\frac{dG_3}{d\ell_4} &= (kL_3^3+W)\frac{\partial Q_3}{\partial g_3}\cdot\frac{\partial U}{\partial Q_3}\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\\
\frac{dg_3}{d\ell_4} &= -(kL_3^3+W)\frac{\partial Q_3}{\partial G_3}\cdot\frac{\partial U}{\partial Q_3}\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\\
\frac{dG_4}{d\ell_4} &= (kL_3^3+W)\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\\
\frac{dg_4}{d\ell_4} &= -(kL_3^3+W)\frac{\partial Q_4}{\partial G_4}\cdot\frac{\partial V}{\partial Q_4}\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)
\end{aligned}
\right.
\tag{4.7}
$$

$$
+\,O\left(\frac{\mu}{|Q_4|^3}+1/\chi^2\right).
$$

We shall use the following notation: $X=(L_3,\ell_3,G_3,g_3)$, $Y=(G_4,g_4)$.

4.2. **A priori bounds.**

4.2.1. *Estimates of positions.* We have the following estimates for the positions.

Lemma 4.1. *Given C and δ there exists C' such that if*

$$|Q_3| < 2-\delta, \quad |Q_{4y}| < C \tag{4.8}$$

then

(a) *we have*

$$\left|\frac{\partial Q_3}{\partial X}\right| < C'; \tag{4.9}$$

(b) *when Q_4 is moving to the right of the section $\{x_4 = -\chi/2\}$ we have*

$$|Q_4| \begin{cases} \geq 2, & \text{if } |\ell_4^*| \leq |\ell_4| \leq C \\ \in \left[\frac{1}{2}, 2\right] L_4^2(\ell_4^*)|\ell_4|, & \text{if } |\ell_4| \geq C, \end{cases} \tag{4.10}$$

where ℓ_4^* is the value of ℓ_4 restricted on $x_4 = -2$;
when Q_4 is moving to the left of the section $\{x_4 = -\chi/2\}$, we have

$$|Q_4 - Q_1| \leq 2L_4^2(\ell_4^*)|\ell_4| + C' \tag{4.11}$$

for some constant C' where ℓ_4 is one of the Delaunay variables in the left and ℓ_4^* is its value on the section $\{x_4 = -\chi/2\}$.

The intuition behind this lemma is the following. Since the total energy of the system is zero and Q_3 and Q_4 interact only weakly with each other, then both particles have energies close to $\frac{1}{2L_4^2(\ell_4^*)}$ in absolute value. Since Q_4 spends most of the time away from Q_1, Q_2 and Q_3 most of its energy is kinetic energy so it moves with approximately constant speed. Since it makes a little progress in y direction its velocity is almost horizontal most of the time. This explains (4.10),(4.11). To give the complete proof we have to use the Hamiltonian equations. See Section 4.3.

Lemma 4.2. *If inequalities* (4.8), (4.10), (4.11) *are valid and in addition*

$$1/C \leq |L_3|, |L_4| \leq C, \quad |G_3|, |G_4| < C, \tag{4.12}$$

then we have

$$\frac{\partial Q_4}{\partial \ell_4} = O(1), \quad \frac{\partial Q_4}{\partial (L_4, G_4, g_4)} = O(\ell_4), \quad \frac{\partial Q_4}{\partial g_4} \cdot Q_4 = 0 \text{ and } \frac{\partial Q_4}{\partial G_4} \cdot Q_4 = O(\ell_4)$$

as $t \to \infty$.

Proof. This follows directly from Lemma A.2 in Appendix A.4. □

4.2.2. *Estimates of potentials.*

Lemma 4.3. *Under the assumptions of Lemma 4.2 we have the following estimates for the potentials U, V, W:*

(a) *When Q_4 is moving to the right of the section $\{x_4 = -\chi/2\}$, we have*

$$V_R,\ U_R,\ W_R = O\left(\frac{1}{\chi} + \frac{\mu}{\ell_4^2+1}\right).$$

(b) *When Q_4 is moving to the left of the section $\{x_4 = -\chi/2\}$, we have*

$$V_L,\ U_L,\ W_L = O\left(\frac{1}{\chi}\right).$$

Proof. This follows directly from equations (4.3), (4.4) and (4.6) and (4.10) in Lemma 4.1. Our choice of the section $\{x_4 = -2\}$ excludes the collision between Q_3 and Q_4. So we put $\frac{\mu}{\ell_4^2+1}$ to stress the fact that the denominator is bounded away from zero. We do the same thing in the following proofs without mentioning it any more. □

4.2.3. *Estimates of gradients of potentials.* To take partial derivatives w.r.t. Delaunay variables, we use the formulas

$$\frac{\partial}{\partial X} = \frac{\partial Q_3}{\partial X}\cdot\frac{\partial}{\partial Q_3},\quad \frac{\partial}{\partial Y} = \frac{\partial Q_4}{\partial Y}\cdot\frac{\partial}{\partial Q_4}.$$

Lemma 4.4. *Under the assumptions of Lemma 4.2 we have the following estimates for the gradients of the potentials U, V*

$$\begin{aligned}&\frac{\partial U_R}{\partial Q_3},\ \frac{\partial Q_4}{\partial(G_4,g_4)}\frac{\partial V_R}{\partial Q_4} = O\left(\frac{1}{\chi^2}+\frac{\mu}{\ell_4^2+1}\right),\quad \frac{\partial V_R}{\partial Q_4} = O\left(\frac{1}{\chi^2}+\frac{\mu}{|\ell_4|^3+1}\right),\\ &\frac{\partial U_L}{\partial Q_3} = O\left(\frac{1}{\chi^2}\right),\quad \frac{\partial V_L}{\partial Q_4} = O\left(\frac{1}{\chi^2}\right),\quad \frac{\partial Q_4}{\partial(G_4,g_4)}\frac{\partial V_L}{\partial Q_4} = O\left(\frac{1}{\chi^2}\right).\end{aligned}\tag{4.13}$$

Proof. The estimates for the $\frac{\partial}{\partial Q_{3,4}}$ terms are straightforward. Indeed, we only need to use the fact $\left|\frac{d}{dx}\frac{1}{|x|^k}\right| = \frac{k}{|x|^{k+1}}$ together with the estimates in Lemma 4.1.

The estimates of all $\frac{\partial}{\partial(G_4,g_4)}$ terms are similar. We consider for instance $\frac{\partial Q_4}{\partial G_4}\frac{\partial V_R}{\partial Q_4}$. We have

$$\frac{\partial Q_4}{\partial G_4}\frac{\partial V_R}{\partial Q_4} = \frac{\partial Q_4}{\partial G_4}\frac{Q_4+(\chi,0)}{|Q_4+(\chi,0)|^3} + O\left(\mu\left|\frac{\partial Q_4}{\partial G_4}\right||Q_4|^{-3}\right).\tag{4.14}$$

The second term here is $O(\mu/(\ell_4^2+1))$ due to (4.10) and Lemma A.2(a). To handle the first term let $\frac{\partial Q_4}{\partial G_4} = (\boldsymbol{a}, \boldsymbol{b})$, $Q_4 = (x, y)$. Note that equations (A.3), (A.4), (4.8), (4.10), and (4.12) show that x, ℓ_4 are all comparable in the sense that the ratios between any two of these qualities are bounded from above and below. On the other hand Lemma A.2(a) tells us that $\boldsymbol{a}x + \boldsymbol{b}y = O(\ell_4)$. Since $\boldsymbol{b}y = O(\boldsymbol{b}) = O(\ell_4)$ we conclude that $\boldsymbol{a}x = O(\ell_4)$ and thus $\boldsymbol{a} = O(1)$. Thus the first term in (4.14) is

$$\frac{\frac{\partial Q_4}{\partial G} \cdot Q_4 + \boldsymbol{a}\chi}{|Q_4 + (\chi, 0)|^3}.$$

The numerator here is $O(\chi)$ while the denominator is at least $(\chi/2)^3$. This completes the estimate of $\frac{\partial Q_4}{\partial G_4}\frac{\partial V_R}{\partial Q_4}$. Other derivatives are similar. □

Plugging the above estimates into (4.7) we obtain the following.

Lemma 4.5. *Under the assumptions of Lemma 4.2 we have the following estimates on the RHS of* (4.7).

(a) *When* $-\frac{\chi}{2} \le x_4 \le -2$ *we have*

$$\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{\ell_4^2+1}\right), \quad \frac{d\ell_3}{d\ell_4} = O(1).$$

(b) *When* Q_4 *is moving to the left of the section* $\{x_4 = -\chi/2\}$, *we have*

$$\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2}\right), \quad \frac{d\ell_3}{d\ell_4} = O(1).$$

In Section 6 we will need the following bounds on the second derivatives.

Lemma 4.6. *Under the assumptions of Lemma 4.2 we have the following estimates for the second derivatives.*

$$\begin{aligned} &\frac{\partial^2 U_R}{\partial Q_3^2} = O\left(\frac{1}{\chi^3} + \frac{\mu}{\ell_4^2+1}\right), \ \frac{\partial^2 V_R}{\partial Q_4^2} = O\left(\frac{1}{\chi^3} + \frac{\mu}{\ell_4^4+1}\right), \\ &\frac{\partial^2 (U_R, V_R)}{\partial Q_3 \partial Q_4} = O\left(\frac{\mu}{|\ell_4|^3+1}\right), \\ &\frac{\partial^2 U_L}{\partial Q_3^2} = O\left(\frac{1}{\chi^3}\right), \quad \frac{\partial^2 V_L}{\partial Q_4^2} = O\left(\frac{1}{\chi^3}\right), \quad \frac{\partial^2 (U_L, V_L)}{\partial Q_3 \partial Q_4} = O\left(\frac{1}{\chi^3}\right). \end{aligned} \tag{4.15}$$

We omit the proof since it is again a direct computation.

4.3. **Proof of Lemma 4.1.**

Proof of Lemma 4.1. Let τ be the maximal time interval such that

$$\frac{3}{4}|L_3(\ell_4^*)| \le |L_3| \le \frac{4}{3}|L_3(\ell_4^*)|, \quad \frac{3}{4}|G_i(\ell_4^*)| \le |G_i(\ell_4)| \le \frac{4}{3}|G_i(\ell_4^*)|,\ i=3,4, \tag{4.16}$$

on $[0,\tau]$ where ℓ_4^* is the value ℓ_4 restricted on $\{x_4=-2\}$. (4.16) implies that $e_4=\sqrt{1+G_4^2/L_4^2}$ is bounded. We always have we have $|Q_4|\ge 2$ since Q_4 is to the left of the section $\{x_4=-2\}$. Therefore (4.6) implies that $L_4=L_3+O(\mu)$ in the right case and $L_4=L_3+O(1/\chi)$ in the left case. Now formula (A.3) and Lemma A.1 allow us replace $\sinh u, \cosh u$ by $(1+o(1))\dfrac{\ell_4}{e_4}$ as $|\ell_4|\to\infty$.

$$\begin{aligned}|Q_4| &= L_4\sqrt{L_4^2(\cosh u-e_4)^2+G_4^2\sinh^2 u}\\ &= L_4\sqrt{(L_4^2+G_4^2)(1+o(1))^2\frac{\ell_4^2}{e_4^2}} = L_4^2(1+o(1))|\ell_4|.\end{aligned}$$

This proves estimate (4.10) for $t\le\min(\tau,\bar\tau)$ where $\bar\tau$ is the first time then x_4 reaches $-\dfrac{\chi}{2}$. Thus for $t\le\min(\tau,\bar\tau)$ the assumptions of Lemma 4.5 are satisfied and hence

$$\frac{dL_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dG_3}{d\ell_4} = O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4-Q_3|^2}\right) \tag{4.17}$$

(note that to prove the estimates in Lemma 4.5 in the right case we do not need the assumption (4.11)). If we integrate (4.17) w.r.t. ℓ_4 on the interval of size $O(\chi)$ we find that the oscillations of L_3, G_4, G_3 are $O(\mu)$. Therefore $\bar\tau<\tau$ and we obtain the estimates of (4.10) up to the time $\bar\tau$.

The analysis of the cases when Q_4 is to the left of the section $\{x_4=-\chi/2\}$ and then it travels back from $\{x_4=-\chi/2\}$ to $\{x_4=-2\}$ is similar once we establish the bounds on the angular momentum at the beginning of the corresponding pieces of the orbit. Let us show, for example, that at the moment then the orbit hits $\{x_4=-\dfrac{\chi}{2}\}$ for the first time, the angular momentum of Q_4 w.r.t. Q_1 is $O(1)$. Indeed we have already established that $G_{4R}=-\dfrac{\chi v_{4y}}{2}-yv_{4x}=O(1)$. Also (4.16) shows that $v=O(1)$ and so (4.8) implies that $yv_{4x}=O(1)$. Accordingly

$$\chi v_{4y} = -G_{4R}-yv_{4y}=O(1)$$

and hence $G_{4L}=G_{4R}+\chi v_{4y}=O(1)$ as claimed. The argument for the second time the orbit hits $\{x_4=-\dfrac{\chi}{2}\}$ is the same. This completes the proof of part (b).

To show part (a), we notice $\dfrac{\partial Q_3}{\partial X}$ depends on ℓ_3, g_3 periodically according to equation (A.1). So part (a) follows since we have already obtained bounds on L_3 and G_3. $\square$

The next lemma gives more information about the Q_4 part of the orbit than Lemma 4.1. It justifies the assumptions of Lemma A.2.

Lemma 4.7. *Under the hypothesis of Lemma 4.2, we have:*

(a) *when Q_4 is moving to the right of the section $\{x=-\chi/2\}$, we have*

$$\tan g_4 = \mathrm{sign}(u)\frac{G_4}{L_4} + O\left(\frac{\mu}{|\ell_4|+1}+\frac{1}{\chi}\right), \quad \text{as } |\ell_4|, \chi \to \infty.$$

(b) *when Q_4 is moving to the left of the section $\{x=-\chi/2\}$, then*

$$G_4, g_4 = O(1/\chi), \text{ as } \chi \to \infty.$$

Proof. We prove part (b) first. From equation (A.5) we see that if ℓ_4 is of order χ and $y=O(1)$ then $G_4\cos g_4 + \mathrm{sign}(u)L_4 \sin g_4 = O(1/\chi)$. Integrating the estimates of Lemma 4.5(b) we see that during the time $x_4 \le -\chi/2$ we have

$$G_4 = G^* + O(1/\chi), \quad L_4 = L^* + O(1/\chi), \quad g_4 = g^* + O(1/\chi) \tag{4.18}$$

where (L^*, G^*, g^*) are the orbit parameters of Q_4 then it first hits $\{x_4=-\chi/2\}$. It follows that both

$$G^*\cos g^* + L^* \sin g^* = O(1/\chi), \quad \text{and} \quad G^* \cos g^* - L^* \sin g^* = O(1/\chi).$$

Since L^* is not too small this is only possible if $G^* = O(1/\chi)$, $g^* = O(1/\chi)$. Now part (b) follows from (4.18).

The proof of part (a) is similar. Consider for example the case when Q_4 moves to the right. Now (4.18) has to be replaced by

$$(G_4, L_4, g_4) = (G^*, L^*, g^*) + O\left(\frac{\mu}{|\ell_4|+1}+\frac{1}{\chi}\right), \tag{4.19}$$

(since we use part (a) of Lemma 4.5 rather than part (b)). As before we have

$$G^*\cos g^* + L^*\sin g^* = O(1/\chi).$$

Since $\cos g^*$ can not be too small (since otherwise $G^*\cos g^* - L^* \sin g^* \approx L^* \sin g$ would not be small) we can divide the last equation by $L^*\cos g$ to get

$$\tan g^* = \frac{G^*}{L^*} + O\left(\frac{1}{\chi}\right).$$

Now part (a) follows from (4.19). □

4.4. **Proof of Lemma 2.4.** We begin by demonstrating that the orbits satisfying the conditions of Lemma 2.4 satisfy the assumptions of Lemma 4.5.

Lemma 4.8. (a) *Given δ, C there exist constants $\hat{C}, \mu_0$ such that for $\mu \le \mu_0$ the following holds. Consider a time interval $[0,T]$ and an orbit satisfying the following conditions*

(i) *$x_4(t) \in (-\chi-1, -2)$ for $t\in(0,T)$, $x_4(0)=-2$, $x_4(T)=-\chi$.*

(ii) *$y_4(0) \le C$, $y_4(T) \le C$.*

(iii) *At time* 0, Q_3 *moves on an elliptic orbit which is completely contained in* $\{x_3 \geq -(2-\delta)\}$.

Then $|y_4(t)| \leq \hat{C}$ *for all* $t \in [0, T]$.

(b) *The result of part (a) remains valid if* (i) *is replaced by*

($\tilde{\text{i}}$) $x_4(t) < -2$ *for* $t \in (0, T)$, $x_4(0) = x_4(T) = -2$.

Proof. To prove part (a) we first establish a preliminary estimate showing that Q_4 travels roughly in the direction of Q_1.

Sublemma 4.9. *Given* $\tilde{\theta} > 0$ *there exists* μ_0, χ_0 *such that the following holds for* $\mu \leq \mu_0$, $\chi > \chi_0$. *If the outgoing asymptote satisfies*

$$|\pi - \theta_4^+(0)| > \tilde{\theta} \tag{4.20}$$

then Q_4 *escapes from the two center system.*

Proof. We consider the case $\theta_4^+(0) < \pi - \tilde{\theta}$, the other case is similar. If we disregard the influence of Q_1 and Q_3 then Q_4 would move on a hyperbolic orbit and its velocity would approach $(\sqrt{2E_4(0)}\cos\theta_4^+(0), \sqrt{2E_4(0)}\sin\theta_4^+(0))$. Accordingly given R we can find $\bar{t}$, μ_0 such that uniformly over all orbits satisfying (i)-(iii) and $\theta_4^+(0) < \pi - \tilde{\theta}$ we have for $\mu \leq \mu_0$

$$y_4(\bar{t}) > R, \quad v_{4y}(\bar{t}) > 0.8\sqrt{E_4(0)}\sin\tilde{\theta}.$$

Let $\tilde{t} = \inf\{t > \bar{t} : v_{4y} < \frac{\sqrt{E_4(0)}}{2}\sin\tilde{\theta}\}$. We shall show that $\tilde{t} = \infty$ which implies the sublemma since for $t \in [\bar{t}, \tilde{t}]$ we have

$$y_4(t) > R + (\tilde{t} - \bar{t})\frac{\sqrt{E_4}}{2}\sin\tilde{\theta}. \tag{4.21}$$

To see that $\tilde{t} = \infty$ note that (4.21) implies that

$$|\dot{v}_{4y}| \leq \frac{1}{(R + (\tilde{t} - \bar{t})\frac{\sqrt{E_4}}{2}\sin\tilde{\theta})^2}$$

and so

$$|v_{4y}(\tilde{t}) - v_{4y}(\bar{t})| \leq \int_0^\infty \frac{ds}{(R + s\frac{\sqrt{E_4}}{2}\sin\tilde{\theta})^2} = \frac{2}{R\sqrt{E_4}\sin\tilde{\theta}}.$$

Hence if R is sufficiently large we have $v_{4y}(\tilde{t}) \geq \frac{\sqrt{E_4}}{2}\sin\tilde{\theta}$ which is only possible if $\tilde{t} = \infty$. □

We now consider the case $|\pi - \theta_4^+| < \tilde{\theta}$. Arguing as above we see that given R, we can find for μ small enough a time $\bar{t}$ such that

$$x_4(t) < -R, \quad v_{4x}(\bar{t}) < -0.8\sqrt{E_4(0)}\cos\tilde{\theta}.$$

Let $\hat{t}$ be the first time after $\bar{t}$ such that $x_4 = -(\chi - R)$. Arguing as in Sublemma 4.9 we see that for $t \in [\bar{t}, \hat{t}]$ we have $|v_{4x}| \geq \frac{\sqrt{E_4(0)}}{2}\cos\tilde{\theta}$. Hence the force from Q_2 and Q_3

is $O(1/t^2)$ and the force from Q_1 is $O(1/(\hat{t}-t)^2)$. Accordingly v_4 remains $O(1)$ so the energy of Q_4 remains bounded. Next if $|y_4(\hat{t})|>R$ then the argument of Sublemma 4.9 shows that $y_4(T)>R/2$ giving a contradiction if $R>2C$. Accordingly we have for $t\in[\hat{t},T]$ that $E_4=O(1)$, $y_4=O(1)$ and $G_{4L}=O(1)$. It remains to show that $|y_4(t)|<\hat{C}$ for $t\in[\bar{t},\hat{t}]$. To this end let t^* be the first time when $x_4=-\dfrac{\chi}{2}$. We first get $E_4=O(1)$ for $t\in[t^*,\hat{t}]$ since by arguing as in the Sublemma we get the oscillation of v_4 is bounded. Next, we have that $G_{4L}(t^*)=O(1)$ since $G_{4L}(\hat{t})=O(1)$ and for $t\in[t^*,\hat{t}]$ we have $\dot{G}_{4L}=O(1/\chi)$,(this estimate of $\dot{G}_{4L}=\ddot{v}_4\times x_4$ does not need any assumption on G_{4L}.) Likewise $G_{4R}=O(1)$. Therefore

$$\chi v_{4y}(t^*)=G_{4R}-G_{4L}=O(1)$$

and so $v_{4y}(t^*)=O(1/\chi)$. Since $G_{4L}(t^*)=\left(\frac{\chi}{2}v_{4y}-yv_{4x}\right)(t^*)$ we have $y(t^*)=O(1)$. Next for $t\in[t^*,\hat{t}]$ we have

$$y_4(t)=y_4(t^*)+v_{4y}(t-t^*)+\int_{t^*}^{t}\int_{t^*}^{u}\ddot{y}_4(s)dsdu.$$

Note that

$$\ddot{y}_4(s)=O\left(\frac{y}{|Q_4-Q_1|^3}\right)=O\left(\frac{y}{(\hat{t}-s+R)^3}\right).$$

Combining the last two estimates we get

$$|y(t)|\le C_1+C_2\sup_s\{|y(s)|\}\int_{t^*}^{t}\int_{t^*}^{u}\frac{dsdu}{(\hat{t}-s+R)^3}\le C+C\left(\frac{1}{R}+\frac{1}{\chi}\right)\sup_s|y(s)|.$$

We choose R large enough to get that $|y|$ is bounded on $[t^*,\hat{t}]$. The argument for $[\bar{t},t^*]$ is the same except that the force from Q_3 is $O\left(\dfrac{\mu y_4}{|Q_4|^3}\right)$. This completes the proof of part (a).

To prove part (b) we note that if $|y_4(\hat{t})|>R^2$ then Q_4 escapes by the argument of Sublemma 4.9. Hence $|y_4(\hat{t})|<R^2$. This implies (via already established part (a) of the lemma) that y is uniformly bounded on $[0,\hat{t}]$. The argument for $[\hat{t},T]$ is the same with the roles of Q_1 and Q_2 interchanged. □

Proof of Lemma 2.4. Initially we have $1/C\le|L_3|\le C$, $|G_3|,|G_4|\le C$ for some constant $C>1$. We assume (4.16) from time 0 to some time τ. Due to the previous lemma, we can use Lemma 4.5 to get the estimates on the time interval $[0,\tau]$

$$\frac{dL_3}{d\ell_4},\frac{dG_3}{d\ell_4},\frac{dg_3}{d\ell_4},\frac{dG_4}{d\ell_4},\frac{dg_4}{d\ell_4}=O\left(\frac{1}{\chi^2}+\frac{\mu}{\ell_4^2+1}\right).$$

We integrate the equations to get $O(\mu)$ oscillations of L_3,G_3,G_4 so that τ can be extended to as large as χ. For part (a) of Lemma 2.4, we integrate the equations of $\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_3}{d\ell_4}$, over time of order χ as Q_4 first moves away from Q_2 and then comes back. Therefore we get

$$O\left(2\int_2^{\chi}\left[\frac{\mu}{\ell_4^2+1}+\frac{1}{\chi^2}\right]d\ell_4\right)=O(\mu)$$

estimate for the change of L_3, G_3 and g_3 proving part (a).

Part (b) of Lemma 2.4 follows from Lemma 4.7. □

5. Derivatives of the Poincaré map

In computing C^1 asymptotics of both local and global maps we will need formulas for the derivatives of Poincaré maps between two sections. Here we give the formulas for such derivatives for the later reference.

Recall our use of notations. X denotes Q_3 part of our system and Y denotes Q_4 part. Thus

$$X=(L_3,\ell_3,G_3,g_3),\quad Y=(G_4,g_4).$$

$(X,Y)^i$ will denote the orbit parameters at the initial section and $(X,Y)^f$ will denote the orbit parameters at the final section. Likewise we denote by ℓ_4^i the initial "time" when Q_4 crosses some section, and by ℓ_4^f final "time" when Q_4 arrives at the next. We abbreviate the RHS of (4.7)) as

$$X'=\mathcal{U},\quad Y'=\mathcal{V}.$$

Here $'$ is the derivative w.r.t. ℓ_4. We also denote $Z=(X,Y)$ and $\mathcal{W}=(\mathcal{U},\mathcal{V})$ to simplify the notations further.

Suppose that we want to compute the derivative of the Poincaré map between the sections S^i and S^f. Assume that on S^i we have $\ell_4=\ell_4^i(Z^i)$ and on S^f we have $\ell_4=\ell_4^f(Z^f)$. We want to compute the derivative $\mathcal{D}$ of the Poincaré map along the orbit starting from (Z_*^i,ℓ_*^i) and ending at (Z_*^f,ℓ_*^f). We have $\mathcal{D}=dF_3dF_2dF_1$ where F_1 is the Poincaré map between S^i and $\{\ell_4=\ell_*^i\}$, F_2 is the flow map between the times ℓ_*^i and ℓ_*^f, and F_3 is the Poincaré map between $\{\ell_4=\ell_*^f\}$ and S^f. We have $F_1=\Phi(Z^i,\ell_4(Z^i),\ell_*^i)$ where $\Phi(Z,a,b)$ denotes the flow map starting from Z at time a and ending at time b. Since

$$\frac{\partial\Phi}{\partial Z}(Z_*^i,\ell_*^i,\ell_*^i)=\mathrm{Id},\quad \frac{\partial\Phi}{\partial a}=-\mathcal{W}$$

we have $dF_1=\mathrm{Id}-\mathcal{W}(\ell_4^i)\otimes\frac{D\ell_4^i}{DZ^i}$. Inverting the time we get

$$dF_3=\left(\mathrm{Id}-\mathcal{W}(\ell_4^f)\otimes\frac{D\ell_4^f}{DZ^f}\right)^{-1}.$$

Finally $dF_2 = \dfrac{DZ(\ell_*^f)}{DZ(\ell_*^i)}$ is just the fundamental solution of the variational equation between the times ℓ_*^i and ℓ_*^f. Thus we get

$$\mathcal{D} = \left(\mathrm{Id} - \mathcal{W}(\ell_4^f) \otimes \frac{D\ell_4^f}{DZ^f}\right)^{-1} \frac{DZ(\ell_4^f)}{DZ(\ell_4^i)} \left(\mathrm{Id} - \mathcal{W}(\ell_4^i) \otimes \frac{D\ell_4^i}{DZ^i}\right). \tag{5.1}$$

Next, we study the fundamental solution $\dfrac{DZ(\ell_*^f)}{DZ(\ell_*^i)}$ of the variational equation. We consider Q_3 and Q_4 individually. The variational equation takes form

$$\left(\frac{\partial X}{\partial X(\ell_*^i)}\right)' = \frac{\partial \mathcal{U}}{\partial X}\frac{\partial X}{\partial X(\ell_*^i)} + \frac{\partial \mathcal{U}}{\partial Y}\frac{\partial Y}{\partial X(\ell_*^i)}, \quad \left(\frac{\partial X}{\partial Y(\ell_*^i)}\right)' = \frac{\partial \mathcal{U}}{\partial X}\frac{\partial X}{\partial Y(\ell_*^i)} + \frac{\partial \mathcal{U}}{\partial Y}\frac{\partial Y}{\partial Y(\ell_*^i)},$$
$$\left(\frac{\partial Y}{\partial X(\ell_*^i)}\right)' = \frac{\partial \mathcal{V}}{\partial Y}\frac{\partial Y}{\partial X(\ell_*^i)} + \frac{\partial \mathcal{V}}{\partial X}\frac{\partial X}{\partial X(\ell_*^i)}, \quad \left(\frac{\partial Y}{\partial Y(\ell_*^i)}\right)' = \frac{\partial \mathcal{V}}{\partial Y}\frac{\partial Y}{\partial Y(\ell_*^i)} + \frac{\partial \mathcal{V}}{\partial X}\frac{\partial X}{\partial Y(\ell_*^i)}.$$

Using the Duhamel principle we see that the solution of the variational equation should satisfy

(5.2)

$$\frac{\partial X(\ell_*^f)}{\partial X(\ell_*^i)} = \mathbb{U}(\ell_*^i, \ell_*^f) + \int_{\ell_*^i}^{\ell_*^f} \mathbb{U}(\ell_4, \ell_*^f)\frac{\partial \mathcal{U}}{\partial Y}\frac{\partial Y}{\partial X(\ell_*^i)}d\ell_4, \quad \frac{\partial X}{\partial Y(\ell_4^i)} = \int_{\ell_*^i}^{\ell_*^f} \mathbb{U}(\ell_4, \ell_*^f)\frac{\partial \mathcal{U}}{\partial Y}\frac{\partial Y}{\partial Y(\ell_*^i)}d\ell_4,$$
$$\frac{\partial Y}{\partial Y(\ell_4^i)} = \mathbb{V}(\ell_*^i, \ell_*^f) + \int_{\ell_*^i}^{\ell_*^f} \mathbb{V}(\ell_4, \ell_*^f)\frac{\partial \mathcal{V}}{\partial X}\frac{\partial X}{\partial Y(\ell_*^i)}d\ell_4, \quad \frac{\partial Y}{\partial X(\ell_4^i)} = \mathbb{U}(\ell_4, \ell_*^f)\frac{\partial \mathcal{V}}{\partial X}\frac{\partial X}{\partial X(\ell_*^i)}d\ell_4$$

where $\mathbb{U}$ and $\mathbb{V}$ denote the fundamental solutions of $\mathbb{U}' = \dfrac{\partial \mathcal{U}}{\partial X}\mathbb{U}$ and $\mathbb{V}' = \dfrac{\partial \mathcal{V}}{\partial Y}\mathbb{V}$ respectively.

6. Variational equation

The next step in the proof is the C^1 analysis of the global map. It occupies sections 6-8. We shall work under the assumptions of Lemma 3.2. In particular we will use the estimates of Section 4 and Appendix A.

The plan of the proof of Proposition 3.6 is the following. Matrices (I), (III) and (V) are treated in Sections 6 and 7. Namely, in Sections 6 we study the variational equation while in Section 7 we describe the contribution of the boundary terms. Finally in Section 8 we compute matrices (II) and (IV) which describe the change of variables between the Delaunay coordinates with different centers which are used to the left and to the right of the line $x = -\dfrac{\chi}{2}$.

6.1. **Estimates of the coefficients.**

Lemma 6.1. *We have the following estimates for the RHS of the variational equation.*

(a) *When Q_4 is moving to the right of the section $\{x = -\chi/2\}$, we have*

$$\left[\begin{array}{c|c} \frac{\partial \mathcal{U}_R}{\partial X} & \frac{\partial \mathcal{U}_R}{\partial Y} \\ \hline \frac{\partial \mathcal{V}_R}{\partial X} & \frac{\partial \mathcal{V}_R}{\partial Y} \end{array}\right] = O\left(\begin{array}{cccc|cc} \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} \\ \frac{1}{\chi} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi} & \frac{1}{\chi} \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} \\ \hline \frac{1}{\chi} & \frac{1}{\chi^3} & \frac{1}{\chi^3} & \frac{1}{\chi^3} & \frac{1}{\chi} & \frac{1}{\chi} \\ \frac{1}{\chi} & \frac{1}{\chi^3} & \frac{1}{\chi^3} & \frac{1}{\chi^3} & \frac{1}{\chi} & \frac{1}{\chi} \end{array}\right) + O\left(\frac{\mu}{|Q_4|^2}\right)$$

In addition we have

$$\frac{\partial \mathcal{V}}{\partial Y} = -\frac{1}{\chi}\left[\begin{array}{cc} \frac{\xi L^4 \mathrm{sign}(\dot{x_4})}{(G^2+L^2)(1-\xi)^3} & \frac{\xi L^3}{(1-\xi)^3} \\ \frac{-\xi L^5}{(G^2+L^2)^2(1-\xi)^3} & \frac{-\xi L^4 \mathrm{sign}(\dot{x_4})}{(G^2+L^2)(1-\xi)^3} \end{array}\right] + O\left(\frac{\mu}{\chi} + \frac{\mu}{|Q_4|^2}\right),$$

$$\frac{\partial \mathcal{V}}{\partial L_3} = -\frac{1}{\chi}\left(\frac{-\xi G_4 L_4^3 \mathrm{sign}(\dot{x}_4)}{(L_4^2+G_4^2)(1-\xi)^3}, \frac{\xi G_4 L_4^4}{(L^2+G_4^2)^2(1-\xi)^3}\right)^T + O\left(\frac{\mu}{\chi} + \frac{\mu}{|Q_4|^2}\right),$$

where $\xi = \dfrac{|Q_4|}{\chi} = \dfrac{|Q_4 - Q_2|}{\chi}$.

(b) *When Q_4 is moving to the left of the section $x = -\chi/2$, we have*

$$\left[\begin{array}{c|c} \frac{\partial \mathcal{U}_L}{\partial X} & \frac{\partial \mathcal{U}_L}{\partial Y} \\ \hline \frac{\partial \mathcal{V}_L}{\partial X} & \frac{\partial \mathcal{V}_L}{\partial Y} \end{array}\right] = O\left(\begin{array}{cccc|cc} \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} \\ \frac{1}{\chi} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} \\ \hline \frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} & \frac{1}{\chi} & \frac{1}{\chi} \\ \frac{1}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} & \frac{\mu}{\chi^2} & \frac{1}{\chi} & \frac{1}{\chi} \end{array}\right)$$

In addition we have

$$\frac{\partial \mathcal{V}}{\partial Y} = -\frac{1}{\chi}\left[\begin{array}{cc} \frac{\xi L^2 \mathrm{sign}(\dot{x_4})}{(1-\xi)^3} & \frac{\xi L^3}{(1-\xi)^3} \\ \frac{-\xi L}{(1-\xi)^3} & \frac{-\xi L^2 \mathrm{sign}(\dot{x_4})}{(1-\xi)^3} \end{array}\right] + O\left(\frac{\mu}{\chi}\right),$$

where $\xi = \dfrac{|Q_4 - Q_1|}{\chi}$.

Proof. (a) We estimate the four blocks of the derivative matrix separately.

• We begin with $\dfrac{\partial \mathcal{U}_R}{\partial X}$ part. We consider first the partial derivatives of ℓ_3' since it is the largest component of $\mathcal{U}$. Opening the brackets in the second line of (4.7) we get

(6.1)
$$\frac{d\ell_3}{d\ell_4} = -k+\frac{1}{L_3^3}W+kL_3^3\frac{\partial Q_3}{\partial L_3}\cdot\frac{\partial U}{\partial Q_3}+k^2L_3^3\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}+2kW\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}+O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^3}\right).$$

Note that by (4.6)

$$\begin{aligned} W_R &= k_R 3L_3^5\left(\frac{1}{|Q_3+(\chi,0)|}+\frac{1}{|Q_4+(\chi,0)|}+\frac{\mu Q_4\cdot Q_3}{|Q_4|^3}\right)+O\left(\frac{\mu}{|Q_4|^3}\right)\\ &= O\left(\frac{1}{\chi}+\frac{\mu}{|Q_4|^2}\right) \end{aligned} \tag{6.2}$$

Observe that the RHS of (6.1) depends on L_3 in three ways. First, in contains several terms of the form L_3^m. Second, Q_3 depends on L_3 via (A.2). Third, Q_4 depends on L_4 via (A.5) and L_4 depends on L_3 via (4.6). In particular we need to consider the contribution to $\frac{\partial}{\partial L_3}\frac{d\ell_3}{d\ell_4}$ coming from

$$\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4} = \frac{\partial L_4}{\partial L_3}\frac{\partial Q_4}{\partial L_4}\frac{\partial}{\partial Q_4}.$$

By Lemma A.2 and equation (4.10) we have $\frac{\partial Q_4}{\partial L_4} = O(|Q_4|)$. Therefore the main contribution to (2,1) entry is $O\left(\frac{1}{\chi}+\frac{\mu}{|Q_4|^2}\right)$ and it comes from $\frac{\partial W_R}{\partial Q_4}\frac{\partial Q_4}{\partial L_4}\frac{\partial L_4}{\partial L_3}$, $W_R\frac{\partial}{\partial L_3}\frac{1}{L_3^3}$ and $\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\left(k^2L_3^3\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)$.

For the $(2,2),(2,3),(2,4)$ entries, the computations are similar. We need to act $\frac{\partial}{\partial \ell_3}, \frac{\partial}{\partial G_3}, \frac{\partial}{\partial g_3}$ on (6.1). (4.6) and (6.2) show that the contribution coming from $\frac{\partial L_4}{\partial(\ell_3, G_3, g_3)}$ is $O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right)$. It remains to consider the contribution coming from $\frac{\partial Q_3}{\partial(\ell_3, G_3, g_3)}\frac{\partial}{\partial Q_3}$. Now the bound for $(2,2),(2,3)$ and $(2,4)$ entries follows directly from Lemmas 4.1, 4.3, 4.4, and 4.6.

Next, consider $(1,1)$ entry. We need to estimate

$$\frac{\partial}{\partial L_3}\left((kL_3^3+W)\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\right).$$

Using the Leibniz rule we see that the leading term comes from $\frac{\partial}{\partial L_3}\left(kL_3^3\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\right)$ and it is of order $O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right)$. The estimates for other entries of the $\frac{\partial \mathcal{U}_R}{\partial X}$ part are similar to the $(1,1)$ entry. This completes the analysis of $\frac{\partial \mathcal{U}_R}{\partial X}$.

• Next, we consider $\dfrac{\partial \mathcal{V}_R}{\partial Y}$.

Using the Leibniz rule again we see that the main contribution to the derivatives of $\mathcal{V}$ comes from differentiating $\begin{bmatrix} L_3^3 \dfrac{\partial Q_4}{\partial g_4} \cdot \dfrac{\partial V}{\partial Q_4} \\ -L_3^3 \dfrac{\partial Q_4}{\partial G_4} \cdot \dfrac{\partial V}{\partial Q_4} \end{bmatrix}$

Consider the $(5,5)$ entry. The main contribution to this entry comes from

$$\frac{\partial}{\partial G_4}\left(L_3^3 \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4}\right) = L_3^3 \left(\frac{\partial^2 Q_4}{\partial G_4 \partial g_4} \cdot \frac{\partial V}{\partial Q_4} + \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial^2 V}{\partial Q_4^2} \cdot \frac{\partial Q_4}{\partial G_4}\right).$$

By Lemmas 4.4 and 4.6 the first term is $|Q_4| \cdot O\left(\dfrac{1}{\chi^2} + \dfrac{\mu}{|Q_4|^3}\right) = O\left(\dfrac{1}{\chi} + \dfrac{\mu}{|Q_4|^2}\right)$ and the second term is $|Q_4|^2 \cdot O\left(\dfrac{1}{\chi^3} + \dfrac{\mu}{|Q_4|^4}\right) = O\left(\dfrac{1}{\chi} + \dfrac{\mu}{|Q_4|^2}\right)$. This gives the desired upper bound of the $(5,5)$ entry. Notice that $O(1/\chi)$ term comes from $L_3^3 \dfrac{\partial}{\partial G_4}\left(\dfrac{\partial Q_4}{\partial g_4} \cdot \dfrac{\partial \tilde{V}}{\partial Q_4}\right)$ where $\tilde{V} = -\dfrac{1}{|Q_4 + (\chi, 0)|}$. Thus we need to find the asymptotics of

$$L_3^3 \frac{\partial}{\partial G_4}\left(\frac{\frac{\partial Q_4}{\partial g_4} \cdot (Q_4 + (\chi, 0))}{|Q_4 + (\chi, 0)|^3}\right). \tag{6.3}$$

Let $\dfrac{\partial Q_4}{\partial g_4} = (\boldsymbol{a}, \boldsymbol{b})$. Arguing in the same way as in the estimation of (4.14) we see that $\boldsymbol{a} = O(1)$. Accordingly the numerator in (6.3) is $O(\chi)$ so if we differentiate the denominator of (6.3) the resulting fraction will be of order $O(\chi)O(\chi^{-3}) = O(\chi^{-2})$. Hence $O(1/\chi)$ term comes from

$$L_3^3 \frac{\frac{\partial}{\partial G_4}\left(\frac{\partial Q_4}{\partial g_4} \cdot (Q_4 + (\chi, 0))\right)}{|Q_4 + (\chi, 0)|^3}.$$

The numerator here equals to

$$\frac{\partial}{\partial G_4}\left(\frac{\partial Q_4}{\partial g_4} \cdot Q_4\right) + \frac{\partial^2 Q_4}{\partial G_4 \partial g_4} \cdot (\chi, 0).$$

The first term is $O(\chi)$ due to Lemma A.2(a) so the main contribution comes from the second term. Using Lemma A.3 we see that $(5,5)$ entry equals to

$$-\frac{L_3^3 L_4^2}{\sqrt{L_4^2 + G_4^2}} \frac{\chi \sinh u}{|Q_4 + (\chi, 0)|^3} + O\left(\frac{\mu}{\chi} + \frac{\mu}{|Q_4|^2}\right).$$

Recall that $L_3 = L_4(1 + o(1))$ (due to (4.6)) and $\sinh u = \text{sign}(u) \dfrac{|\ell_4| L_4}{\sqrt{L_4^2 + G_4^2}}$ (due to (A.4)). Since Lemma 4.1 implies that $|Q_4| = |\ell_4|/L_4^2(1 + o(1))$ we obtain that

$O(1/\chi)$-term in $(5,5)$ is asymptotic to

$$-\frac{L^4\mathrm{sign}(u)}{L^2+G^2}\frac{\chi|Q_4|}{(\chi-|Q_4|)^3}.$$

Since u and $\dot{x}_4$ have opposite signs we obtain the asymptotics of $O(1/\chi)$-term claimed in part (a) of the Lemma 6.1. The analysis of other entries of $\dfrac{\partial\mathcal{V}_R}{\partial Y}$ is similar.

• Next, consider the $\dfrac{\partial\mathcal{U}_R}{\partial Y}$ term.

The analysis of $(2,5)$ entry is similar to the analysis of $(2,2)$ entry except that $\dfrac{\partial}{\partial G_4}\left(k^2L_3^3\dfrac{\partial Q_4}{\partial L_4}\dfrac{\partial V}{\partial Q_4}\right)$ contains the term $k^2L_3^3\dfrac{\partial^2 Q_4}{\partial L_4\partial G_4}\dfrac{\partial V}{\partial Q_4}$ which is of order $O(1/\chi)$ due to Lemmas 4.6 and A.3 and this term provides the leading contribution for large t. The analysis of $(2,6)$ is similar to $(2,5)$.

The estimate of the remaining entries of $\dfrac{\partial\mathcal{U}_R}{\partial Y}$ is similar to the analysis of $(1,1)$ entry.

• Thus to complete the proof of (a) it remains to consider $\dfrac{\partial\mathcal{V}}{\partial X}$. We begin with $(5,1)$ entry. We need to act by $\dfrac{\partial}{\partial L_3}+\dfrac{\partial L_4}{\partial L_3}\dfrac{\partial}{\partial L_4}$ on

$$(kL_3^3+W)\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right).$$

The leading term for the estimate of $(5,1)$ comes from

$$\begin{aligned}&\left(\frac{\partial}{\partial L_3}+\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\right)\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)\\ &=\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)+O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right)=O\left(\frac{1}{\chi}+\frac{\mu}{|Q_4|^2}\right).\end{aligned}$$

Observe that $O(1/\chi)$ term here comes from $\dfrac{\partial}{\partial L_4}\left(\dfrac{\partial Q_4}{\partial g_4}\cdot\dfrac{\partial V}{\partial Q_4}\right)$ which can be analyzed in the same way as $(5,5)$ term. The analysis of $(6,1)$ is the same as of $(5,1)$.

The $(5,2)$ entry is equal to $\left(\dfrac{\partial}{\partial \ell_3}+\dfrac{\partial L_4}{\partial \ell_3}\dfrac{\partial}{\partial L_4}\right)\left[\left(\dfrac{\partial Q_4}{\partial g_4}\cdot\dfrac{\partial V}{\partial Q_4}\right)\Gamma\right]$ where

$$\Gamma=kL_3^3+W+k^2L_3^6\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}+2kL_3^3W\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}+W^2\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}.$$

Now the estimate of the $(5,2)$ entry follows from the following estimates

$$\Gamma=O(1),\quad\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)=O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right),$$

$$\left(\frac{\partial}{\partial \ell_3}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right)\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)=\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial}{\partial \ell_3}\frac{\partial V}{\partial Q_4}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{L_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)$$
$$=O\left(\frac{\mu}{|Q_4|^2}+\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right)\left(\frac{1}{\chi}+\frac{\mu}{|Q_4|^2}\right)\right)=O\left(\frac{1}{\chi^3}+\frac{\mu}{|Q_4|^2}\right),$$

and

$$\left(\frac{\partial}{\partial \ell_3}+\frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right)\Gamma=O\left(\frac{1}{\chi^2}+\frac{\mu}{|Q_4|^2}\right).$$

The remaining entries of $\frac{\partial \mathcal{V}}{\partial X}$ are similar to the $(5,2)$ entry. This completes the proof of part (a).

(b)• The estimate of $\frac{\partial \mathcal{V}_L}{\partial Y}$ and $\frac{\partial \mathcal{U}_L}{\partial X}$ are the same as in part (a) however, now $|Q_4|$ is of order χ so $O(\mu/|Q_4|^2)$ is dominated by other terms. In addition to compute the leading part we need to use part (c) Lemma A.3 rather than part (b). Moreover, in order to be able to use the formulas of that Lemma we need to shift the origin to Q_1. Therefore the coordinates of Q_2 become $(\chi,0)$. Then we have

$$\frac{\partial \mathcal{V}_L}{\partial Y}=L_3^3\begin{bmatrix}\frac{\partial^2 Q_4}{\partial G\partial g}\cdot\frac{(-\chi,0)}{|Q_4-(\chi,0)|^3} & \frac{\partial^2 Q_4}{\partial g^2}\cdot\frac{(-\chi,0)}{|Q_4-(\chi,0)|^3}\\ -\frac{\partial^2 Q_4}{\partial G^2}\cdot\frac{(-\chi,0)}{|Q_4-(\chi,0)|^3} & -\frac{\partial^2 Q_4}{\partial G\partial g}\cdot\frac{(-\chi,0)}{|Q_4-(\chi,0)|^3}\end{bmatrix}+O\left(\frac{\mu}{\chi}\right). \tag{6.4}$$

Now the asymptotic expression of $\frac{\partial \mathcal{V}_L}{\partial Y}$ follows directly from Lemma A.3(c). We point out that the "$-$" sign in front of the matrices of $\frac{\partial V}{\partial Y}$ and $\frac{\partial V}{\partial L_3}$ comes from the fact that the new time ℓ_4 that we are using satisfies $\frac{d\ell_4}{dt}=-\frac{1}{L_4^3}+o(1)$ as $\mu\to 0,\chi\to\infty$.

• Next, we consider the $\frac{\partial \mathcal{U}_L}{\partial Y}$ term.

First consider $(1,5)$. We need to find G_4 derivative of

$$\left[\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\right](kL_3^3+W)\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right).$$

Differentiating the first factor we get using Lemma 4.6

$$\frac{\partial}{\partial G_4}\left(\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial U}{\partial Q_3}\right)=\frac{\partial Q_3}{\partial \ell_3}\cdot\frac{\partial^2 U}{\partial Q_3\partial Q_4}\frac{\partial Q_4}{\partial G_4}=O\left(\frac{\mu}{\chi^2}\right). \tag{6.5}$$

When we differentiate the product of the remaining factors then the main contribution comes from

$$\frac{\partial}{\partial G_4}\left(\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)=\frac{\partial^2 Q_4}{\partial L_4\partial G_4}\cdot\frac{\partial V}{\partial Q_4}+\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial}{\partial G_4}\left(\frac{\partial V}{\partial Q_4}\right). \tag{6.6}$$

To bound the last expression we use Lemma A.3. Namely, the second derivative $\dfrac{\partial^2 Q_4}{\partial G_4 \partial L_4} = O(1) + \ell_4(0,1)$, is almost vertical and $\dfrac{\partial V_L}{\partial Q_4} = \dfrac{Q_4}{|Q_4|^3} + \dfrac{\mu(Q_4-Q_3)}{|Q_4-Q_3|^3}$ is almost horizontal. This shows that $\dfrac{\partial^2 Q_4}{\partial G_4 \partial L_4} \cdot \dfrac{\partial V}{\partial Q_4} = \dfrac{1}{\chi^2}$. The main contribution to the second summand in (6.6) comes from $\dfrac{\partial}{\partial G_4}\left(\nabla\left(\dfrac{1}{Q_4}\right)\right)$. Using Lemma A.2, we get

$$\frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial}{\partial G_4}\left(\nabla\left(\frac{1}{Q_4}\right)\right) = (\ell_4(1,0)+O(1))\left(\frac{-\mathrm{Id}}{|Q_4|^3} + 3\frac{Q_4\otimes Q_4}{|Q_4|^5}\right)(\ell_4(0,1)+O(1)) = \frac{1}{\chi^2}.$$

Since $\dfrac{\partial Q_3}{\partial \ell_3} \cdot \dfrac{\partial U}{\partial Q_3} = O(1/\chi^2)$ we get the required estimate for $(1,5)$ entry.

The estimates of other $\dfrac{\partial \mathcal{U}_L}{\partial Y}$ terms are similar to the estimate of $(1,5)$ entry, except for $(2,5)$ and $(2,6)$ entries which are different because $\dfrac{d\ell_3}{d\ell_4}$ is larger than the other coordinates of $\mathcal{U}$.

Now consider $(2,5)$ entry. We need to compute

(6.7)
$$\begin{aligned}
&-\frac{\partial}{\partial G_4}\left((kL_3^3+W)(\frac{1}{L_3^3} + \frac{\partial Q_3}{\partial L_3}\cdot\frac{\partial U}{\partial Q_3})\left(1+(kL_3^3+W)\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}\right)\right)\\
&= -\frac{\partial}{\partial G_4}\left(k + \frac{1}{L_3^3}W + kL_3^3\frac{\partial Q_3}{\partial L_3}\cdot\frac{\partial U}{\partial Q_3} + k^2L_3^3\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + 2kW\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + \frac{1}{\chi^3}\right)\\
&= 0 + \frac{1}{\chi^2} + \frac{\mu}{\chi^2} + \frac{1}{\chi^2} + \frac{1}{\chi^3} + \frac{1}{\chi^3} = O\left(\frac{1}{\chi^2}\right)
\end{aligned}$$

where the analysis of the leading terms is similar to (6.5), (6.6).

- Finally, we consider $\dfrac{\partial \mathcal{V}_L}{\partial X}$. We begin with $(5,1)$ entry. We need to compute

$$\left[\frac{\partial}{\partial L_3} + \frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\right]\left(\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)\Gamma\right)$$

where

$$\Gamma = kL_3^3 + W + k^2L_3^6\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + 2kL_3^3W\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4} + W^2\frac{\partial Q_4}{\partial L_4}\cdot\frac{\partial V}{\partial Q_4}.$$

The main contribution to $\left[\dfrac{\partial}{\partial L_3} + \dfrac{\partial L_4}{\partial L_3}\dfrac{\partial}{\partial L_4}\right]\left(\dfrac{\partial Q_4}{\partial g_4}\cdot\dfrac{\partial V}{\partial Q_4}\right)$ comes from

$$\frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right) = \frac{\partial L_4}{\partial L_3}\frac{\partial^2 Q_4}{\partial L_4\partial g_4}\cdot\frac{\partial V}{\partial Q_4} + \frac{\partial L_4}{\partial L_3}\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial^2 V}{\partial Q_4^2}\frac{\partial Q_4}{\partial L_4}.$$

The two summands above can be estimated by $O(1/\chi^2)$ by the argument used to bound (6.6). Next a direct calculation shows that

$$\Gamma = O(1), \quad \left[\frac{\partial}{\partial L_3} + \frac{\partial L_4}{\partial L_3}\frac{\partial}{\partial L_4}\right]\Gamma = O(1)$$

while $\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right) = O(1/\chi^2)$ by Lemma 4.4 This gives the required bound for the $(5,1)$ entry. The bound for the $(6,1)$ entry is similar.

Next, consider $(5,2)$. It equals to

$$\left[\frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right]\left(\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)\Gamma\right)$$

The main contribution to $\left[\frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right]\left(\frac{\partial Q_4}{\partial g_4}\cdot\frac{\partial V}{\partial Q_4}\right)$ comes from $\frac{\partial}{\partial \ell_3}\left(\frac{\partial Q_4}{\partial g_4}\cdot\nabla\left(\frac{\mu}{|Q_4-Q_3|}\right)\right)$ and it is of order $O\left(\frac{\mu}{\chi^2}\right)$. On the other hand the main contribution to $\left[\frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3}\frac{\partial}{\partial L_4}\right]\Gamma$ comes from $\frac{\partial W}{\partial \ell_3}$ and it is of order $O\left(\frac{1}{\chi^2}\right)$. Combining this with C^0 bounds mentioned used in the analysis of $(5,1)$ we obtain the required estimate on the $(5,2)$ entry. The remaining entries of $\frac{\partial \mathcal{V}_L}{\partial X}$ are similar to $(5,2)$. □

6.2. **Estimates of the solutions.** We integrate the variational equations to get the $\frac{\partial(X,Y)(\ell_4^f)}{\partial(X,Y)(\ell_4^i)}$ in equation (5.1).

Recall that map (I) describes the transition between sections $\{x=-2\}$ and $\left\{x=-\frac{\chi}{2}\right\}$, map (III) describes the transition between sections $\left\{x=-\frac{\chi}{2},\ \dot{x}<0\right\}$ and $\left\{x=-\frac{\chi}{2},\ \dot{x}>0\right\}$ and map (V) describes the transition between sections $\left\{x=-\frac{\chi}{2}\right\}$, and $\{x=-2\}$.

Lemma 6.2. *The following estimates are valid*

(a) *For maps (I) and (V),*

(6.8)

$$\frac{\partial(X,Y)(\ell_4^f)}{\partial(X,Y)(\ell_4^i)} = \left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu)\\ O(1) & 1+O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)\\ O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu) & O(\mu)\\ O(\mu) & O(\mu) & O(\mu) & 1+O(\mu) & O(\mu) & O(\mu)\\ \hline O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)\\ O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \end{array}\right].$$

(b) *For map (III),*

(6.9)
$$\frac{\partial(X,Y)(\ell_4^f)}{\partial(X,Y)(\ell_4^i)}=\left[\begin{array}{cccc|cc} 1+O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi})\\ O(1) & 1+O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi})\\ O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & 1+O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi})\\ O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & 1+O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi})\\ \hline O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi}) & O(1) & O(1)\\ O(\frac{1}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi}) & O(\frac{\mu}{\chi}) & O(1) & O(1) \end{array}\right].$$

(c) $\dfrac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^i)}$ *has the following limits as $\mu\to 0,\chi\to\infty$*

$$\text{Map (I)}:\left[\begin{array}{cc} 1+\dfrac{\tilde L_4^2}{2(\tilde L_4^2+\tilde G_4^2)} & -\dfrac{\tilde L_4}{2}\\ \dfrac{\tilde L_4^3}{2(\tilde L_4^2+\tilde G_4^2)^2} & 1-\dfrac{\tilde L_4^2}{2(\tilde L_4^2+\tilde G_4^2)}\end{array}\right],\ \text{Map (III)}:\left[\begin{array}{cc}\dfrac{1}{2} & -\dfrac{L_4}{2}\\ \dfrac{3}{2L_4} & \dfrac{1}{2}\end{array}\right],$$

$$\text{Map (V)}:\left[\begin{array}{cc} 1-\dfrac{\hat L_4^2}{2(\hat L_4^2+\hat G_4^2)} & -\dfrac{\hat L_4}{2}\\ \dfrac{\hat L_4^3}{2(\hat L_4^2+\hat G_4^2)^2} & 1+\dfrac{\hat L_4^2}{2(\hat L_4^2+\hat G_4^2)}\end{array}\right].$$

In addition for map (I) we have $\dfrac{\partial Y}{\partial L_3}\to\left(-\dfrac{\tilde G_4\tilde L_4}{2(\tilde L_4^2+\tilde G_4^2)},-\dfrac{\tilde G_4\tilde L_4^2}{2(\tilde L_4^2+\tilde G_4^2)^2}\right)^T.$

Parts (a) and (b) of this lemma claim that we can integrate the estimates of Lemma 6.1 over ℓ_4-interval of size $O(\chi)$.

Proof. (a) We divide the proof into several steps.

Step 1. Keeping in mind the integrals

$$\int_0^{\chi/2}\frac{1}{\chi}d\ell_4=O(1),\ \text{and}\ \int_0^{\chi/2}\frac{\mu}{\ell_4^2+1}d\ell_4=O(\mu)$$

we conclude using the Gronwall inequality that if $(\delta X,\delta Y)(\ell_4^i)=O(1)$ then $(\delta X,\delta Y)(\ell_4)=O(1)$ for all $\ell_4\in[\ell_4^i,\ell_4^f]$.

Step 2. Plugging the estimate of step 1 back into the variational equation we see that $(\delta L_3,\delta G_3,\delta g_3)(\ell_4)-(\delta L_3,\delta G_3,\delta g_3)(\ell_4^i)=O(\mu)$. This proves the required bound for $(\delta L_3,\delta G_3,\delta g_3)$.

Step 3. Steps 1 and 2 imply that

$$(\delta Y)'(\ell_4)=\frac{\partial\mathcal{V}}{\partial Y}(\ell_4)\delta Y(\ell_4)+\frac{\partial\mathcal{V}}{\partial L_3}(\ell_4)\delta L_3(\ell_4^i)+O\left(\frac{\mu}{\ell_4^2+1}\right).$$

We treat this as a nonhomogeneous linear equation for δY. Thus

(6.10)
$$\delta Y(\ell_4) = \mathbb{V}(\ell_4^i, \ell_4)\delta Y(\ell_4^i) + \left(\int_{\ell_4^0}^{\ell_4} \mathbb{V}(s,\ell_4)\frac{\partial \mathcal{V}}{\partial L_3}(s)ds\right)\delta L_3(\ell_4^i) + \int_{\ell_4^i}^{\ell_4} O\left(\|\mathbb{V}(s,\ell_4)\|\frac{\mu}{1+s^2}\right)ds$$

where $\mathbb{V}(s,\ell_4)$ denotes the fundamental solution of the corresponding homogeneous equation. (6.10) immediately implies the required bound for δY.

Step 4. Plugging the estimates of steps 2 and 3 into the equation for $\delta\ell_3$ we see that if $(\delta L_3, \delta G_3, \delta g_3)(\ell_4^i) = 0$ and hence $(\delta L_3, \delta G_3, \delta g_3)(\ell_4) = O(\mu)$ for all ℓ_4 then $(\delta\ell_3)' = O\left(\frac{\mu}{1+\ell_4^2}\right)$ proving the required bound for $\delta\ell_3$.

(b) We use the same steps as in part (a). On step 1 we show that $(\delta X, \delta Y)(\ell_4) = O(1)$ for all ℓ_4. On step 2 we conclude that $(\delta L_3, \delta G_3, \delta g_3)(\ell_4) - (\delta L_3, \delta G_3, \delta g_3)(\ell_4^i) = O(1/\chi)$. On step 3 we prove the result of part (b) for δY. On step 4 we use the results of step 3 to show that if $\delta X(\ell_4^i) = 0$ then $(\delta L_3, \delta G_3, \delta g_3)(\ell_4) = O(\mu/\chi)$ and $\delta\ell_3(\ell_4) = O(1/\chi)$.

To prove (c) we need to find the asymptotics of $\mathbb{V}$. Consider map (I) first. $\mathbb{V}$ satisfies
$$\mathbb{V}' = \frac{\partial \mathcal{V}}{\partial Y}\mathbb{V}.$$
By already established part (a) $\mathbb{V} = O(1)$ so the above equation can be rewritten as
$$\mathbb{V}' = \frac{\xi L^2}{\chi(1-\xi)^3}A\mathbb{V} + O\left(\frac{\mu}{\ell_4^2+1} + \frac{\mu}{\chi}\right).$$
where $A = \begin{bmatrix} -\frac{L^2}{(G^2+L^2)} & L \\ -\frac{L^3}{(G^2+L^2)^2} & \frac{L^2}{(G^2+L^2)} \end{bmatrix}$. Now Gronwall Lemma gives $\mathbb{V} \approx \tilde{\mathbb{V}}$ where $\tilde{\mathbb{V}}$ is the fundamental solution of $\tilde{\mathbb{V}}' = \frac{\xi L^2}{\chi(1-\xi)^3}A\tilde{\mathbb{V}}$. Using ξ as the independent variable we get $\frac{d\tilde{\mathbb{V}}}{d\xi} = -\frac{\xi}{(1-\xi)^3}A\tilde{\mathbb{V}}$. Note that $\xi(\ell_4^i) = o(1)$, $\xi(\ell_4^f) = \frac{1}{2} + o(1)$. Making a further time change $d\tau = \frac{\xi d\xi}{(1-\xi)^3}$ we obtain the constant coefficient linear equation $\frac{d\tilde{\mathbb{V}}}{d\tau} = -A\tilde{\mathbb{V}}$. Observe that $\mathrm{Tr}(A) = \det(A) = 0$ and so $A^2 = 0$. Therefore

(6.11)
$$\tilde{\mathbb{V}}(\sigma,\tau) = \mathrm{Id} - (\tau - \sigma)A.$$

Since $\tau = \frac{\xi^2}{2(1-\xi)^2}$ we have $\tau(0) = 0$, $\tau\left(\frac{1}{2}\right) = \frac{1}{2}$. Plugging this into (6.11) we get the claimed asymptotics for map (I). The analysis of map (V) is similar. To analyze map (III) we split
$$\frac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^i)} = \frac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^m)}\frac{\partial Y(\ell_4^m)}{\partial Y(\ell_4^i)}$$

where $\ell_4^m = \dfrac{\ell_4^i + \ell_4^f}{2}$. Using the argument presented above we obtain

$$\frac{\partial Y(\ell_4^m)}{\partial Y(\ell_4^i)} = \begin{bmatrix} \frac{3}{2} & -\frac{L}{2} \\ \frac{1}{2L} & \frac{1}{2} \end{bmatrix}, \quad \frac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^m)} = \begin{bmatrix} \frac{1}{2} & -\frac{L}{2} \\ \frac{1}{2L} & \frac{3}{2} \end{bmatrix}.$$

Multiplying the above matrices we obtain the required asymptotics for map (III).

Next using the same argument as in analysis of $\dfrac{\partial Y(\ell_4^f)}{\partial Y(\ell_4^i)}$ we obtain $\dfrac{\partial Y}{\partial L_3} \approx \mathbb{W}$ where

$$\mathbb{W}' = \frac{\xi L^2}{\chi(1-\xi)^3}\left[A\mathbb{W} + \left(\frac{GL}{(L^2+G^2)}, \frac{GL^2}{(L^2+G^2)^2}\right)^T\right].$$

In terms of the new time this equation reads

$$\frac{d\mathbb{W}}{d\tau} = -\left[A\mathbb{W} + \left(\frac{GL}{(L^2+G^2)}, \frac{GL^2}{(L^2+G^2)^2}\right)^T\right].$$

Solving this equation using (6.11) and initial condition $(0,0)^T$, we obtain the asymptotics of $\dfrac{\partial Y}{\partial L_3}$. □

7. Boundary contributions and the proof of Proposition 3.6

According to (5.1) we need to work out the boundary contributions in order to complete the proof of Proposition 3.6.

7.1. **Dependence of ℓ_4 on variables (X, Y).** To use the formula (5.1) we need to work out $(\mathcal{U}, \mathcal{V})(\ell_4^i) \otimes \dfrac{\partial \ell_4^i}{\partial (X,Y)^i}$ and $(\mathcal{U}, \mathcal{V})(\ell_4^f) \otimes \dfrac{\partial \ell_4^f}{\partial (X,Y)^f}$. Consider x_4 component of Q_4 (see equation (A.5)).

$$x_4 = \cos g_4(L_4^2 \sinh u_4 - e_4) - \sin g_4(L_4 G_4 \cosh u_4).$$

For fixed $x_4 = -\chi/2$ or -2, we can solve for ℓ_4 as a function of L_4, G_4, g_4. From the calculations in the Appendix A.2, Lemma A.2, and the implicit function theorem, we get

for the section $x_4 = -\chi/2$, $\left(\dfrac{\partial \ell_4}{\partial L_4}, \dfrac{\partial \ell_4}{\partial G_4}, \dfrac{\partial \ell_4}{\partial g_4}\right)\Big|_{x_4=-\chi/2} = (O(\chi), O(1), O(1))$,

for the section $x_4 = -2$, $\left(\dfrac{\partial \ell_4}{\partial L_4}, \dfrac{\partial \ell_4}{\partial G_4}, \dfrac{\partial \ell_4}{\partial g_4}\right)\Big|_{x_4=-2} = (O(1), O(1), O(1))$.

Using equation (4.6) which relates L_4 to L_3, we obtain for the section $\{x_4 = -\chi/2\}$,

$$\begin{aligned}
&\frac{\partial \ell_4}{\partial (X,Y)}\Big|_{x_4=-\chi/2} = (O(\chi), O(1/\chi), O(1/\chi), O(1/\chi), O(1), O(1)),\\
&(\mathcal{U},\mathcal{V})\Big|_{x_4=-\chi/2} = (O(1/\chi^2), O(1), O(1/\chi^2), O(1/\chi^2), O(1/\chi^2), O(1/\chi^2))^T,
\end{aligned} \tag{7.1}$$

For the section $\{x_4 = -2\}$,

$$\begin{aligned}
&\frac{\partial \ell_4}{\partial L_3}\Big|_{x_4=-2} = (O(1), O(\mu), O(\mu), O(\mu), O(1), O(1)),\\
&(\mathcal{U},\mathcal{V})\Big|_{x_4=-2} = (O(\mu), O(1), O(\mu), O(\mu), O(\mu), O(\mu))^T.
\end{aligned} \tag{7.2}$$

The matrix $(\mathcal{U},\mathcal{V}) \otimes \frac{\partial \ell_4}{\partial (X,Y)}\Big|_{x_4=-\chi/2}$ has rank 1 and the only nonzero eigenvalue is $O(1/\chi)$, and $(\mathcal{U},\mathcal{V}) \otimes \frac{\partial \ell_4}{\partial (X,Y)}\Big|_{x_4=-2}$ has rank 1 and the only nonzero eigenvalue is $O(\mu)$. So the inversion appearing in (5.1) is valid.

7.2. **Asymptotics of matrices $(I), (III), (V)$ from the Proposition 3.6.** Here we complete the computations of matrices (I), (III) and (V).

The boundary contribution to (I). In this case, ℓ_4^i stands for the section $\{x_4 = -2\}$ and ℓ_4^f stands for the section $\{x_4 = -\chi/2\}$. So we use equation (7.2) to form $(\mathcal{U},\mathcal{V})(\ell_4^i) \otimes \frac{\partial \ell_4^i}{\partial (X,Y)^i}$ in equation (5.1) and equation (7.1) to form $(\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^f}{\partial (X,Y)^f}$. We have

$$\begin{aligned}
&\left(\mathrm{Id} - (\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^i}{\partial (X,Y)^i}\right)^{-1} = \mathrm{Id} + \sum_{k=1}^{\infty}\left((\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^i}{\partial (X,Y)^i}\right)^k\\
&= \mathrm{Id} + \left((\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^i}{\partial (X,Y)^i}\right)\sum_{k=0}^{\infty}\left(\frac{\partial \ell_4^i}{\partial (X,Y)^i}\cdot (\mathcal{U},\mathcal{V})(\ell_4^f)\right)^k\\
&= \mathrm{Id} + \left((\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^i}{\partial (X,Y)^i}\right)(1+O(1/\chi)) =
\end{aligned} \tag{7.3}$$

$$\left[\begin{array}{cccc|cc}
1+O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^2) & O(1/\chi^2)\\
O(\chi) & 1+O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1) & O(1)\\
O(1/\chi) & O(1/\chi^3) & 1+O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^2) & O(1/\chi^2)\\
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & 1+O(1/\chi^3) & O(1/\chi^2) & O(1/\chi^2)\\
\hline
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & 1+O(1/\chi^2) & O(1/\chi^2)\\
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^2) & 1+O(1/\chi^2)
\end{array}\right]$$

Now we use equation (5.1) and Lemma 6.2 to obtain the asymptotics of the matrix (I) stated in Proposition 3.6.

The boundary contribution to (III)

This time we use equation (7.1) to form both $(\mathcal{U},\mathcal{V})(\ell_4^i)\otimes\dfrac{\partial \ell_4^i}{\partial (X,Y)^i}$ and $(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial \ell_4^f}{\partial (X,Y)^f}$ in equation (5.1).

The matrix $\left(\mathrm{Id}-(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial \ell_4^f}{\partial (X,Y)^f}\right)^{-1}$ has the same form as (7.3). Now we use equation (5.1) and Lemma 6.2 to obtain the asymptotics of the matrix (III) stated in Proposition 3.6.

The boundary contribution to (V)

This time we use equation (7.1) to form $(\mathcal{U},\mathcal{V})(\ell_4^i)\otimes\dfrac{\partial \ell_4^i}{\partial (X,Y)^i}$ and equation (7.2) to form $(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial \ell_4^f}{\partial (X,Y)^f}$ in equation (5.1).

The matrix $\left(\mathrm{Id}-(\mathcal{U},\mathcal{V})(\ell_4^f)\otimes\dfrac{\partial \ell_4^f}{\partial (X,Y)^f}\right)^{-1}$ has the form

$$\left[\begin{array}{cccc|cc} 1+O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(\mu) & O(\mu)\\ O(1) & 1+O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)\\ O(\mu) & O(\mu^2) & 1+O(\mu^2) & O(\mu^2) & O(\mu) & O(\mu)\\ O(\mu) & O(\mu^2) & O(\mu^2) & 1+O(\mu^2) & O(\mu) & O(\mu)\\ \hline O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & 1+O(\mu) & O(\mu)\\ O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(\mu) & 1+O(\mu) \end{array}\right]$$

Now we use equation (5.1) and Lemma 6.2 to obtain the asymptotics of the matrix (V) stated in Proposition 3.6.

8. Switching foci

Recall that we treat the motion of Q_4 as a Kepler motion focused at Q_2 when it is moving to the right of the section $\{x=-\chi/2\}$ and treat it as a Kepler motion focused at Q_1 when it is moving to the left of the section $\{x=-\chi/2\}$. Therefore, we need to make a change of coordinates when Q_4 crosses the section $\{x_4=-\chi/2\}$. These are described by the matrices (II) and (IV). Under this coordinate change the Q_3 part of the Delaunay variables does not change. The change of G_4 is given by the difference of angular momentums w.r.t. different reference points (Q_1 or Q_2). To handle it we introduce an auxiliary variable v_{4y}-the y component of the velocity of Q_4. Relating g_4 with respect to the different reference points to v_{4y} we complete the computation.

8.1. **From the right to the left.** We have

$$(II) = \frac{\partial(L_3, \ell_3, G_3, g_3, G_{4L}, g_{4L})}{\partial(L_3, \ell_3, G_3, g_3, G_{4R}, g_{4R})}\Big|_{x_4=-\chi/2} = (iii)(ii)(i)$$

where matrices $(i), (ii)$ and (iii) correspond to the following coordinate changes restricted to the section $\{x_4 = -\chi/2\}$.

$$(G, g)_{4R} \xrightarrow{(i)} (G, v_y)_{4R} \xrightarrow{(ii)} (G, v_y)_{4L} \xrightarrow{(iii)} (G, g)_{4L}.$$

Computation of matrices (i) and (iii)(ii) in Proposition 3.6. (i) is given by the relation

$$v_{4y} = \frac{\frac{1}{L_{4R}} \sinh u_{4R} \sin g_{4R} - \frac{G_{4R}}{L_{4R}^2} \cos g_{4R} \cosh u_{4R}}{1 - e_{4R} \cosh u_{4R}} < 0, \quad L_{4R} = k_R L_3 - \frac{W_R}{3L_3^2}.$$

where last relation follows from (4.6). Recall that by Lemma 4.7

$$g_{4R} = \arctan \frac{G_{4R}}{L_{4R}} + O(1/\chi).$$

In addition (8.1) below and the fact that G_{4R} and G_{4L} are $O(1)$ implies $v_{4y} = O(\frac{1}{\chi})$. Now the asymptotics of (i) is obtained by a direct computation. We compute $\frac{dv_{4y}}{dL_3}$ the other derivatives are similar but easier. We have $\frac{dv_{4y}}{dL_3} = \frac{dv_{4y}}{dL_{4R}} \frac{\partial L_{4R}}{\partial L_3}$. The second term is $k_R + O(1/\chi)$. On the other hand

$$\frac{dv_{4y}}{dL_4} = \frac{\frac{\partial}{\partial L_{4R}} \left(\frac{1}{L_{4R}} \sinh u_{4R} \sin g_{4R} - \frac{G_{4R}}{L_{4R}^2} \cos g_{4R} \cosh u_{4R} \right)}{1 - e_{4R} \cosh u_{4R}}$$
$$+ v_{4R} \frac{\frac{\partial e_{4R}}{\partial L_{4R}} \cosh u_{4R}}{1 - e_{4R} \cosh u_{4R}} + \frac{\partial v_{4R}}{\partial \ell_{4R}} \frac{\partial \ell_{4R}}{\partial L_{4R}}.$$

The main contribution comes from the first term which equals

$$-\frac{G_{4R}}{L_{4R}(L_{4R}^2 + G_{4R}^2)} + O(1/\chi).$$

The second term is $O(1/\chi)$ since $v_{4R} = O(1/\chi)$. Next rewriting

$$v_{4y} = \frac{\frac{1}{L_{4R}} \tanh u_{4R} \sin g_{4R} - \frac{G_{4R}}{L_{4R}^2} \cos g_{4R}}{(1/\cosh u_{4R}) - e_{4R}}$$

we see that

$$\frac{\partial v_{4y}}{\partial \ell_{4R}} \frac{\partial \ell_{4R}}{\partial L_{4R}} = O(1/\chi^2) \times O(\chi) = O(1/\chi)$$

since $\frac{\partial \ell_{4R}}{\partial L_{4R}} = O(\chi)$ by (7.1).

(ii) is given by

$$G_L = G_R - \chi v_{4y}, \tag{8.1}$$

which comes from the simple relation $v_4 \times Q_4 = v_4 \times (Q_4 - Q_1) + v_4 \times Q_1$. Here G_{4R} and v_{4y} are independent variables so the computation of the derivative of (ii) is straightforward.

To compute the derivative of (iii) we use the relation

$$v_{4y} = \frac{\frac{1}{L_{4L}} \sinh u_{4L} \sin g_{4L} - \frac{G_{4L}}{L_{4L}^2} \cos g_{4L} \cosh u_{4L}}{1 - e_{4L} \cosh u_{4L}}$$

where $u_L < 0$. Arguing the same way as for (i) and using the fact that by Lemma 4.7, $G_L, g_L = O(1/\chi)$, $-\sinh u_L, \cosh u_L \simeq \dfrac{\ell_{4L}}{e_L}$ we obtain

$$\delta v_{4y} = \frac{\delta G_{4L}}{k_R^2 L_3^2} + \frac{\delta g_{4L}}{k_R L_3} + HOT$$

Hence

$$\delta g_{4R} = -\frac{\delta G_{4L}}{k_R L_3} + k_R L_3 \delta v_{4y} + HOT = -\frac{(\delta G_{4R}/k_R) + \chi \delta v_{4y}}{k_R L_3} + HOT$$

completing the proof of the lemma. □

8.2. **From the left to the right.** At this step we need to compute

$$(IV) = \frac{\partial(L_3, \ell_3, G_3, g_3, G_{4R}, g_{4R})}{\partial(L_3, \ell_3, G_3, g_3, G_{4L}, g_{4L})}\Big|_{x_4 = -\chi/2} = (iii')(ii')(i').$$

where the matrices (iii'), (ii') and (i') correspond to the following changes of variables restricted to the section $\{x_4 = -\chi/2\}$.

$$(G, g)_L \xrightarrow{(i')} (G, v_{4y})_L \xrightarrow{(ii')} (G, v_{4y})_R \xrightarrow{(iii')} (G, g)_R.$$

Computation of matrices (iii') and $(ii')(i')$ in Proposition 3.6. (i') is given by

$$v_{4y} = \frac{\frac{1}{L_{4L}} \sinh u_{4L} \sin g_{4L} - \frac{G_{4L}}{L_{4L}^2} \cos g_{4L} \cosh u_{4L}}{1 - e_{4L} \cosh u_{4L}} < 0.$$

Here $u_L > 0$ and $G_{4L}, g_{4L} = O(1/\chi)$.

(ii') is given by

$$G_R = G_L + \chi v_{4yL}.$$

Now the analysis is similar to Subsection 8.1. In particular the main contribution to $[(ii')(i')]_{44}$ comes from

$$\frac{\partial(G_{4R}, v_{4y})}{\partial(G_{4L}, g_{4L})} = \frac{\partial(G_{4R}, v_{4y})}{\partial(G_{4L}, v_{4y})} \frac{\partial(G_{4L}, v_{4y})}{\partial(G_{4L}, g_{4L})} = \begin{bmatrix} 1 & \chi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{L_3^2} + O\left(\frac{1}{\chi}\right) & -\frac{1}{L_3} + O\left(\frac{1}{\chi}\right) \end{bmatrix}.$$

The analysis of (43) part is similar.

(iii') is given by

$$G_R = G_R, \qquad v_{4y} = \frac{\frac{1}{L_{4R}}\sinh u_{4R}\sin g_{4R} - \frac{G_{4R}}{L_{4R}^2}\cos g_{4R}\cosh u_{4R}}{1 - e_{4R}\cosh u_{4R}} < 0.$$

Here $u_{4R} < 0$, and by Lemma 4.7, $\tan g_{4R} = -\dfrac{G_{4R}}{L_{4R}} + O(1/\chi)$. To get the asymptotics of the derivative we first show that similarly to Subsection 8.1, we have

$$dv_{4y} = \left(-\frac{G_{4R}}{L_3(k_R^2L_3^2 + G_{4R}^2)}, 0, 0, 0, \frac{1}{k_R^2L_3^2 + G_{4R}^2}, \frac{1}{k_RL_3}\right) + O\left(\frac{1}{\chi}, \frac{1}{\chi^2}, \frac{1}{\chi^2}, \frac{1}{\chi^2}, \frac{1}{\chi}, \frac{1}{\chi}\right)$$

and then take the inverse. □

9. Approaching close encounter

In this paper we choose to separate local and global maps by section $\{x_4 = -2\}$. We could have used instead $\{x_4 = -10\}$, or $\{x_4 = -100\}$. Our first goal is to show that the arbitrariness of this choice does not change the asymptotics of derivative of the local map (we have already seen in Sections 6.2 and 7 that it does not in change the asymptotics of the derivative of the global map).

We choose the section $\{|Q_3 - Q_4| = \mu^\kappa\}$, $1/3 < \kappa < 1/2$. Outside the section the orbits are treated as perturbed Kepler motions and inside the section the orbits are treated as two body scattering. We shall estimate the errors of this approximation. We break the orbit into three pieces: from $\{x_4 = -2, \dot{x}_4 > 0\}$ to $\{|Q_3^- - Q_4^-| = \mu^\kappa\}$, from $\{|Q_3^- - Q_4^-| = \mu^\kappa\}$ to $\{|Q_3^+ - Q_4^+| = \mu^\kappa\}$ and from $\{|Q_3^+ - Q_4^+| = \mu^\kappa\}$ to $\{x_4 = -2, \dot{x}_4 > 0\}$. Here and below, we use the following convention.

Convention: *A variable with superscript $-$ (reap. $+$) means its value measured on the section $|Q_3 - Q_4| = \mu^\kappa$ before (resp. after) Q_3, Q_4 coming to close encounter.*

In this section we consider the two pieces of orbit outside the section $\{|Q_3 - Q_4| = \mu^\kappa\}$. We use Hamiltonian (4.1). Then we convert the Cartesian coordinates to Delaunay coordinates. The resulting Hamiltonian is

$$H = -\frac{1}{2L_3^2} + \frac{1}{2L_4^2} - \frac{1}{|Q_4 + (\chi, 0)|} - \frac{1}{|Q_3 + (\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}. \tag{9.1}$$

The difference with the Hamiltonian (4.2) is that we do not do the Taylor expansion to the potential $-\dfrac{1}{|Q_3 - Q_4|}$.

The next lemma and the remark after it tell us that we can neglect those two pieces.

Lemma 9.1. *Consider the orbits satisfying the conditions of Lemma 3.1. For the pieces of orbit from $x_4 = -2, \dot{x}_4 > 0$ to $|Q_3^- - Q_4^-| = \mu^\kappa$ and from $|Q_3^+ - Q_4^+| = \mu^\kappa$ to $x_4 = -2, \dot{x}_4 > 0$, the derivative matrices have the following form in Delaunay* *coordinates* $\dfrac{\partial(X,Y)^-}{\partial(X,Y)|_{x_4=-2}}, \dfrac{\partial(X,Y)|_{x_4=-2}}{\partial(X,Y)^+} =$

$$\left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ O(1) & O(1) & O(1) & O(1) & O(1) & O(1) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] + O\left(\mu^{1-2\kappa} + \frac{1}{\chi^3}\right).$$

Proof. The proof follows the plan in Section 5. We first consider the integration of the variational equation. We treat the orbit as Kepler motions perturbed by Q_1 and interaction between Q_3 and Q_4. Consider first the perturbation coming from the interaction of Q_3 and Q_4. The contribution of this interaction to the variational equation is of order $\dfrac{\mu}{|Q_3 - Q_4|^3}$. If we integrate the variational equation along an orbit such that $|Q_3 - Q_4|$ goes from -2 to μ^κ, then the contribution has the order

$$O\left(\int_{-2}^{\mu^\kappa} \frac{\mu}{|t|^3} dt\right) = O(\mu^{1-2\kappa}). \tag{9.2}$$

Similar consideration shows that the perturbation from Q_1 is $O(1/\chi^3)$.
On the other hand absence of perturbation, all Delaunay variables except ℓ_3 are constants of motion. The $(2,1)$ entry is also $o(1)$ following from the same estimate as the $(2,1)$ entry of the matrix in Lemma 6.1. After integrating over time $O(1)$, the solutions to the variational equations have the form

$$\mathrm{Id} + O(\mu^{1-2\kappa} + 1/\chi^3).$$

Next we compute the boundary contributions. The analysis is the same as Section 7. The derivative is given by formula (5.1). We need to work out $(\mathcal{U},\mathcal{V})(\ell_4^i) \otimes \dfrac{\partial \ell_4^i}{\partial(X,Y)^i}$ and $(\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \dfrac{\partial \ell_4^f}{\partial(X,Y)^f}$. In both cases we have

$$(\mathcal{U},\mathcal{V}) = (0,1,0,0,0,0) + O(\mu^{1-2\kappa}).$$

For the section $\{x_4 = -2\}$, we use (7.2). For the section $\{|Q_3 - Q_4| = \mu^\kappa\}$, we have

$$\frac{\partial \ell_4}{\partial(X,Y)} = -\left(\frac{\partial|Q_3 - Q_4|}{\partial \ell_4}\right)^{-1} \frac{\partial|Q_3 - Q_4|}{\partial(X,Y)} = -\frac{(Q_3 - Q_4)\cdot \frac{\partial(Q_3 - Q_4)}{\partial(X,Y)}}{(Q_3 - Q_4)\cdot \frac{\partial(Q_3 - Q_4)}{\partial \ell_4}} \tag{9.3}$$

We will prove in Lemma 10.2(c) below that the angle formed by $Q_3 - Q_4$ and $v_3 - v_4$ is $O\left(\mu^{1-\kappa}\right)$ (the proof of Lemma 10.2 does not rely on section 9). Thus in (9.3) we can replace $Q_3 - Q_4$ by $v_3 - v_4$ making $O\left(\mu^{1-\kappa}\right)$ mistake. Hence

$$\frac{\partial \ell_4}{\partial(X,Y)} = \frac{(v_3 - v_4)\cdot \frac{\partial(Q_3-Q_4)}{\partial(X,Y)}}{(v_3 - v_4)\cdot \frac{\partial Q_4}{\partial \ell_4}} + O(\mu^{1-\kappa}),$$

Note that $\dfrac{\partial Q_4}{\partial \ell_4}$ is parallel to v_4. Using the information about v_3 and v_4 from Appendix B.1 we see that $\langle v_3, v_4\rangle \neq \langle v_4, v_4\rangle$. Therefore the denominator in (9.3) is bounded away from zero and so

$$\frac{\partial \ell_4}{\partial(X,Y)} = (O(1), O(1), O(1), O(1), O(1), O(1)).$$

We also need to make sure the second component $\dfrac{\partial \ell_4}{\partial \ell_3}$ is not close to 1, so that $\mathrm{Id} - (\mathcal{U},\mathcal{V})(\ell_4^f) \otimes \dfrac{\partial \ell_4^f}{\partial(X,Y)^f}$ is invertible when $|Q_3 - Q_4| = \mu^\kappa$ serves as the final section. In fact, due to (4.7), $\dfrac{\partial \ell_4}{\partial \ell_3} \simeq -1$. Using formula (5.1), we get the asymptotics stated in the lemma. □

Remark 9.1. *Using the explicit value of the vectors* $\hat{\bar{\mathbf{l}}}_2,\ \hat{\bar{\mathbf{l}}}_3,\ w,\ \tilde{w}$ *in equations (3.3), we find that in the limit* $\mu \to 0, \chi \to \infty$

$$\left(\frac{\partial(X,Y)^-}{\partial(X,Y)|_{x_4=-2}}\right) span\{w, \tilde{w}\} = span\{w, \tilde{w}\}$$

and

$$\hat{\bar{\mathbf{l}}}_2\left(\frac{\partial(X,Y)|_{x_4=-2}}{\partial(X,Y)^+}\right) = \hat{\bar{\mathbf{l}}}_2, \quad \hat{\bar{\mathbf{l}}}_3\left(\frac{\partial(X,Y)|_{x_4=-2}}{\partial(X,Y)^+}\right) = \hat{\bar{\mathbf{l}}}_3$$

This tells us that we can neglect the derivative matrices corresponding to the pieces of orbit from $x_4 = -2, \dot{x}_4 > 0$ *to* $|Q_3^- - Q_4^-| = \mu^\kappa$ *and from* $|Q_3^+ - Q_4^+| = \mu^\kappa$ *to* $x_4 = -2, \dot{x}_4 > 0$. *We thus can identify* $d\mathbb{L}$ *with*

$$\frac{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^-} + O(\mu^{1-2\kappa})$$

where $(L_3, \ell_3, G_3, g_3, G_4, g_4)^\pm$ *denote the Delaunay variables measured on the section* $\{|Q_3^\pm - Q_4^\pm| = \mu^\kappa\}$.

10. C^0 estimate for the local map

In Sections 10 and 12 we consider the piece of orbit from $|Q_3^- - Q_4^-| = \mu^\kappa$ to $|Q_3^+ - Q_4^+| = \mu^\kappa$. Because of Remark 9.1, we simply write $d\mathbb{L}$ for the derivative for this piece.

10.1. **Justifying Gerver's asymptotics.** It is convenient to use the coordinates of relative motion and the motion of mass center. We define

$$v_\pm = v_3 \pm v_4, \quad Q_\pm = \frac{Q_3 \pm Q_4}{2}. \tag{10.1}$$

Here "-" refers to the relative motion and "+" refers to the center of mass motion. To study the relative motion, we make the following rescaling:

$$q_- := Q_-/\mu, \quad \tau := t/\mu \text{ and } v_- \text{ remains unchanged.} \tag{10.2}$$

In this way, we zoom in the picture of Q_3 and Q_4 by a factor $1/\mu$.

Then we have the following lemma.

Lemma 10.1. *Inside the sphere* $|Q_-| = \mu^\kappa, 1/3 < \kappa < 1/2$,

(a) *the equation governing the motion of the center of mass is a Kepler motion focused at* Q_2 *perturbed by* $O(\mu^{2\kappa})$,

$$\dot{Q}_+ = \frac{v_+}{2}, \quad \dot{v}_+ = -\frac{2Q_+}{|Q_+|^3} + O(\mu^{2\kappa}). \tag{10.3}$$

(b) *In the rescaled variables, the equation governing the relative motion is a Kepler motion focused at the origin perturbed by* $O(\mu^{1+2\kappa})$,

$$\frac{dq_-}{d\tau} = \frac{v_-}{2}, \quad \frac{dv_-}{d\tau} = \frac{q_-}{2|q_-|^3} + O(\mu^{1+2\kappa}). \tag{10.4}$$

Proof. Note that (10.1) preserves the symplectic form.

$$dv_3 \wedge dQ_3 + dv_4 \wedge dQ_4 = dv_- \wedge dQ_- + dv_+ \wedge dQ_+,$$

The Hamiltonian becomes

$$\begin{aligned} H = & \frac{|v_-|^2}{4} - \frac{\mu}{2|Q_-|} + \frac{|v_+|^2}{4} - \frac{1}{|Q_+ + Q_-|} - \frac{1}{|Q_+ - Q_-|} \\ & - \frac{1}{|Q_+ + Q_- + (\chi, 0)|} - \frac{1}{|Q_+ - Q_- + (\chi, 0)|} \\ = & \frac{|v_-|^2}{4} - \frac{\mu}{2|Q_-|} + \frac{|v_+|^2}{4} - \frac{2}{|Q_+|} + \frac{|Q_-|^2}{2|Q_+|^3} - \frac{3|Q_+ \cdot Q_-|^2}{2|Q_+|^5} + O(\mu^{3\kappa}) + O(1/\chi), \end{aligned} \tag{10.5}$$

where the $O(\mu^{3\kappa})$ includes the $|Q_-|^3$ and higher order terms. In the following, we drop $O(1/\chi)$ terms since $1/\chi \ll \mu$. So the Hamiltonian equations for the motion of the mass center part are

$$\dot{Q}_+ = \frac{v_+}{2}, \quad \dot{v}_+ = -\frac{2Q_+}{|Q_+|^3} + O(\mu^{2\kappa})$$

proving part (a) of the lemma.

Next, we study the relative motion. From equation (10.5), we get the equations of motion for the center of mass

$$\dot{Q}_- = \frac{v_-}{2}, \quad \dot{v}_- = -\frac{\mu Q_-}{2|Q_-|^3} - \frac{Q_-}{|Q_+|^3} + \frac{3|Q_+ \cdot Q_-|Q_+}{|Q_+|^5} + O(\mu^{2\kappa}),$$

as $\mu \to 0$, where $O(\mu^{2\kappa})$ includes quadratic and higher order terms of $|Q_-|$. After making the rescaling according to (10.2) the equations for the relative motion part become

$$\frac{dq_-}{d\tau} = \frac{v_-}{2}, \quad \frac{dv_-}{d\tau} = \frac{q_-}{2|q_-|^3} + \frac{\mu^2 q_-}{|Q_+|^3} - \frac{3\mu^2|Q_+ \cdot q_-|Q_+}{|Q_+|^5} + O(\mu^{1+2\kappa}). \tag{10.6}$$ □

Lemma 10.1 implies the following C^0 estimate.

Lemma 10.2. (a) *We have the following equations*

$$\begin{cases} v_3^+ = \frac{1}{2}R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3} + \mu^{3\kappa-1}), \\ v_4^+ = -\frac{1}{2}R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3} + \mu^{3\kappa-1}), \\ Q_3^+ + Q_4^+ = Q_3^- + Q_4^- + O(\mu^k), \\ |Q_3^- - Q_4^-| = \mu^\kappa, \quad |Q_3^+ - Q_4^+| = \mu^\kappa, \end{cases} \tag{10.7}$$

where $R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$,

$$\alpha = \pi + 2\arctan\left(\frac{G_{in}}{\mu\mathcal{L}_{in}}\right), \quad \text{and } \frac{1}{4\mathcal{L}_{in}^2} = \frac{v_-^2}{4} - \frac{\mu}{2|Q_-|}, \quad G_{in} = 2v_- \times Q_-. \tag{10.8}$$

(b) *We have $\mathcal{L}_{in} = O(1)$. If α is bounded away from 0 and 2π by an angle independent of μ then $G_{in} = O(\mu)$ and the closest distance between Q_3 and Q_4 is bounded away from zero by $\delta\mu$ and from above by μ/δ for some $\delta > 0$ independent of μ.*

(c) *Also if α is bounded away from π by an angle independent of μ, then the angle formed by Q_- and v_- is $O(\mu^{1-\kappa})$.*

(d) *The time interval during which the orbit stays in the sphere $|Q_-| = \mu^\kappa$ is*

$$\Delta t = \mu\Delta\tau = O(\mu^\kappa).$$

Remark 10.1. *Part (d) is very intuitive. The radius of the sphere $|Q_-| = \mu^\kappa$ is μ^κ. The relative velocity is $O(1)$ and it gets larger when Q_- gets closer to the origin. So the total time for the relative motion to stay inside the sphere is $O(\mu^\kappa)$.*

Proof. In the proof, we omit the subscript *in* standing for the variables *inside* the sphere $|Q_-| = \mu^\kappa$ without leading to confusion.

The idea of the proof is to treat the relative motion as a perturbation of Kepler motion and then approximate the relative velocities by their asymptotic values for the Kepler motion.

Fix a small number δ_1. Below we derive several estimates valid for the first δ_1 units of time the orbit spends in the set $|Q_-| \le \mu^k$. We then show that $\Delta t \ll \delta_1$. It will be convenient to measure time from the orbit enters the set $|Q_-| < \mu^k$.

Using the formula in the Appendix A.1, we decompose the Hamiltonian (10.5) as $H = H_{rel} + \mathfrak{h}(Q_+, v_+)$ where

$$H_{rel} = \frac{\mu^2}{4L^2} + \frac{|Q_-|^2}{2|Q_+|^2} - \frac{|Q_+ \cdot Q_-|^2}{2|Q_+|^5} + O(\mu^{3\kappa}),\ \text{as } \mu \to 0,$$

and $\mathfrak{h}$ depends only on Q_+ and v_+.

Note that H is preserved and $\dot{\mathfrak{h}} = O(1)$ which implies that $\frac{L}{\mu}$ is $O(1)$ and moreover that ratio does not change much for $t \in [0, \delta_1]$. Using the identity $\frac{\mu^2}{4L^2} = \frac{v_-^2}{4} - \frac{\mu}{2|Q_-|}$ we see that initially $\frac{L}{\mu}$ is uniformly bounded from below for the orbits from Lemma 2.2. Thus there is a constant δ_2 such that for $t \in [0, \delta_1]$ we have $\delta_2\mu \le L(t) \le \frac{\mu}{\delta_2}$.

Expressing the Cartesian variables via Delaunay variables (c.f. equation (A.3) in Section A.2) we have up to a rotation by g

(10.9)

$$\begin{aligned}
&q_1 = \frac{1}{\mu} L^2(\cosh u - e), \quad q_2 = \frac{1}{\mu} LG \sinh u,\\
&O(\mu^\kappa) = |Q_-| = \sqrt{|q_1|^2 + |q_2|^2} = \frac{L}{\mu}\sqrt{L^2(\cosh^2 u - 2\cosh u e + e^2) + G^2 \sinh u}\\
&= \frac{L}{\mu}\sqrt{L^2(\cosh^2 u - 2e\cosh u + e^2) + (L^2e^2 - L^2)\sinh u}\\
&= \frac{L^2}{\mu}\sqrt{1 - 2e\cosh u + e^2 + e^2 \sinh u} = \frac{L^2}{\mu}(e\cosh u - 1),
\end{aligned}$$

where $u - e\sinh u = \ell$. This gives $\ell = O(\mu^{\kappa-1})$.

Next

$$\dot{\ell} = -\frac{\partial H}{\partial L} = -\frac{\mu^2}{2L^3} - \frac{\partial H_{rel}}{\partial Q_-}\frac{\partial Q_-}{\partial L} = -\frac{\mu^2}{2L^3} + O(\mu^\kappa)O(\mu^{\kappa-1}) = -\frac{\mu^2}{2L^3} + O(\mu^{2\kappa-1}).$$

Since the leading term here is at least $\frac{\delta_2^3}{2\mu}$ while $\ell = O(\mu^{\kappa-1})$ we obtain part (d) of the lemma. In particular the estimates derived above are valid for the time the orbits spend in $|Q_-| \le \mu^\kappa$. Next, without using any control on G (using the inequality $\left|\frac{\partial e}{\partial G}\right| = \frac{1}{L}\frac{G/L}{e} \le \frac{1}{L}$), we have

$$\dot{G} = \frac{\partial H}{\partial Q_-}\frac{\partial Q_-}{\partial g} = O(|Q_-|^2) = O(\mu^{2\kappa}), \quad \dot{L} = \frac{\partial H}{\partial Q_-}\frac{\partial Q_-}{\partial \ell} = O(\mu^{\kappa+1}), \tag{10.10}$$

$$\dot{g} = \frac{\partial H}{\partial Q_-}\frac{\partial Q_-}{\partial G} = O(\mu^\kappa)O(\mu^{\kappa-1}) = O(\mu^{2\kappa-1}). \tag{10.11}$$

Integrating over time $\Delta t = O(\mu^\kappa)$ we get the oscillation of g and $\arctan\frac{G}{L}$ are $O(\mu^{3\kappa-1})$.

We are now ready to derive the first two equations of (10.7). It is enough to show $v_-^+ = R(\alpha)v_-^- + O(\mu^{(1-2\kappa)/3} + \mu^{3\kappa-1})$ where $\alpha = 2\arctan\frac{G}{L}$ is the angle formed by the two asymptotes of the Kepler hyperbolic motion. We first have $|v_-^+| = |v_-^-| + O(\mu^\kappa)$ using the total energy conservation. It remains to show the expression of α. Let us denote till the end of the proof $\phi = \arctan\frac{G}{L}$, $\gamma = \frac{(1/2)-\kappa}{3}$. Recall (see (A.3)) that for $v_- = (p_1, p_2)$,

$$p_1 = \tilde{p}_1\cos g + \tilde{p}_2 \sin g, \quad p_2 = -\tilde{p}_1 \sin g + \tilde{p}_2\cos g \text{ where} \tag{10.12}$$

$$\tilde{p}_1 = \frac{\mu}{L}\frac{\sinh u}{1-e\cosh u}, \quad \tilde{p}_2 = \frac{\mu G}{L^2}\frac{\cosh u}{1-e\cosh u}.$$

Consider two cases.

(I) $G \le \mu^{\kappa+\gamma}$. In this case on the boundary of the sphere $|Q_-| = \mu^\kappa$ we have $\ell > \delta_3\mu^{-\gamma}$ for some constant δ_3. Thus

$$\frac{p_2}{p_1} = \frac{\frac{\mu G}{L^2}\cosh u\cos g + \frac{\mu}{L}\sinh u \sin g}{-\frac{\mu G}{L^2}\cosh u \sin g + \frac{\mu}{L}\sinh u\cos g} = \frac{\frac{G}{L} \pm \tan g}{\pm 1 - \frac{G}{L}\tan g} + O(e^{-2|u|}) = \tan(g\pm\phi) + O(\mu^{2\gamma}).$$

where the plus sign is taken if $u > 0$ and the minus sign is taken if $u < 0$. Since arctan is globally Lipschitz, this completes the proof in case (I) by choosing $\alpha = 2\phi$.

(II) $G > \mu^{\kappa+\gamma}$. In this case $\frac{G}{L} \gg 1$ and so it suffices to show that $\frac{p_2}{p_1}$ (or $\frac{p_1}{p_2}$) changes little during the time the orbit is inside the sphere. Consider first the case where $|g^-| > \frac{\pi}{4}$ so $\sin g$ is bounded from below. Then

$$\frac{p_2}{p_1} = \cot g + O(\mu^{1-(\kappa+\gamma)})$$

proving the claim of part (a) in that case. The case $|g^-| \le \frac{\pi}{4}$ is similar but we need to consider $\frac{p_1}{p_2}$. This completes the proof in case (II).

Combining equation (10.3) and Lemma 10.1(c) we obtain

$$Q_+^+ = Q_+^- + O(\mu^\kappa). \tag{10.13}$$

We also have $Q_-^+ = Q_-^- + O(\mu^\kappa)$ due to to the definition of the sections $\{|Q_-^\pm| = \mu^\kappa\}$. This proves the last two equation in (10.7). Plugging (10.13) into (10.3) we see that

$$v_+^+ = v_+^- + O(\mu^\kappa).$$

This completes the proof of part (a).

The first claim of part (b) has already been established. The estimate of G follows from the formula for α. The estimate of the closest distance follows from the fact that if α is bounded away from 0 and 2π then the Q_- orbit of $Q_-(t)$ is a small perturbation of Kepler motion and for Kepler motion the closest distance is of order G. We integrate the $\dot{G}$ equation (10.10) over time $O(\mu^\kappa)$ to get the total variation ΔG is at most $\mu^{3\kappa}$, which is much smaller than μ. So G is bounded away from 0 by a quantity of order $O(\mu)$.

Finally part (c) follows since we know $G = \mu^\kappa |v_-| \sin \measuredangle(v_-, Q_-) = O(\mu)$. □

Proof of Lemma 2.2. Letting $\mu = 0$ in the first two equations of (10.7) we obtain the equations of elastic collisions. Namely, both the kinetic energy conservation

$$|v_3^+|^2 + |v_4^+|^2 = |v_3^-|^2 + |v_4^-|^2$$

and momentum conservation

$$v_3^+ + v_4^+ = v_3^- + v_4^-$$

laws hold. On the other hand, the Gerver's map $\mathbf{G}$ in Lemma 2.2 is also defined through elastic collisions. The assumption $\bar{\theta}^+ = \pi + O(\mu)$ implies that α in (10.7) is μ close to its value in Gerver's case. As a result, Lemma 10.2 says actually the same thing as Lemma 2.2 up to a change of variables going from Cartesian coordinates to the set of variables $E_3, \ell_3, G_3, g_3, G_4, g_4$. □

10.2. **Proof of Lemma 2.3.**

Proof. We follow the same argument as in the proof of Lemma 2.2 to get that the orbit of Q_3 is a small (of order $O(\tilde{\theta})$) deformation of Gerver's Q_3 ellipse. So we only need to prove this lemma in Gerver's setting. Since the Q_3 ellipse has semimajor 1 in Gerver's case, the distance from the apogee to the focus is strictly less than 2. Therefore we can find some $\delta > 0$ such that $|Q_3| \leq 2 - 2\delta$ in the Gerver case. The rest of the proof is similar to the proof of Lemma 4.1. □

11. Consequences of C^0 estimates

Here we obtain corollaries C^0 estimates for the local end global maps. Namely, in subsection 11.1 we show that the orbits we construct are collision free. In subsection 11.2 we show that the angular momentum can be prescribed freely during the consecutive iterations of the inductive scheme, that is, we prove Sublemma 3.4.

11.1. **Avoiding collisions.**
Here we exclude the possibility of collisions. The possible collisions may occur for the pair Q_3, Q_4 and the pair Q_1, Q_4. The fact that there is no collision between Q_4 and Q_1 is a consequence of the following result.

Lemma 11.1. *If an orbit satisfies the conditions of Lemma 4.1 and there is a collision between Q_4 and Q_1 then we have $\bar{G}_4 + G_4 = O(\mu)$ where $\bar{G}_4$ denotes the angular momentum of Q_4 after the application of the global map.*

Proof. Since we are concerned with the orbit of Q_4 it is convenient to use L_4 instead of L_3. L_4 satisfies the equation $L_4' = -\dfrac{dt}{d\ell_4}\dfrac{\partial V_L}{\partial \ell_4} = O(1/\chi^2)$, where "$'$" means the $\dfrac{d}{d\ell_4}$ derivative. We write the equations of motion as $\mathbf{Y}' = \mathbf{V}$, where $\mathbf{Y} = (L_4, G_4, g_4)$ and $\mathbf{V}$ is the RHS of the Hamiltonian equations (4.5).

We run the orbit coming to a collision backward so that we can compare it to the orbits exiting collision. We shall use the subscript *in* to refer to the orbit coming to collision with time direction reversed the subscript *out* for the orbit exiting collision. We have

$$(\mathbf{Y}_{in} - \mathbf{Y}_{out})' = O\left(\left\|\frac{\partial \mathbf{V}}{\partial \mathbf{Y}}\right\| \quad |\mathbf{Y}_{in} - \mathbf{Y}_{out}|\right) + O\left(\frac{\mu}{|Q_4 - Q_3|^2}\right)$$

where the last term comes from the $\dfrac{\mu}{|Q_4 - Q_3|}$ term in the potential V_L. We integrate this estimate for ℓ_4 starting from the collision and ending when the outgoing orbit hits the section $\{x_4 = -\chi/2\}$. The initial condition is $\mathbf{Y}_{in} - \mathbf{Y}_{out} = 0$ since L_4, G_4, g_4 assume the same values before and after the Q_4-Q_1 collision. Next, $\left\|\dfrac{\partial \mathbf{V}}{\partial \mathbf{Y}}\right\| = O\left(\dfrac{1}{\chi}\right)$ (this is proven in Lemma 6.1(b) for the case when L_4 is replaced by L_3, and the proof in the present case is similar; the applicability of Lemma 6.1 as well as Lemma 4.1 used below follows from Lemma 4.8). Now the estimates

$$\int_{\ell_4^i}^{\ell_4^f} \frac{\partial \mathbf{V}}{\partial \mathbf{Y}} d\ell_4 = O(1), \quad \int_{\ell_4^i}^{\ell_4^f} O\left(\frac{\mu}{|Q_4 - Q_3|^2}\right) d\ell_4 = O(\mu/\chi)$$

and the Gronwall Lemma imply that

$$\mathbf{Y}_{in}(\ell_4^f) - \mathbf{Y}_{out}(\ell_4^f) = O(\mu/\chi). \tag{11.1}$$

Next we estimate the angular momentum of Q_4 w.r.t. Q_2. We have

$$G_{4R} = G_{4L} + v_4 \times (-\chi, 0) = G_{4L} + v_{4y}\chi, \tag{11.2}$$

where v_{4y} is the y component of the velocity of Q_4 at the time the orbit hits the section $\{x_4 = -\chi/2\}$. Using the equation (A.5) in the Appendix A.2 and Lemma 4.7 we see that for the orbits of interest

$$v_{4y} = \frac{k}{L_4^2}(L_4 \sin g_4 - G_4 \cos g_4) + O\left(\frac{1}{\chi^2}\right).$$

Now (11.1) shows that $v_{4y,in} - v_{4y,out} = O(\mu/\chi)$. Hence (11.2) implies that

$$G_{4R,in} - G_{4R,out} = O(\mu)$$

Finally the proof of Lemma 4.1 shows that the angular momentum of Q_4 w.r.t. Q_2 changes by $O(\mu)$ during the time the orbits moves from the section $\{x_4 = -\chi/2\}$ to the section $\{x_4 = -2\}$. □

Now we exclude the possibility of collisions between Q_3 and Q_4. Note that Q_3 and Q_4 have two potential collision points corresponding to two intersections of the ellipse of Q_3 and the branch of the hyperbola utilized by Q_4. See Fig 1 and 2 in Section 2.3. Now it follows from Lemma 10.2(b) that Q_3 and Q_4 do not collide near the intersection where they have the close encounter. We need also to rule out the collision near the second intersection point. This was done by Gerver in [G2]. Namely he shows that the time for Q_3 and Q_4 to move from one crossing point to the other are different. As a result, if Q_3 and Q_4 come to the correct intersection points nearly simultaneously, they do not collide at the wrong points. To see that the travel times are different recall that by second Kepler's law the area swiped by the moving body in unit time is a constant for the two-body problem. In terms of Delaunay coordinates, this fact is given by the equation $\dot{\ell} = \pm\frac{1}{L^3}$ where $-$ is for hyperbolic motion and $+$ for elliptic. In our case, we have $L_3 \approx L_4$ when $\mu \ll 1, \chi \gg 1$. Therefore in order to collide Q_3 and Q_4 must swipe nearly the same area within the unit time. We see from Fig 1 and Fig 2, the area swiped by Q_4 is a proper subset of that by Q_3 between the two crossing points. Therefore the travel time for Q_4 is shorter.

11.2. **Choosing angular momentum.**

Proof of the Sublemma 3.4. The idea is to apply the strong expansion of the Poincaré map in a neighborhood of the collisional orbit studied in Lemma 11.1. Notice Delaunay coordinates regularizes double collisions, so that our estimate of $d\mathbb{G}$ holds also for collisional orbits.

Step 1. We first show that there is a collisional orbit as ℓ_3 varies. The proof of Lemma 11.1 shows that Q_4 nearly returns back to its initial position. Sublemma 4.9 shows that if after the application of the local map we have $\theta_4^+(0) = \pi + \tilde{\theta}$ then the orbit hits the line $x_4 = -\chi$ so that its y_4 coordinate is a large positive number and if $\theta_4^+(0) = \pi - \tilde{\theta}$ then the orbit hits the line $x_4 = -\chi$ so that its y_4 coordinate is a large negative number. Therefore due to the Intermediate Value Theorem it suffices to show that our surface $S_j,\ j = 1, 2$, contains points $\boldsymbol{x}_1, \boldsymbol{x}_2$ such that $\theta_4^+(\boldsymbol{x}_1) = \pi - \tilde{\theta}$, $\theta_4^+(\boldsymbol{x}_2) = \pi + \tilde{\theta}$. We have the expression $\theta_4^+ = g_4^+ - \arctan\frac{G_4^+}{L_4^+}$. By direct calculation we find $d\theta^+ = L_4^+\hat{\bar{\mathbf{l}}}$ (see also item (2) of Remark 3.1). Since $TS_j \subset \mathcal{K}_j$ and the cone $\mathcal{K}_j$ is centered at the plane $span\{\bar{\mathbf{u}}_{3-j}, \bar{\bar{\mathbf{u}}}_{3-j}\}$. At Gerver's collision point $\theta^+ = 0$. It is enough to show that as we vary ℓ_3 close to Gerver's collision point such that θ^+ changes significantly. Note that $\bar{\bar{\mathbf{u}}}_{3-j} \to \tilde{w} = \frac{\partial}{\partial \ell_3}$. We get using Lemma 3.8

$$d\theta^+ \cdot (d\mathbb{L}\bar{\bar{\mathbf{u}}}_{3-j}) = L_4^+\hat{\bar{\mathbf{l}}}_j \cdot \left(\frac{1}{\mu}(\hat{\mathbf{u}}_j(\hat{\mathbf{l}}_j\tilde{w}) + o(1)) + O(1)\right) = c_j(\boldsymbol{x})/\mu,\ c_j(\boldsymbol{x}_j) \neq 0.$$

We choose $\tilde{\theta} \ll 1$ but independent of μ such that the assumption of Lemma 3.1 and Sublemma 4.9 is satisfied.

Step 2. We show that there exists ℓ_3 such that $\bar{e}_4(\mathcal{P}(\ell_3, \tilde{e}_4))$ is close to e_4^{**}. We fix $\tilde{e}_4$ then $\mathcal{P}$ becomes a function of one variable ℓ_3. As we vary ℓ_3, the same calculation as in Step 1 gives $\hat{\bar{\mathbf{l}}}_j \cdot (d\mathbb{L}\bar{\mathbf{u}}_{3-j}) = \bar{c}_j(\boldsymbol{x})/\mu,\ \bar{c}_j(\boldsymbol{x}_j) \neq 0$ and that $\bar{\mathbf{u}}_j$ contains nonzero $\partial/\partial e_4$ component. Therefore the projection of $\mathcal{P} = \mathbb{G} \circ \mathbb{L}$ to the e_4 component, i.e. $\bar{e}_4(\ell_3, \tilde{e}_4)$ as a function of ℓ_3 is strongly expanding with derivative bounded from below by $\dfrac{\bar{c}\chi^2}{\mu}$ provided that the assumptions of Lemma 4.1 are satisfied (for the orbits of interest this will always be the case according to Lemma 4.8). Since the map $\bar{e}_4(\ell_3, \tilde{e}_4)$ is not injective, we study $\bar{G}_4(\ell_3, \tilde{e}_4)$ instead of $\bar{e}_4(\ell_3, \tilde{e}_4)$ using the relation $e = \sqrt{1 + 2(G/L)^2}$. We have the same strong expansion for $\bar{G}_4(\ell_3, \tilde{e}_4)$ since our estimates of the $d\mathbb{L}, d\mathbb{G}$ are done using G_4 instead of e_4. Thus it follows from the strong expansion of the map $\bar{G}_4(\ell_3, \tilde{e}_4)$ that a R-neighborhood of G_4^{**} (corresponding to e_4^{**}) is covered if e_4 varies in a $\dfrac{R\mu}{\bar{c}\chi^2}$-neighborhood. Taking R large we can ensure that $\bar{G}_4$ changes from a large negative number to a large positive number. Then we use the intermediate value theorem to find e_4 such that $|\bar{G}_4 - G_4^{**}| < \sqrt{\delta}$, hence $|\bar{e}_4 - e_4^{**}| < \sqrt{\delta}$.

Step 3. We show that for the orbit just constructed $\mathcal{P}(\ell_3, \tilde{e}_4) \in U_2(\delta)$. Since e_4 changes substantially Q_4 must pass close to Q_1 and hence $\mathbb{L}(\tilde{e}_4, \tilde{l}_3)$ must have θ_4^+ small. Therefore by Lemma 2.2 $\mathbb{L}(\tilde{e}_4, \tilde{l}_3)$ has (E_3, e_3, g_3) close to $\mathbf{G}_{\tilde{e}_4,2,4}(E_3(\tilde{e}_4, \tilde{l}_3), e_3(\tilde{e}_4, \tilde{l}_3), g_3(\tilde{e}_4, \tilde{l}_3))$. It follows that

$$|E_3 - E_3^{**}| < K\delta, \quad |e_3 - e_3^{**}| < K\delta, \quad |g_3 - g_3^{**}| < K\delta.$$

Next Lemma 2.4 shows that after the application of $\mathbb{G}$, (E_3, e_3, g_3) change little and θ_4^- becomes $O(\mu)$. □

12. Derivative of the local map

12.1. **Justifying the asymptotics.** Here we give the proof of Lemma 3.1. Our goal is to show that the main contribution to the derivative comes from differentiating the main term in Lemma 10.2.

Proof of Lemma 3.1. Since the transformation from Delaunay to Cartesian variables is symplectic and the norms of the transformation matrices are independent of μ, it is sufficient to prove the lemma in terms of Cartesian coordinates. To go to the coordinates system used in Lemma 3.1, we only need to multiply the Cartesian derivative matrix by $O(1)$ matrices, namely, by $\dfrac{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial(Q_3, v_3, Q_4, v_4)^+}$ on the left and by $\dfrac{\partial(Q_3, v_3, Q_4, v_4)^-}{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^-}$ on the right. This does not change the form of the $d\mathbb{L}$ stated in Lemma 3.1.

As before we use the formula (5.1). We need to consider the integration of the variational equations and also the boundary contribution. The proof is organized as follows. The main part of the proof is the study of the relative motion part, while controlling the motion of the mass center is easier.

For the relative motion part, we use the Delaunay variables (L, ℓ, G, g). Using ℓ as the time variable we get from (10.5) that the equations for relative motion take the following form (recall that the scale for ℓ is $O(\mu^{\kappa-1})$):

$$\begin{cases} \dfrac{\partial L}{\partial \ell} = -2\mu^{-2}L^3\dfrac{\partial H}{\partial \ell}\left(1 - 2\mu^{-2}L^3\dfrac{\partial H}{\partial L} + \dots\right) = O(\mu^{2+\kappa}), \\ \dfrac{\partial G}{\partial \ell} = -2\mu^{-2}L^3\dfrac{\partial H}{\partial g}\left(1 - 2\mu^{-2}L^3\dfrac{\partial H}{\partial L} + \dots\right) = O(\mu^{1+2\kappa}), \\ \dfrac{\partial g}{\partial \ell} = 2\mu^{-2}L^3\dfrac{\partial H}{\partial G}\left(1 - 2\mu^{-2}L^3\dfrac{\partial H}{\partial L} + \dots\right) = O(\mu^{2\kappa}), \end{cases} \tag{12.1}$$

where $\dots$ denote the lower order terms. The estimates of the last two equations follow from (10.10) and (10.11) while the estimate of the first equation is similar to the last two.

Then we analyze the variational equations.

$$\left[\begin{array}{c} \dfrac{d\delta L}{d\ell} \\ \dfrac{d\delta G}{d\ell} \\ \dfrac{d\delta g}{d\ell} \end{array}\right] = O\begin{pmatrix} \mu^{1+\kappa} & \mu^{1+\kappa} & \mu^{1+2\kappa} \\ \mu^{1+\kappa} & \mu^{2\kappa} & \mu^{1+2\kappa} \\ \mu^{2\kappa-1} & \mu^{2\kappa-1} & \mu^{2\kappa} \end{pmatrix}\left[\begin{array}{c} \delta L \\ \delta G \\ \delta g \end{array}\right] + O\begin{pmatrix} \mu^{2+\kappa} & 0 \\ \mu^{1+2\kappa} & 0 \\ \mu^{2\kappa} & 0 \end{pmatrix}\left[\begin{array}{c} \delta Q_+ \\ \delta v_+ \end{array}\right]. \tag{12.2}$$

In the following, we first set $\delta Q_+ = 0$ and work with the fundamental solution of the homogeneous equation. Then we will prove that δQ_+ is negligible.

Introducing $\delta\mathscr{G} = \dfrac{\delta g}{\mu^{3\kappa-2}}$ we need the asymptotics of the fundamental solution of

$$\left[\begin{array}{c} \dfrac{d\delta L}{d\ell} \\ \dfrac{d\delta G}{d\ell} \\ \dfrac{d\delta \mathscr{G}}{d\ell} \end{array}\right] = O\begin{pmatrix} \mu^{1+\kappa} & \mu^{1+\kappa} & \mu^{5\kappa-1} \\ \mu^{1+\kappa} & \mu^{2\kappa} & \mu^{5\kappa-1} \\ \mu^{1-\kappa} & \mu^{1-\kappa} & \mu^{2\kappa} \end{pmatrix}\left[\begin{array}{c} \delta L \\ \delta G \\ \delta \mathscr{G} \end{array}\right]. \tag{12.3}$$

Integrating this equation over time $\mu^{\kappa-1}$ we see that the fundamental solution is $O(1)$. Now arguing the same way as in Section 6.2 we see that the fundamental solution takes form

$$\mathrm{Id} + O\begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa} & \mu^{6\kappa-2} \\ \mu^{2\kappa} & \mu^{3\kappa-1} & \mu^{6\kappa-2} \\ 1 & 1 & \mu^{3\kappa-1} \end{pmatrix}. \tag{12.4}$$

In the following it is convenient to use variables $L = \mu\mathcal{L}$, G and g. In these variables fundamental solution of the variational equation is

$$\mathrm{Id} + O\begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} \\ \mu^{3\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{pmatrix}. \tag{12.5}$$

Next, we compute the boundary contribution. In terms of the Delaunay variables inside the sphere $|Q_-| = \mu^\kappa$, we have

$$\frac{\partial \ell}{\partial(\mathcal{L}, G, g)} = -\left(\frac{\partial |Q_-|}{\partial \ell}\right)^{-1} \frac{\partial |Q_-|}{\partial(\mathcal{L}, G, g)} = (O(\mu^{\kappa-1}), O(\mu^{\kappa-2}), 0). \tag{12.6}$$

Indeed, due to (10.9) we have $\dfrac{\partial |Q_-|}{\partial g} = 0$, $\dfrac{\partial |Q_-|}{\partial \ell} = O(\mu)$, $\dfrac{\partial |Q_-|}{\partial \mathcal{L}} = O(\mu^\kappa)$ and $\dfrac{\partial |Q_-|}{\partial G} = O(\mu^{\kappa-1})$. Combining this with (12.1) we get

$$\left(\frac{\partial \mathcal{L}}{\partial \ell}, \frac{\partial G}{\partial \ell}, \frac{\partial g}{\partial \ell}\right) \otimes \frac{\partial \ell}{\partial(\mathcal{L}, G, g)} = O\begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{pmatrix}. \tag{12.7}$$

Using (5.1) we obtain the derivative matrix

$$\begin{aligned} \frac{\partial(\mathcal{L}, G, g)^+}{\partial(\mathcal{L}, G, g)^-} &= \left(\mathrm{Id} + O\begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{pmatrix}\right)^{-1} \times \\ &\left(\mathrm{Id} + O\begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} \\ \mu^{3\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{pmatrix}\right)\left(\mathrm{Id} - O\begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{pmatrix}\right) \\ &= \mathrm{Id} + O\begin{pmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} \\ \mu^{3\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1} \end{pmatrix} := \mathrm{Id} + P. \end{aligned} \tag{12.8}$$

We are now ready to compute the relative motion part of the derivative of the Poincaré map. For the space variables, we are only interested in the angle $\Theta := \arctan\left(\dfrac{q_2}{q_1}\right)$ since the length $|(q_1, q_2)|$ is fixed when restricted on the sphere.

We split the derivative matrix as follows:

$$\frac{\partial(\Theta_-, v_-)^+}{\partial(\Theta_-, v_-)^-} = \frac{\partial(\Theta_-, v_-)^+}{\partial(\mathcal{L}, G, g)^+} \frac{\partial(\mathcal{L}, G, g)^+}{\partial(\mathcal{L}, G, g)^-} \frac{\partial(\mathcal{L}, G, g)^-}{\partial(\Theta_-, v_-)^-} = \tag{12.9}$$

$$\frac{\partial(\Theta_-,v_-)^+}{\partial(\mathcal{L},G,g)^+}\frac{\partial(\mathcal{L},G,g)^-}{\partial(\Theta_-,v_-)^-}+\frac{\partial(\Theta_-,v_-)^+}{\partial(\mathcal{L},G,g)^+}P\frac{\partial(\mathcal{L},G,g)^-}{\partial(\Theta_-,v_-)^-}=I+II.$$

Using equations (10.9) and (10.12) we obtain

(12.10)
$$\frac{\partial(\Theta_-,v_-)^+}{\partial(\mathcal{L},G,g)^+}=O\begin{pmatrix}1&\mu^{-1}&1\\1&\mu^{-1}&1\\1&\mu^{-1}&1\end{pmatrix}.$$

Next, we consider the first term in (12.9).

(12.11)
$$I=\frac{\partial(\Theta_-,v_-)^+}{\partial\mathcal{L}^+}\otimes\frac{\partial\mathcal{L}^-}{\partial(\Theta_-,v_-)^-}+\frac{\partial(\Theta_-,v_-)^+}{\partial G^+}\otimes\frac{\partial G^-}{\partial(\Theta_-,v_-)^-}+\frac{\partial(\Theta_-,v_-)^+}{\partial g^+}\otimes\frac{\partial g^-}{\partial(\Theta_-,v_-)^-}.$$

Using the expressions

$$\frac{1}{4\mathcal{L}^2}=\frac{v_-^2}{4}-\frac{\mu}{2|Q_-|},\quad G=v_-\times Q_-=|v_-|\cdot|Q_-|\sin\measuredangle(v_-,Q_-)$$

we see that

(12.12)
$$\frac{\partial\mathcal{L}^-}{\partial(\Theta_-,v_-)^-}=O(1),\quad \frac{\partial G^-}{\partial(\Theta_-,v_-)^-}=(O(\mu^\kappa),O(\mu^\kappa)).$$

Next, we have $\frac{\partial(\Theta_-,v_-)^+}{\partial g^+}=(O(1),O(1))$ from equations (10.9) and (10.12). To obtain the derivatives of g we use the fact that

$$\frac{p_2}{p_1}=\frac{\sin g\sinh u\pm\frac{G}{\mu\mathcal{L}}\cos g\cosh u}{\cos g\sinh u\mp\frac{G}{\mu\mathcal{L}}\sin g\cosh u}=\frac{\tan g\pm\frac{G}{\mu\mathcal{L}}}{1\mp\frac{G}{\mu\mathcal{L}}\tan g}+e^{-2|u|}E(G/\mu\mathcal{L},g,u),$$

where E is a smooth function satisfying $\frac{\partial E}{\partial g}=O(1)$ as $\ell\to\infty$. Therefore we get

$$g=\arctan\left(\frac{p_2}{p_1}-e^{-2|u|}E(G/\mu\mathcal{L},g)\right)\mp\arctan\frac{G}{\mu\mathcal{L}}\text{ as }\ell\to\infty.$$

We choose the $+$ when considering the incoming orbit parameters. Thus

$$\frac{\partial g}{\partial(\Theta_-,v_-)}\left(1+O(e^{-2|u|})\right)=\frac{\partial\arctan\frac{p_2}{p_1}}{\partial(\Theta_-,v_-)}+\frac{\partial\arctan\frac{G}{\mu\mathcal{L}}}{\partial\mathcal{L}}\frac{\partial\mathcal{L}}{\partial(\Theta_-,v_-)}$$
$$+\left(\frac{\partial\arctan\frac{G}{\mu\mathcal{L}}}{\partial G}+O(e^{-2|u|}/\mu)\right)\frac{\partial G}{\partial(\Theta_-,v_-)}+O(e^{-2|u|}).$$

Hence

(12.13)
$$\frac{\partial g}{\partial(\Theta_-,v_-)}=O\left(\frac{1}{\mu}\right)\frac{\partial G}{\partial(\Theta_-,v_-)}+O(1),$$

where the $1/\mu$ comes from $\dfrac{\partial \arctan \dfrac{G}{\mu\mathcal{L}}}{\partial G}$ and all other terms are $O(1)$ or even smaller. Therefore

(12.14)

$$I = \frac{1}{\mu}\left(\mu\frac{\partial(\Theta_-,v_-)^+}{\partial G^+} + \mu\frac{\partial \arctan \frac{G^-}{\mu\mathcal{L}^-}}{\partial G^-}\frac{\partial(\Theta_-,v_-)^+}{\partial g^+} + O(e^{-2|u|})\right)\otimes\frac{\partial G^-}{\partial(\Theta_-,v_-)^-}+$$

$$\left(\frac{\partial(\Theta_-,v_-)^+}{\partial\mathcal{L}^+}\otimes\frac{\partial\mathcal{L}^-}{\partial(\Theta_-,v_-)^-} + \frac{\partial(\Theta_-,v_-)^+}{\partial g^+}\otimes\left(\frac{\partial\arctan\frac{p_2^-}{p_1^-}}{\partial(\Theta_-,v_-)} + \frac{\partial\arctan\frac{G^-}{\mu\mathcal{L}^-}}{\partial\mathcal{L}^-}\frac{\partial\mathcal{L}^-}{\partial(\Theta_-,v_-)}\right)\right)$$

$+O(e^{-2|u|})$. Since the expression in parenthesis of the first term is $O(1)$, I has the rate of growth required in Lemma 3.1.

Now we study the second term in (12.9)

$$(12.15)\quad \begin{aligned} II &= O\begin{pmatrix} 1 & \mu^{-1} & 1\\ 1 & \mu^{-1} & 1\\ 1 & \mu^{-1} & 1\end{pmatrix}\cdot O\begin{pmatrix}\mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa}\\ \mu^{3\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa}\\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1}\end{pmatrix}\frac{\partial(\mathcal{L},G,g)^-}{\partial(\theta_-,v_-)^-}\\ &= O\begin{pmatrix}\mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1}\\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1}\\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & \mu^{3\kappa-1}\end{pmatrix}\frac{\partial(\mathcal{L},G,g)^-}{\partial(\Theta_-,v_-)^-}\\ &= O\begin{pmatrix}\mu^{3\kappa-1}\\ \mu^{3\kappa-1}\\ \mu^{3\kappa-1}\end{pmatrix}\otimes\frac{\partial\mathcal{L}^-}{\partial(\Theta_-,v_-)^-}\\ &\quad + O\left(\begin{bmatrix}\mu^{3\kappa-2}\\ \mu^{3\kappa-2}\\ \mu^{3\kappa-2}\end{bmatrix}\right)\otimes\frac{\partial G^-}{\partial(\Theta_-,v_-)^-} + O\begin{pmatrix}\mu^{3\kappa-1}\\ \mu^{3\kappa-1}\\ \mu^{3\kappa-1}\end{pmatrix}\otimes\frac{\partial g^-}{\partial(\Theta_-,v_-)^-}\end{aligned}$$

where we use that $\mu^{2\kappa} < \mu^{3\kappa-1}$ and $\mu^{2\kappa-1} < \mu^{3\kappa-2}$ since $\kappa < 1/2$. The first summand in (12.15) is $O(\mu^{3\kappa-1})$.Therefore (12.13) implies that

$$(12.16)\qquad II = \frac{1}{\mu}O\begin{pmatrix}\mu^{3\kappa-1}\\ \mu^{3\kappa-1}\\ \mu^{3\kappa-1}\end{pmatrix}\otimes\frac{\partial G^-}{\partial(\Theta_-,v_-)^-} + O(\mu^{3\kappa-1}).$$

Now we combine (12.14) and (12.16) to get

(12.17)
$$\frac{\partial(\Theta_-,v_-)^+}{\partial(\Theta_-,v_-)^-}=\frac{1}{\mu}\left(\mu\frac{\partial(\Theta_-,v_-)^+}{\partial G^+}+\mu\frac{\partial\arctan\frac{G^-}{\mu\mathcal{L}^-}}{\partial G^-}\frac{\partial(\Theta_-,v_-)^+}{\partial g^+}+O(\mu^{3\kappa-1})\right)$$
$$\otimes\frac{\partial G^-}{\partial(\Theta_-,v_-)^-}+\left(\frac{\partial(\Theta_-,v_-)^+}{\partial\mathcal{L}^+}\otimes\frac{\partial\mathcal{L}^-}{\partial(\Theta_-,v_-)^-}\right.$$
$$\left.+\frac{\partial(\Theta_-,v_-)^+}{\partial g^+}\otimes\left(\frac{\partial\arctan\frac{p_2^-}{p_1^-}}{\partial(\Theta_-,v_-)^-}+\frac{\partial\arctan\dfrac{G^-}{\mu\mathcal{L}^-}}{\partial\mathcal{L}^-}\frac{\partial\mathcal{L}^-}{\partial(\Theta_-,v_-)^-}\right)+O(\mu^{3\kappa-1})\right).$$

(12.17) has the structure stated in the lemma. In (12.17), we use the variable Θ_- for the relative position Q_- and we have $\dfrac{\partial G^-}{\partial(\Theta_-,v_-)^-}=O(\mu^\kappa)$. To get back to Q_-, i.e. to obtain $\dfrac{\partial(Q_-,v_-)^+}{\partial(Q_-,v_-)^-}$, we use $Q_-=\mu^\kappa(\cos\Theta_-,\sin\Theta_-)$. So we have the estimate $\dfrac{\partial Q_-^+}{\partial(\mathcal{L}_-,G_-,g_-)^+}=O(\mu^\kappa)\dfrac{\partial\Theta_-^+}{\partial(\mathcal{L}_-,G_-,g_-)^+}=O(\mu^{\kappa-1})$. To get $\dfrac{\partial-}{\partial Q_-^-}$, we use the transformation from polar coordinates to Cartesian, $\dfrac{\partial-}{\partial Q_-^-}=\dfrac{\partial-}{\partial(r_-,\Theta_-)^-}\dfrac{\partial(r_-,\Theta_-)^-}{\partial Q_-^-}$, where $r_-=|Q_-^-|=\mu^\kappa$. Therefore we have

$$\frac{\partial r_-^-}{\partial Q_-^-}=0,\qquad\frac{\partial-}{\partial Q_-^-}=\frac{1}{\mu^\kappa}\frac{\partial-}{\partial\Theta_-^-}(-\sin\Theta_-^-,\cos\Theta_-^-).$$

So we have the estimate $\dfrac{\partial G^-}{\partial Q_-^-}=O(1)$, and $\dfrac{\partial\mathcal{L}^-}{\partial Q_-^-}=\dfrac{\partial\mathcal{L}^-}{\partial\Theta_-^-}=0$ since in the expression $\dfrac{1}{4\mathcal{L}^2}=\dfrac{v_-^2}{4}-\dfrac{\mu}{2|Q_-|}$, the angle Θ_- plays no role. Finally, we have $\dfrac{\partial\arctan\frac{p_2^-}{p_1^-}}{\partial Q_-^-}=0$. So we get

$$\frac{\partial(Q_-,v_-)^+}{\partial(Q_-,v_-)^-}=\frac{1}{\mu}(O(\mu^\kappa)_{1\times2},O(1)_{1\times2})\otimes(O(1)_{1\times2},O(\mu^\kappa)_{1\times2})+O(1)_{4\times4}+O(\mu^{3\kappa-1}).$$

It remains to show that other entries of the derivative matrix are $O(1)$.

Consider the following decomposition

$$
\begin{aligned}
&\frac{\partial(Q_-,v_-,Q_+,v_+)^+}{\partial(Q_-,v_-,Q_+,v_+)^-}=\frac{\partial(Q_-,v_-,Q_+,v_+)^+}{\partial(\mathcal{L},G,g,Q_+,v_+)^+}\frac{\partial(\mathcal{L},G,g,Q_+,v_+)^+}{\partial(\mathcal{L},G,g,Q_+,v_+)(\ell^f)}\\
&\frac{\partial(\mathcal{L},G,g,Q_+,v_+)(\ell^f)}{\partial(\mathcal{L},G,g,Q_+,v_+)(\ell^i)}\frac{\partial(\mathcal{L},G,g,Q_+,v_+)(\ell^i)}{\partial(\mathcal{L},G,g,Q_+,v_+)^-}\frac{\partial(\mathcal{L},G,g,Q_+,v_+)^-}{\partial(Q_-,v_-,Q_+,v_+)^-}\\
&:=\begin{bmatrix} M & 0\\ 0 & \mathrm{Id}\end{bmatrix}\begin{bmatrix} A & 0\\ B & \mathrm{Id}\end{bmatrix}\begin{bmatrix} C & D\\ E & F\end{bmatrix}\begin{bmatrix} A' & 0\\ B' & \mathrm{Id}\end{bmatrix}\begin{bmatrix} N & 0\\ 0 & \mathrm{Id}\end{bmatrix}\\
&=\begin{bmatrix} MACA'N+MADB'N & MAD\\ (BC+E)A'N+(BD+F)B'N & BD+F\end{bmatrix}
\end{aligned}\tag{12.18}
$$

We have already computed M, A, C, A' and N (see (12.10), (12.7), (12.12), (12.13)), where C is (12.5) and $ACA'=\mathrm{Id}+P$ is given by (12.8). We still need to compute B,B',D,E,F.

From the Hamiltonian (10.5), we have $\dot{\ell}=-\dfrac{1}{2\mu\mathcal{L}^3}+O(\mu^{2\kappa})$. We need to supplement (12.1) and (12.2) by the following equations.

$$
\begin{aligned}
&\frac{dQ_+}{d\ell}=-\frac{v_+}{2}(2\mu\mathcal{L}^3)(1+O(\mu^{2\kappa+1}))=O(\mu)\\
&\frac{dv_+}{d\ell}=\left(\frac{2Q_+}{|Q_+|^3}+O(\mu^{2\kappa})\right)(2\mu\mathcal{L}^3)(1+O(\mu^{2\kappa+1}))=O(\mu).
\end{aligned}\tag{12.19}
$$

$$
\begin{bmatrix} \dfrac{d\delta Q_+}{d\ell}\\ \dfrac{d\delta v_+}{d\ell}\end{bmatrix}=\begin{bmatrix} \mu^{2\kappa+2} & \mu\\ \mu & 0\end{bmatrix}\begin{bmatrix} \delta Q_+\\ \delta v_+\end{bmatrix}+\begin{bmatrix} \mu & \mu^{2\kappa+1} & \mu^{2\kappa+2}\\ \mu & \mu^{2\kappa+1} & \mu^{2\kappa+2}\end{bmatrix}\begin{bmatrix} \delta\mathcal{L}\\ \delta G\\ \delta g\end{bmatrix}.\tag{12.20}
$$

It follows from (12.6) and (12.19) that

$$
B,B'=O\begin{pmatrix}\mu\\ \mu\end{pmatrix}\otimes O\left(\mu^{\kappa-1},\mu^{2\kappa-2},0\right).\tag{12.21}
$$

Next, we obtain

$$
D=O\begin{pmatrix}\mu^{2\kappa} & \mu^{2\kappa}\\ \mu^{3\kappa} & \mu^{3\kappa}\\ \mu^{3\kappa-1} & \mu^{3\kappa-1}\end{pmatrix},\quad E=O\begin{pmatrix}\mu^{\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa}\\ \mu^{\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa}\end{pmatrix},\tag{12.22}
$$

$$
F=\mathrm{Id}+O\begin{pmatrix}\mu^{3\kappa} & \mu^{\kappa}\\ \mu^{\kappa} & \mu^{3\kappa}\end{pmatrix}.\tag{12.23}
$$

It is a straightforward computation that CA' dominates DB', so ADB' is provides a small correction to the P in $ACA'=\mathrm{Id}+P$ in (12.8). Therefore

$$
MACA'N+MADB'N
$$

in (12.18) has the same structure as $MACA'N$ obtained in (12.14) and (12.15). Next (12.21), (12.22), (12.23) give

$$BD + F = \mathrm{Id} + O\begin{pmatrix} \mu^{5\kappa-1} & \mu^{\kappa} \\ \mu^{\kappa} & \mu^{5\kappa-1} \end{pmatrix}.$$

Accordingly

$$(BC+E)A'N + (BD+F)B'N = \frac{1}{\mu}[O(\mu^{2\mu})]_{1\times 4} \otimes \frac{\partial G^-}{\partial(\theta, v)_-^-} + O(\mu^\kappa). \tag{12.24}$$

Finally, we have $MAD = [O(\mu^{3\kappa-1})]_{3\times 2}$.
These estimates of the matrix (12.18) are enough to conclude the Lemma. To summarize, we get the resulting derivative estimate as
(12.25)

$$(12.18) = \frac{1}{\mu}O(\mu^{\kappa}_{1\times 2}, 1_{1\times 2}, \mu^{2\kappa}_{1\times 4}) \otimes O\left(1_{1\times 2}, \mu^{\kappa}_{1\times 2}, 0_{1\times 4}\right) + O\begin{pmatrix} (O(1))_{4\times 4} & \mu^{3\kappa-1} \\ \mu^{\kappa} & \mathrm{Id}_4 + \mu^{\kappa} \end{pmatrix}.$$

□

The above proof actually gives us more information. Below we use the Delaunay variables $(L_3, \ell_3, G_3, g_3, G_4, g_4)^{\pm}$ as the orbit parameters *outside* the sphere $|Q_-| = \mu^\kappa$ and add a subscript *in* to the Delaunay variables *inside* the sphere. We relate C^0 estimates of Lemma 10.2 to the C^1 estimates obtained above. Namely consider the following equation which is obtained by discarding the $O(\mu^{3\kappa-1})$ and $O(\mu^\kappa)$ errors in (10.7)

$$Q_-^+ = 0,\ v_-^+ = R(\alpha)v_-^-, \quad Q_+^+ = Q_+^-,\ v_+^+ = v_+^-, \tag{12.26}$$

where α is given in (10.8). We have the following corollary saying that $d\mathbb{L}$ can be obtained by taking derivative directly in (12.26).

Corollary 12.1. *The derivative of the local map has the following form*

$$d\mathbb{L} = \frac{1}{\mu}(\hat{\mathbf{u}} + O(\mu^\kappa)) \otimes \mathbf{l} + \hat{B} + O(\mu^{3\kappa-1}), \tag{12.27}$$

where $\hat{\mathbf{u}}, \mathbf{l}$ *and* $\hat{B}$ *are computed from* (12.26). *In particular,*

(12.28)

$$\hat{\mathbf{u}} = \frac{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial(Q_3, v_3, Q_4, v_4)^+}\frac{\partial(Q_3, v_3, Q_4, v_4)^+}{\partial(Q_-, v_-, Q_+, v_+)^+}\frac{\partial(Q_-, v_-, Q_+, v_+)^+}{\partial\alpha}\left(\mu\frac{\partial\alpha}{\partial G_{in}}\right),$$

$$\mathbf{l} = \frac{\partial G_{in}}{\partial(Q_-, v_-, Q_+, v_+)^-}\frac{\partial(Q_-, v_-, Q_+, v_+)^-}{\partial(Q_3, v_3, Q_4, v_4)^-}\frac{\partial(Q_3, v_3, Q_4, v_4)^-}{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^-}.$$

Proof. In (12.28), the derivatives $\dfrac{\partial(L_3,\ell_3,G_3,g_3,G_4,g_4)^+}{\partial(Q_3,v_3,Q_4,v_4)^+}\dfrac{\partial(Q_3,v_3,Q_4,v_4)^+}{\partial(Q_-,v_-,Q_+,v_+)^+}$ in $\hat{\mathbf{u}}$ and $\dfrac{\partial(Q_-,v_-,Q_+,v_+)^-}{\partial(Q_3,v_3,Q_4,v_4)^-}\dfrac{\partial(Q_3,v_3,Q_4,v_4)^-}{\partial(L_3,\ell_3,G_3,g_3,G_4,g_4)^-}$ in $\mathbf{l}$ are obvious. We focus on the remaining part.

We have

$$\frac{\partial(Q_+,v_+)^+}{\partial(Q_+,v_+)^-}=\mathrm{Id}_4,\ \frac{\partial(Q_+,v_+)^+}{\partial(Q_-,v_-)^-}=\frac{\partial(Q_-,v_-)^+}{\partial(Q_+,v_+)^-}=0,\ \frac{\partial(Q_+,v_+)^+}{\partial\alpha}=\frac{\partial G_{in}}{\partial(Q_+,v_+)^-}=0$$

from (12.26) and (10.8) for G_{in}, which agrees with the corresponding blocks in (12.25) up to an $o(1)$ error as $\mu\to 0$. It remains to compare $\dfrac{\partial(Q_-,v_-)^+}{\partial(Q_-,v_-)^-}$.

It is easy to see from (12.17) that the expression for $\mathbf{l}$ in (12.28) is true.

We take derivative directly in (12.26) to get $\dfrac{\partial(Q_-,v_-)^+}{\partial\alpha}=\left(0,\dfrac{\partial v_-^+}{\partial\alpha}\right)$. To get the expression of $\hat{\mathbf{u}}$ in (12.28), it is enough to show the following compared with (12.17)

$$\frac{\partial v_-^+}{\partial\alpha}\left(\frac{\partial\alpha}{\partial G_{in}}\right)=\left(\frac{\partial v_-^+}{\partial G^+}+\frac{\partial\arctan\frac{G^-}{\mu\mathcal{L}^-}}{\partial G^-}\frac{\partial v_-^+}{\partial g^+}\right),\quad G_{in}=G_-,\tag{12.29}$$

Actually we have using (10.9) and geometric consideration

$$v_-^+=R(\alpha)\,v_-^-+O(e^{-2|u|})=R(\beta)\,(|v_-^-|,0)+O(e^{-2|u|}),\quad e^{-|u|}\simeq\mu^\kappa,\text{ where}$$

$$\alpha=2\arctan\frac{G_{in}}{\mu\mathcal{L}},\ \beta=g+\arctan\frac{G_{in}}{\mu\mathcal{L}},\ g=\arctan\frac{G_{in}}{\mu\mathcal{L}}+\arctan\frac{p_2^-}{p_1^-}+O(\mu^{2\kappa}),v_-=(p_1,p_2).$$

We take the G_{in} derivative directly and neglect $e^{-2|u|}$ term in the v_-^+ expression above to get (12.29). The $e^{-2|u|}$ term is negligible as we did in the proof of Lemma 3.1. In (12.26), v_-^+ also depends on v_- explicitly. When we take partial derivative with respect to the explicit dependence, we get a $O(1)$ matrix that goes into $\hat{B}$. We again compare with (12.17) to show the equivalence of $\hat{B}$ obtained in two different ways. However, we will not need any information from $\hat{B}$ except its boundedness in the paper. The proof is now complete.

□

Corollary 12.2. *Let $\gamma(s):(-\varepsilon,\varepsilon)\to\mathbb{R}^6$ be a C^1 curve such that $\Gamma=\gamma'(0)=O(1)$ and $\dfrac{\partial G_{in}^-}{\partial s}=O(\mu)$ then when taking derivative with respect to s in equations*

$$\begin{cases}|v_3^+|^2+|v_4^+|^2=|v_3^-|^2+|v_4^-|^2+o(1),\\ v_3^++v_4^+=v_3^-+v_4^-+o(1),\\ Q_3^++Q_4^+=Q_3^-+Q_4^-+o(1),\end{cases}$$

obtained from equation (10.7), the $o(1)$ terms are small in the C^1 sense.

Proof. For the motion of the mass center, it follows from Corollary 12.1 that $\dfrac{\partial(Q_+,v_+)^+}{\partial(Q_-,v_-,Q_+,v_+)^-} = \dfrac{1}{\mu}\dfrac{\partial(Q_+,v_+)^+}{\partial\alpha}\otimes \mathbf{l} + (0_{4\times 4},\mathrm{Id}_{4\times 4}) + o(1)$. We already obtained that $\dfrac{\partial(Q_+,v_+)^+}{\partial\alpha} = O(\mu^{2\kappa})$ (see equation (12.24)). Due to Corollary 12.1 our assumption that $\dfrac{\partial G^-_{in}}{\partial s} = O(\mu)$ implies that

$$\mathbf{l}\cdot\Gamma = O(\mu) \tag{12.30}$$

which suppresses the $1/\mu$ term. This proves the corollary for the last two identities. To derive the first equation we use the fact that the Hamiltonian (10.5) is preserved. Namely we use the fact that RHS (10.5) is the same in + and − variables. It is enough to show $\dfrac{d}{ds}(|v_-^+|^2-|v_-^-|^2) = o(1)$ since we already have the required estimate for the velocity of the mass center. In (10.5), the terms involving only Q_+, v_+ are handled using the result of the previous paragraph. The term $-\dfrac{\mu}{|Q_-|}$ vanishes when taking derivative since $|Q_-| = \mu^\kappa$ is constant. All the remaining terms have Q_- to the power 2 or higher. We have $\dfrac{\partial Q_-^-}{\partial s} = O(1)$ since $\Gamma = O(1)$. We also have $\dfrac{\partial Q_-^+}{\partial s} = O(1)$ due to (12.30). Therefore after taking the s derivative, any term involving Q_- is of order $O(\mu^\kappa)$. This completes the proof of the energy conservation part. □

12.2. **Proof of the Lemma 3.8.** In this section we work out the $O(1/\mu)$ term in the local map.

Proof. The proof is relies on a numerical computation.

Before collision, $\mathbf{l} = \dfrac{\partial G_{in}}{\partial -}$**.** According to Corollary 12.1 we can differentiate the asymptotic expression of Lemma 10.2. We have $\left(\dfrac{\partial G_{in}}{\partial G_4^-}, \dfrac{\partial G_{in}}{\partial g_4^-}\right) =$

$$-(v_3^- - v_4^-)\times\left(\frac{\partial}{\partial G_4^-},\frac{\partial}{\partial g_4^-}\right)Q_4 - (v_3^- - v_4^-)\times\left(\frac{\partial Q_4}{\partial \ell_4^-}\right)\cdot\left(\frac{\partial \ell_4^-}{\partial G_4^-},\frac{\partial \ell_4^-}{\partial g_4^-}\right) + O(\mu^\kappa+\mu^{1-2\kappa}),$$

where $O(\mu^\kappa)$ comes from $\left(\dfrac{\partial}{\partial -}(v_3^- - v_4^-)\right)\times(Q_3-Q_4)$ and $O(\mu^{1-2\kappa})$ comes from $\dfrac{\partial Q_4}{\partial L_4^-}\dfrac{\partial L_4^-}{\partial -}$ where L_4 is solved from the Hamiltonian (9.1) $H=0$.

We need to eliminate ℓ_4 using the relation $|Q_3-Q_4| = \mu^\kappa$.

$$\left(\frac{\partial \ell_4^-}{\partial G_4^-},\frac{\partial \ell_4^-}{\partial g_4^-}\right) = -\left(\frac{\partial|Q_3-Q_4|}{\partial \ell_4^-}\right)^{-1}\left(\frac{\partial|Q_3-Q_4|}{\partial G_4^-},\frac{\partial|Q_3-Q_4|}{\partial g_4^-}\right)$$

$$= -\frac{(Q_3 - Q_4)\cdot\left(\frac{\partial Q_4}{\partial G_4^-}, \frac{\partial Q_4}{\partial g_4^-}\right)}{(Q_3 - Q_4)\cdot\frac{\partial Q_4}{\partial \ell_4^-}} = -\frac{(v_3^- - v_4^-)\cdot\left(\frac{\partial Q_4}{\partial G_4^-}, \frac{\partial Q_4}{\partial g_4^-}\right)}{(v_3^- - v_4^-)\cdot\frac{\partial Q_4}{\partial \ell_4^-}} + O(\mu^{1-\kappa}).$$

Here we replaced $Q_3^- - Q_4^-$ by $v_3^- - v_4^-$ using the fact that the two vectors form an angle of order $O(\mu^{1-\kappa})$ (see Lemma 10.2(c)). Therefore

$$\left(\frac{\partial G_{in}}{\partial G_4^-}, \frac{\partial G_{in}}{\partial g_4^-}\right) = -(v_3^- - v_4^-)\times\left(\frac{\partial}{\partial G_4^-}, \frac{\partial}{\partial g_4^-}\right) Q_4$$

$$+(v_3^- - v_4^-)\times\frac{\partial Q_4}{\partial \ell_4^-}\left(\frac{(v_3^- - v_4^-)\cdot\left(\frac{\partial Q_4}{\partial G_4^-}, \frac{\partial Q_4}{\partial g_4^-}\right)}{(v_3^- - v_4^-)\cdot\frac{\partial Q_4}{\partial \ell_4^-}}\right) + O(\mu^\kappa + \mu^{1-2\kappa}).$$

Similarly, we get

$$\frac{\partial G_{in}}{\partial \ell_3^-} = (v_3^- - v_4^-)\times\frac{\partial Q_3}{\partial \ell_3^-} + (v_3^- - v_4^-)\times\frac{\partial Q_4}{\partial \ell_4^-}\left(\frac{(v_3^- - v_4^-)\cdot\frac{\partial Q_3}{\partial \ell_3^-}}{(v_3^- - v_4^-)\cdot\frac{\partial Q_4}{\partial \ell_4^-}}\right) + O(\mu^\kappa + \mu^{1-2\kappa}).$$

We use Mathematica and the data in the Appendix B.2 to work out $\frac{\partial G_{in}}{\partial -}$. The results are : for the first collision, $\hat{\mathbf{l}}_1 = [*, -0.8, *, *, 3.42, -2.54]$, and for the second collision: $\hat{\mathbf{l}}_2 = [*, -0.35, *, *, 3.44, -0.47]$. We can check directly that $\hat{\mathbf{l}}_i\cdot w_{3-i}\neq 0$ and $\hat{\mathbf{l}}_i\cdot\tilde{w}\neq 0$ for $i = 1, 2$ using (3.3).

After collision, $\hat{\mathbf{u}} = \frac{\partial -}{\partial\alpha}$. In equation (10.7), we let $\mu\to 0$. Applying the implicit function theorem to (10.7) with $\mu = 0$ we obtain

$$\left(\frac{\partial(Q_3^+, v_3^+, Q_4^+, v_4^+)}{\partial(X^+, Y^+)} + \frac{\partial(Q_3^+, v_3^+, Q_4^+, v_4^+)}{\partial\ell_4^+}\otimes\frac{\partial\ell_4^+}{\partial(X^+, Y^+)}\right)\cdot\frac{\partial(X^+, Y^+)}{\partial\alpha}$$
$$= \frac{1}{2}\left(0, 0, R\left(\frac{\pi}{2}+\alpha\right)(v_3^- - v_4^-), 0, 0, -R\left(\frac{\pi}{2}+\alpha\right)(v_3^- - v_4^-)\right)^T$$
$$= \frac{1}{2}\left(0, 0, R\left(\frac{\pi}{2}\right)(v_3^+ - v_4^+), 0, 0, -R\left(\frac{\pi}{2}\right)(v_3^+ - v_4^+)\right)^T.$$

where $R(\pi/2+\alpha) = \frac{dR(\alpha)}{d\alpha}$ and $\frac{\partial\ell_4^+}{\partial(X^+, Y^+)}$ is given by (9.3). Again we use Mathematica to work out the $\frac{\partial -}{\partial\alpha}$. The results are: for the first collision $\hat{\mathbf{u}}_1 = [-0.49, *, *, *, -0.20, -0.64]$ and for the second collision $\hat{\mathbf{u}}_2 = [-1.00, *, *, *, 0.34, -0.50]$. We can check directly that $\bar{\mathbf{l}}_i\cdot\hat{\mathbf{u}}_i\neq 0$ for $i = 1, 2$ using (3.3).

To obtain a symbolic sequence with any order of symbols 3, 4 as claimed in the main theorem, we notice that the only difference is that the outgoing relative velocity changes sign $(v_3^+ - v_4^+)\to -(v_3^+ - v_4^+)$. So we only need to send $\hat{\mathbf{u}}\to -\hat{\mathbf{u}}$. □

12.3. **Proof of the Lemma 3.9.** In this section, we prove Lemma 3.9, which guarantees the non degeneracy condition Lemma 3.3 (see the proof of Lemma 3.3). Since we have already obtained $\mathbf{l}$ and $\mathbf{u}$ in $d\mathbb{L}$ and $\bar{\mathbf{l}}, \bar{\bar{\mathbf{l}}}, \bar{\mathbf{u}}, \bar{\bar{\mathbf{u}}}$ in $d\mathbb{G}$, one way to prove Lemma 3.3 is to work out the matrix B explicitly using Corollary 12.1 on computer. In that case, the current section is not necessary. However, in this section, we use a different approach, which simplifies the computation and has several advantages. The first advantage is that this treatment has clear physical and geometrical meaning. Second, we use the same way to control the shape of the ellipse in Appendix B.3. Third, this method gives us a way to deal with the singular limit $d\mathbb{L}$ as $\mu \to 0$.

Recall that Lemmas 3.1 and 3.2 give the following form for the derivatives of local map and global maps

$$d\mathbb{L} = \frac{1}{\mu}\mathbf{u}_j \otimes \mathbf{l}_j + B + O(\mu^\kappa), \quad d\mathbb{G} = \chi^2 \bar{\mathbf{u}}_j \otimes \bar{\mathbf{l}}_j + \chi \bar{\bar{\mathbf{u}}}_j \otimes \bar{\bar{\mathbf{l}}}_j + O(\mu^2\chi),$$

where $j = 1, 2$ standing for the first or second collision. Moreover, in the limit $\chi \to \infty, \mu \to 0$,

$$span\{\bar{\mathbf{u}}_j, \bar{\bar{\mathbf{u}}}_j\} \to span\{w_j, \tilde{w}\}, \quad \mathbf{l}_j \to \hat{\mathbf{l}}_j,\ \bar{\mathbf{l}}_j \to \bar{\mathbf{l}}_j,\ \bar{\bar{\mathbf{l}}}_j \to \bar{\bar{\mathbf{l}}}_j, \quad j = 1, 2.$$

We first prove an abstract lemma that reduces the study of the local map of the $\mu > 0$ case to $\mu = 0$ case. It shows that we can find a direction in span$\{\bar{\mathbf{u}}, \bar{\bar{\mathbf{u}}}\}$, along which the directional derivative of $d\mathbb{L}$ is not singular.

Lemma 12.3. *Suppose the vector $\tilde{\Gamma}_\mu \in span\{\bar{\mathbf{u}}_{3-j}, \bar{\bar{\mathbf{u}}}_{3-j}\}$ satisfies $\bar{\mathbf{l}}_j(d\mathbb{L}\tilde{\Gamma}_\mu) = 0$ and $\|\tilde{\Gamma}_\mu\|_\infty = 1$. Then we have $\mathbf{l}_j(\tilde{\Gamma}_\mu) = O(\mu)$ as $\mu \to 0$ and the following limits exist*

$$\Gamma_{3-j} = \lim_{\mu\to 0} \tilde{\Gamma}_\mu \text{ and } \lim_{\mu\to 0} d\mathbb{L}\tilde{\Gamma}_\mu = \Delta_j,$$

and the Δ_j satisfies $\hat{\bar{\mathbf{l}}}_j(\Delta_j) = 0$.

Proof. Denote $\Gamma'_\mu = \mathbf{l}_j(\bar{\mathbf{u}}_{3-j})\bar{\bar{\mathbf{u}}}_{3-j} - \mathbf{l}_j(\bar{\bar{\mathbf{u}}}_{3-j})\bar{\mathbf{u}}_{3-j} \in Ker\mathbf{l}_j$ and let v_μ be a vector in span$(\bar{\mathbf{u}}_{3-j}, \bar{\bar{\mathbf{u}}}_{3-j})$ such that $v_\mu \to v$ as $\mu \to 0$ and $\mathbf{l}_j(v_\mu) = 1$. Suppose that

$$\tilde{\Gamma}_\mu = a_\mu v_\mu + b_\mu \Gamma'_\mu$$

then

$$d\mathbb{L}(\tilde{\Gamma}_\mu) = \frac{a_\mu}{\mu}\mathbf{l}_j(v_\mu)\mathbf{u}_j + a_\mu B_j(v_\mu) + b_\mu B_j \Gamma'_\mu + o(1). \tag{12.31}$$

So $\bar{\mathbf{l}}_j(d\mathbb{L}(\tilde{\Gamma}_\mu)) = 0$ implies that

$$a_\mu = -\mu \frac{b_\mu \bar{\mathbf{l}}_j(B_j\Gamma'_\mu) + o(1)}{\mathbf{l}_j(v_\mu)\bar{\mathbf{l}}_j(\mathbf{u}_j) + \mu \bar{\mathbf{l}}_j B_j(v_\mu)}. \tag{12.32}$$

The denominator is not zero since $\mathbf{l}_j(v_\mu) = 1$ and $\bar{\mathbf{l}}_j(\mathbf{u}_j)$ using Lemma 3.8. Therefore $a_\mu = O(\mu)$ and hence $\tilde{\Gamma}_\mu = b_\mu \Gamma'_\mu + O(\mu)$ and $\mathbf{l}_j(\tilde{\Gamma}_\mu) = O(\mu)$. Now the remaining statements of the lemma follow from equations (12.31) and (12.32). □

To compute the numerical values it is more convenient for us to work with polar coordinates. We need the following quantities.

Definition 12.1.
- *ψ: polar angle, related to u by $\tan\frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$ for ellipse. We choose the positive y axis as the axis $\psi = 0$. E: energy; e : eccentricity; G: angular momentum, g: argument of periapsis.*
- *The subscripts $3, 4$ stand for Q_3 or Q_4. The superscript $\pm$ refers to before or after collision. Recall that all quantities are evaluated on the sphere*

$$|Q_3 - Q_4| = \mu^\kappa.$$

Recall the formula $r = \frac{G^2}{1 - e\cos\psi}$ for conic sections in which the perigee lies on the axis $\psi = \pi$. In our case we have

$$\left\{\begin{aligned} r_3^\pm &= \frac{(G_3^\pm)^2}{1 - e_3^\pm \sin(\psi_3^\pm + g_3^\pm)} + o(1),\\ r_4^\pm &= \frac{(G_4^\pm)^2}{1 - e_4^\pm \sin(\psi_4^\pm - g_4^\pm)} + o(1). \end{aligned}\right. \tag{12.33}$$

$o(1)$ terms are small when $\mu \to 0$ (recall that we always assume that $\chi \gg 1/\mu$).

Lemma 12.4. *Under the assumptions of Corollary 12.2 we have*

$$\frac{dr_3^+}{ds} = \frac{dr_4^+}{ds} + o(1), \quad \frac{dr_3^-}{ds} = \frac{dr_4^-}{ds} + o(1), \quad \frac{d\psi_3^+}{ds} = \frac{d\psi_4^+}{ds} + o(1), \quad \frac{d\psi_3^-}{ds} = \frac{d\psi_4^-}{ds} + o(1).$$

Moreover in (12.33) *the $o(1)$ terms are also C^1 small when taking the s derivative.*

Proof. To prove the statement about (12.33), we use the Hamiltonian (4.1). The $r_{3,4}$ obey the Hamiltonian system (4.1). The estimate (9.2) shows the $\frac{-\mu}{|Q_3 - Q_4|}$ gives small perturbation to the variational equations. The two $O(1/\chi)$ terms in (4.1) are also small. This shows that the perturbations to Kepler motion is C^1 small.

Next we consider the derivatives $\frac{\partial r_{3,4}^\pm}{\partial s}$. We consider first the case of "$-$". From the condition $|\vec r_3 - \vec r_4| = \mu^\kappa$, for the Poincaré section we get

$$(\vec r_3 - \vec r_4)\cdot\frac{d}{ds}(\vec r_3 - \vec r_4) = 0.$$

This implies $(\vec r_3 - \vec r_4) \perp \frac{d}{ds}(\vec r_3 - \vec r_4)$.

We also know the angular momentum for the relative motion is

$$G_{in} = (\dot{\vec r}_3 - \dot{\vec r}_4) \times (\vec r_3 - \vec r_4) = O(\mu),$$

which implies $\dot{\vec r}_3-\dot{\vec r}_4$ is almost parallel to $\vec r_3-\vec r_4$. The condition $\dfrac{\partial G_{in}^-}{\partial s}=O(\mu)$ reads

$$\left(\frac{d}{ds}(\dot{\vec r}_3-\dot{\vec r}_4)\right)\times(\vec r_3-\vec r_4)+(\dot{\vec r}_3-\dot{\vec r}_4)\times\left(\frac{d}{ds}(\vec r_3-\vec r_4)\right)=O(\mu).$$

Since the first term is $O(\mu^\kappa)$ due to our choice of the Poincare section we see that

$$(\dot{\vec r}_3-\dot{\vec r}_4)\times\left(\frac{d}{ds}(\vec r_3-\vec r_4)\right)=o(1).$$

Since $\dfrac{d}{ds}(\vec r_3-\vec r_4)$ is almost perpendicular to $(\dot{\vec r}_3-\dot{\vec r}_4)$ by the analysis presented above we get $\dfrac{d}{ds}(\vec r_3-\vec r_4)=o(1)$. Taking the radial and angular part of this vector identity and using that $r_4=r_3+o(1)$, $\psi_4=\psi_3+o(1)$ we get "$-$" part of the lemma.

To repeat the above argument for "+" variables, we first need to establish $\dfrac{\partial G_{in}^-}{\partial s}=O(\mu)$. Indeed, using equations (12.8) and (12.18) we get

$$\begin{aligned}\frac{\partial G_{in}^+}{\partial\psi}&=\frac{\partial G_{in}^+}{\partial(\mathcal{L},G_{in},g,Q_+,v_+)^-}\frac{\partial(\mathcal{L},G_{in},g,Q_+,v_+)^-}{\partial\psi}\\&=O(\mu^{3\kappa},1,\mu^{3\kappa},\mu^{3\kappa}_{1\times2},\mu^{3\kappa}_{1\times2})\cdot O(1,\mu,1,1_{1\times2},1_{1\times2})=O(\mu).\end{aligned}$$

It remains to show $\left(\dfrac{d}{ds}(\dot{\vec r}_3-\dot{\vec r}_4)\right)=O(1)$ in the " + " case. Since we know it is true in the "-" case, the "+" case follows, because the directional derivative of the local map $d\mathbb{L}\Gamma$ is bounded due to our choice of Γ. □

We are now ready to describe the computation of Lemma 3.9. The reader may notice that the computations in the proofs of Lemmas 3.9 and 2.1 are quite similar. Note however that Lemma 3.9 describes the *subleading* term for the derivative of the local map. By contrast the *leading* term can not be understood in terms of the Gerver map since it comes from the possibility of varying the closest distance between Q_3 and Q_4 and this distance is assumed to be zero in Gerver's model.

We will use the following set of equations which follows from (12.26).

$$E_3^++E_4^+=E_3^-+E_4^-, \tag{12.34}$$

$$G_3^++G_4^+=G_3^-+G_4^-, \tag{12.35}$$

$$\frac{e_3^+}{G_3^+}\cos(\psi_3^++g_3^+)+\frac{e_4^+}{G_4^+}\cos(\psi_4^--g_4^-)=\frac{e_3^-}{G_3^-}\cos(\psi_3^-+g_3^-)+\frac{e_4^-}{G_4^-}\cos(\psi_4^--g_4^-), \tag{12.36}$$

$$\frac{(G_3^+)^2}{1-e_3^+\sin(\psi_3^++g_3^+)}=\frac{(G_3^-)^2}{1-e_3^-\sin(\psi_3^-+g_3^-)}, \tag{12.37}$$

$$\psi_3^+ = \psi_3^- \tag{12.38}$$

$$\frac{(G_3^+)^2}{1 - e_3^+ \sin(\psi_3^+ + g_3^+)} = \frac{(G_4^+)^2}{1 - e_4^+ \sin(\psi_4^+ - g_4^+)}, \tag{12.39}$$

$$\frac{(G_3^-)^2}{1 - e_3^- \sin(\psi_3^- + g_3^-)} = \frac{(G_4^-)^2}{1 - e_4^- \sin(\psi_4^- - g_4^-)}, \tag{12.40}$$

$$\psi_4^- = \psi_3^- \tag{12.41}$$

$$\psi_4^+ = \psi_3^+ \tag{12.42}$$

In the above equations we have dropped $o(1)$ terms for brevity. We would like to emphasize that the above approximations hold not only in C^0 sense but also in C^1 sense when we take the derivatives along the directions satisfying the conditions of Corollary 12.2. (12.34) is the approximate conservation of the energy, (12.35) is the approximate conservation of the angular momentum and (12.36) follows from the approximate momentum conservation (see the derivation of (B.2) in Appendix B.3). The possibility of differentiating these equations is justified in Corollary 12.2. The remaining equations reflect the fact that $Q_3^\pm$ and $Q_4^\pm$ are all close to each other. The possibility of differentiating these equations is justified by Lemma 12.4.

We set the total energy to be zero. So we get $E_4^\pm = -E_3^\pm$. This eliminates $E_4^\pm$. Then we also eliminate $\psi_4^\pm$ by setting them to be equal $\psi_3^\pm$.

Proof of the Lemma 3.9. Lemma 12.3 and Corollary 12.1 show that the assumption of Lemma 3.9 implies that the direction Γ along which we take the directional derivative satisfies $\frac{\partial G_{in}}{\partial \Gamma} = O(\mu)$. So we can directly take derivatives in equations (12.34)-(12.34). Recall that we need to compute $dE_3^+(d\mathbb{L}\Gamma)$ where $\Gamma \in Ker \mathbf{l}_j \cap span\{w_{3-j}, \tilde{w}\}$. (3.3) tells us that in in Delaunay coordinates we have

$$\tilde{w} = (0,1,0,0,0,0), \quad w = (0,0,0,0,1,a) \text{ where } a = \frac{-L_4^-}{(L_4^-)^2 + (G_4^-)^2}. \tag{12.43}$$

The formula $\tan\frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$ which relates ψ to ℓ through u shows that (12.43) also holds if we use $(L_3, \psi_3, G_3, g_3, G_4, g_4)$ as coordinates. Hence Γ has the form $(0,1,0,0,c,ca)$. To find the constant c we use (12.40).

Note that the expression $dE_3^+(d\mathbb{L}\Gamma)$ does not involve $d\psi_3^+$. Therefore we can eliminate ψ_3^+ from consideration by setting $\psi_3^+ = \psi_3^- = \psi$ (see (12.38)). Let $\boldsymbol{L}$ denote the projection of our map to $(L_3, G_3, g_3, G_4, g_4)$ variables. Thus we need to find $dE_3^+(d\boldsymbol{L}\Gamma)$. To this end write the remaining equations ((12.35), (12.36), (12.37), and (12.39)) formally as $\mathbf{F}(Z^+, Z^-) = 0$, where in $Z^+ = (E_3^+, G_3^+, g_3^+, G_4^+, g_4^+)$ and $Z^- = (E_3^-, \psi, G_3^-, g_3^-, G_4^-, g_4^-)$.

We have

$$\frac{\partial \mathbf{F}}{\partial Z^+} d\boldsymbol{L}\Gamma + \frac{\partial \mathbf{F}}{\partial Z^-}\Gamma = 0.$$

However, $\frac{\partial \mathbf{F}}{\partial Z^+}$ is not invertible since $\mathbf{F}$ involves only four equations of $\mathbf{F}$ while Z^+ has 5 variables. To resolve this problem we use that by definition of Γ we have $\bar{\mathbf{l}}\cdot\frac{\partial Z^+}{\partial \psi} = 0$, where $\bar{\mathbf{l}} = \left(\frac{G_4^+/L_4^+}{(L_4^+)^2 + (G_4^+)^2}, 0, 0, 0, \frac{-1}{(L_4^+)^2 + (G_4^+)^2}, \frac{1}{L_4^+}\right)$ by (3.3). Thus we get

$$\begin{bmatrix} \bar{\mathbf{l}} \\ \frac{\partial \mathbf{F}}{\partial Z^+} \end{bmatrix} d\boldsymbol{L}\Gamma = -\begin{bmatrix} 0 \\ \frac{\partial \mathbf{F}}{\partial Z^-}\Gamma \end{bmatrix}$$

and so

$$d\boldsymbol{L}\Gamma = -\begin{bmatrix} \bar{\mathbf{l}} \\ \frac{\partial \mathbf{F}}{\partial Z^+} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{\partial \mathbf{F}}{\partial Z^-}\Gamma \end{bmatrix}.$$

We use computer to complete the computation. We only need the entry $\frac{\partial E_3^+}{\partial \psi}$ to prove Lemma 3.9. It turns out this number is 1.855 for the first collision and -1.608 for the second collision. Both are nonzero as needed. □

Appendix A. Delaunay coordinates

A.1. **Elliptic motion.** The material of this section could be found in [Al]. Consider the two-body problem with Hamiltonian

$$H(P,Q) = \frac{|P|^2}{2m} - \frac{k}{|Q|}, \quad (P,Q) \in \mathbb{R}^4.$$

This system is integrable in the Liouville-Arnold sense when $H < 0$. So we can introduce the action-angle variables (L, ℓ, G, g) in which the Hamiltonian can be written as

$$H(L,\ell,G,g) = -\frac{mk^2}{2L^2}, \quad (L,\ell,G,g) \in T^*\mathbb{T}^2.$$

The Hamiltonian equations are

$$\dot{L} = \dot{G} = \dot{g} = 0, \quad \dot{\ell} = \frac{mk^2}{L^3}.$$

We introduce the following notation E-energy, M-angular momentum, e-eccentricity, a-semimajor axis, b-semiminor axis. Then we have the following relations which explain the physical and geometrical meaning of the Delaunay coordinates.

$$a = \frac{L^2}{mk},\ b = \frac{LG}{mk},\ E = -\frac{k}{2a},\ M = G,\ e = \sqrt{1 - \left(\frac{G}{L}\right)^2}.$$

Moreover, g is the argument of periapsis and ℓ is called the mean anomaly, and ℓ can be related to the polar angle ψ through the equations

$$\tan\frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}}\cdot\tan\frac{u}{2}, \quad u - e\sin u = \ell.$$

We also have the Kepler's law $\dfrac{a^3}{T^2} = \dfrac{1}{(2\pi)^2}$ which relates the semimajor axis a and the period T of the ellipse.

Denoting particle's position by (q_1, q_2) and its momentum (p_1, p_2) we have the following formulas in case $g = 0$.

$$\begin{cases} q_1 = a(\cos u - e), \\ q_2 = a\sqrt{1-e^2}\sin u, \end{cases} \qquad \begin{cases} p_1 = -\sqrt{mk}a^{-1/2}\dfrac{\sin u}{1-e\cos u}, \\ p_2 = \sqrt{mk}a^{-1/2}\dfrac{\sqrt{1-e^2}\cos u}{1-e\cos u}, \end{cases}$$

where u and l are related by $u - e\sin u = \ell$.

Expressing e and a in terms of Delaunay coordinates we obtain the following

$$\text{(A.1)} \qquad \begin{aligned} q_1 &= \frac{L^2}{mk}\left(\cos u - \sqrt{1-\frac{G^2}{L^2}}\right), \quad q_2 = \frac{LG}{mk}\sin u. \\ p_1 &= -\frac{mk}{L}\frac{\sin u}{1-\sqrt{1-\frac{G^2}{L^2}}\cos u}, \quad p_2 = \frac{mk}{L^2}\frac{G\cos u}{1-\sqrt{1-\frac{G^2}{L^2}}\cos u}. \end{aligned}$$

Here g does not enter because the argument of perihelion is chosen to be zero. In general case, we need to rotate the (q_1, q_2) and (p_1, p_2) using the matrix $\begin{bmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{bmatrix}$.

Notice that the equation (A.1) describes an ellipse with one focus at the origin and the other focus on the negative x-axis. We want to be consistent with [G2], i.e. we want $g = \pi/2$ to correspond to the "vertical" ellipse with one focus at the origin and the other focus on the positive y-axis (see Appendix B.2). Therefore we rotate the picture clockwise. So we use the Delaunay coordinates which are related to the Cartesian ones through the equation

$$\text{(A.2)} \qquad \begin{aligned} q_1 =& \frac{1}{mk}\left(L^2\left(\cos u - \sqrt{1-\frac{G^2}{L^2}}\right)\cos g + LG\sin u\sin g\right), \\ q_2 =& \frac{1}{mk}\left(-L^2\left(\cos u - \sqrt{1-\frac{G^2}{L^2}}\right)\sin g + LG\sin u\cos g\right). \end{aligned}$$

A.2. **Hyperbolic motion.** The above formulas can also be used to describe hyperbolic motion, where we need to replace "sin → sinh, cos → cosh"(c.f.[Al, F]). Namely, we have

$$
\begin{aligned}
&q_1 = \frac{L^2}{mk}\left(\cosh u - \sqrt{1+\frac{G^2}{L^2}}\right), \quad q_2 = \frac{LG}{mk}\sinh u,\\
&p_1 = -\frac{mk}{L}\frac{\sinh u}{1-\sqrt{1+\frac{G^2}{L^2}}\cosh u}, \quad p_2 = -\frac{mk}{L^2}\frac{G\cosh u}{1-\sqrt{1+\frac{G^2}{L^2}}\cosh u}.
\end{aligned} \tag{A.3}
$$

where u and l are related by

$$u - e\sinh u = \ell, \text{ where } e = \sqrt{1+\left(\frac{G}{L}\right)^2}. \tag{A.4}$$

This hyperbola is symmetric w.r.t. the x-axis, opens to the right and the particle moves clockwise on it when u increases (ℓ decreases). When the particle moves to the right of $x = -\frac{\chi}{2}$ line we have a hyperbola opening to the left and the particle moves anti-clockwise. To achieve this we first reflect (q_1, q_2) around the y-axis, then rotate it by an angle g. If we restrict $|g| < \pi/2$, then the particle moves anti-clockwise on the hyperbola as u increases (ℓ decreases) due to the reflection. Thus we have

$$
\begin{aligned}
q_1 =& -\frac{1}{mk}\left(\cos gL^2(\cosh u - e) + \sin gLG\sinh u\right),\\
q_2 =& \frac{1}{mk}\left(-\sin gL^2(\cosh u - e) + \cos gLG\sinh u\right),\\
P =& \frac{mk}{1-e\cosh u}\left(\frac{1}{L}\sinh u\cos g + \frac{G}{L^2}\sin g\cosh u,\right.\\
&\left.\frac{1}{L}\sinh u\sin g - \frac{G}{L^2}\cos g\cosh u\right).
\end{aligned} \tag{A.5}
$$

If the incoming asymptote is horizontal, (see the arrows in Figure 1 for "incoming" and "outgoing"), then the particle comes from the left, and as u tends to $-\infty$, the y-coordinate is bounded and x-coordinate is negative. In this case we have $\tan g = -\frac{G}{L}$, $g \in (-\pi/2, 0)$.

If the outgoing asymptote is horizontal, then the particle escapes to the left, and as u tends to $+\infty$, the y-coordinate is bounded and x-coordinate is negative. In this case we have $\tan g = +\frac{G}{L}, g \in (0, \pi/2)$.

When the particle Q_4 is moving to the left of the section $\{x = -\chi/2\}$, we treat the motion as hyperbolic motion focused at Q_1. We move the origin to Q_1. The hyperbola opens to the right. The orbit has the following parametrization

$$\begin{aligned} q_1 =& \frac{1}{mk}\left(\cos gL^2(\cosh u - e) - \sin gLG \sinh u\right), \\ q_2 =& \frac{1}{mk}(\sin gL^2(\cosh u - e) + \cos gLG \sinh u). \end{aligned} \tag{A.6}$$

A.3. **Large ℓ asymptotics: auxiliary results.** In the remaining part of Appendix A we study the first and second order derivatives of Q w.r.t. the hyperbolic Delaunay variables (L, ℓ, G, g). These computations are used in our proof. The next lemma allows us to simplify the computations. Since the hyperbolic motion approaches a linear motion, this lemma shows that, we can replace u by $\ln(\mp\ell/e)$ when taking first and second order derivatives.

Lemma A.1. *Let u be the function of ℓ, G and L given by* (A.4)*. Then we can approximate u by $\ln(\mp\ell/e)$ in the following sense.*

$$u \mp \ln\frac{\mp\ell}{e} = O(\ln|\ell|/\ell), \quad \frac{\partial u}{\partial \ell} = \pm 1/\ell + O(1/\ell^2),$$
$$\left(\frac{\partial}{\partial L}, \frac{\partial}{\partial G}\right)(u \pm \ln e) = O(1/|\ell|), \quad \left(\frac{\partial}{\partial L}, \frac{\partial}{\partial G}\right)^2 (u \pm \ln e) = O(1/|\ell|),$$

Here the first sign is taken if $u > 0$ and the second sign is taken then $u < 0$. The estimates above are uniform as long as $|G| \le K$, $1/K \le L \le K$, $\ell > \ell_0$ and the implied constants in $O(\cdot)$ depend only on K and ℓ_0.

Proof. We see from formula (A.4) that $\sinh u \simeq \cosh u = -\dfrac{\ell}{e} + O(\ln|\ell|)$ when $u > 0$ and $\sinh u \simeq -\cosh u \simeq -\dfrac{\ell}{e} + O(\ln|\ell|)$ when $u < 0$ and $|u|$ large enough. This proves C^0 estimate.

Now we consider the first order derivatives. We assume that $u > 0$ to fix the notation. Differentiating (A.4) with respect to ℓ we get

$$\frac{\partial u}{\partial \ell} - e\cosh u \frac{\partial u}{\partial \ell} = 1, \quad \frac{\partial u}{\partial \ell} = 1/\ell + O(1/\ell^2).$$

Next, we differentiate (A.4) with respect to L to obtain

$$\frac{\partial u}{\partial L} - \frac{\partial e}{\partial L}\sinh u - e\cosh u \frac{\partial u}{\partial L} = 0.$$

Therefore,

$$\frac{\partial u}{\partial L} = \frac{\sinh u}{1 - e\cosh u}\frac{\partial e}{\partial L} = -\frac{1}{e}\frac{\partial e}{\partial L} + O(e^{-|u|}) = -\frac{\partial}{\partial L}\ln(e) + O(1/|\ell|).$$

The same argument holds for $\dfrac{\partial}{\partial G}$. This proves C^1 part of the Lemma.

Now we consider second order derivatives. We take $\dfrac{\partial^2}{\partial L^2}$ as example. Combining

$$\frac{\partial^2 u}{\partial L^2}-\frac{\partial^2 e}{\partial L^2}\sinh u-2\cosh u\frac{\partial e}{\partial L}\frac{\partial u}{\partial L}-e\cosh u\frac{\partial^2 u}{\partial L^2}-e\sinh u\left(\frac{\partial u}{\partial L}\right)^2=0.$$

with C^1 estimate proven above we get

$$\begin{aligned}\frac{\partial^2 u}{\partial L^2}&=-\frac{1}{e}\frac{\partial^2 e}{\partial L^2}-\frac{2\partial e}{e\partial L}\frac{\partial u}{\partial L}+\left(\frac{\partial u}{\partial L}\right)^2+O\left(\frac{1}{\ell}\right)\\ &=-\frac{1}{e}\frac{\partial^2 e}{\partial L^2}+\left(\frac{1}{e}\frac{\partial e}{\partial L}\right)^2+O\left(\frac{1}{\ell}\right)=\frac{\partial^2}{\partial L^2}\ln e+O\left(\frac{1}{\ell}\right).\end{aligned}$$

This concludes the C^2 part of the lemma. □

In the estimate of the derivatives presented in the next two subsections we shall often use the following facts. Let $f=\ln e$. Then

$$f_G=\frac{G}{L^2+G^2},\quad f_L=-\frac{G^2}{L(L^2+G^2)},\tag{A.7}$$

$$(f)_{GG}=\frac{L^2-G^2}{(L^2+G^2)^2},\quad f_{LG}=-\frac{2GL}{(L^2+G^2)^2}.\tag{A.8}$$

A.4. **First order derivatives.** In the following computations, we assume for simplicity that $m=k=1$. To get the general case we only need to divide positions by mk.

Lemma A.2. *Under the same conditions as in Lemma A.1 we have the following result for the first order derivatives*

(a)
$$\left|\frac{\partial Q}{\partial \ell}\right|=O(1),\quad \left|\frac{\partial Q}{\partial (L,G,g)}\right|=O(\ell),\quad \frac{\partial Q}{\partial g}\cdot Q=0,$$
$$\frac{\partial Q}{\partial G}\cdot Q=O_{C^2(L,G,g)}(\ell).$$

(b) *If in addition we have* $\left|g\mp\arctan\dfrac{G}{L}\right|\leq C/\ell$ *where* $-$ *sign is taken for* $u>0$ *and* $+$ *sign is taken for* $u<0$ *then we have the following bounds for* (A.5)

$$\frac{\partial Q}{\partial G}=\sinh u\left(0,\frac{L^2}{\sqrt{L^2+G^2}}\right)+O(1),\ \frac{\partial Q}{\partial L}=-\sinh u\left(2\sqrt{L^2+G^2},\frac{GL}{\sqrt{L^2+G^2}}\right)+O(1).$$

(c) *If in addition to the conditions of Lemma A.1 we have* $G,g=O(1/\chi)$ *and* $\ell=O(\chi)$*, then we have the following bounds for* (A.6)

$$\frac{\partial Q}{\partial G}=\sinh u(0,1)+O(1),\quad \frac{\partial Q}{\partial L}=\sinh u(2,0)+O(1).$$

Remark A.1. *The assumptions of the lemma and the next lemma hold in our situation due to Lemma 4.7.*

Proof. We consider only the case $u > 0$. We have

$$Q = O(1) - \sinh u(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG), \text{ as } \ell \to \infty. \tag{A.9}$$

Now the first three estimates of part (a) follow easily. Next $\frac{\partial Q}{\partial G} =$
$-(\cosh u)u'_G(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) - \sinh u(\sin gL, -\cos gL) + O(1)$.
Using Lemma A.1 we obtain

$$\begin{aligned} Q \cdot \frac{\partial Q}{\partial G} &= \frac{1}{2}(\sinh 2u)u'_G|(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG)|^2 + \\ &(\sinh u)^2(\sin gL, -\cos gL) \cdot (\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) + O(\ell) \\ &= \frac{1}{2}(\sinh 2u)(-\ln e)'_G(L^4 + L^2G^2) + L^2G(\sinh u)^2 + O(\ell) = O(\ell) \end{aligned}$$

where the last equality relies on (A.7).

We prove (b) in the + case, the - case being similar. Assume first that g is exactly equal to $\arctan \frac{G}{L}$. Using (A.9) and (A.7) we obtain

$$\begin{aligned} \frac{\partial Q}{\partial G} &= (\cosh u)f_G(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) \\ &- \sinh u(\sin gL, -\cos gL) + O(1) \\ &= \sinh u\left(\frac{G}{L^2+G^2}\left(\frac{L^3+LG^2}{\sqrt{L^2+G^2}}, 0\right) - \left(\frac{GL}{\sqrt{L^2+G^2}}, -\frac{L^2}{\sqrt{L^2+G^2}}\right)\right) + O(1) \\ &= \sinh u\left(0, \frac{L^2}{\sqrt{L^2+G^2}}\right) + O(1). \end{aligned}$$

(A.10)

$$\begin{aligned} \frac{\partial Q}{\partial L} &= (\cosh u)f_L(\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) \\ &- \sinh u(2\cos gL + \sin gG, 2\sin gL - \cos gG) + O(1) \\ &= -\sinh u\left(\frac{G^2/L}{L^2+G^2}\left(\frac{L^3+LG^2}{\sqrt{L^2+G^2}}, 0\right) + \left(\frac{2L^2+G^2}{\sqrt{L^2+G^2}}, \frac{GL}{\sqrt{L^2+G^2}}\right)\right) + O(1) \\ &= -\sinh u\left(2\sqrt{L^2+G^2}, \frac{GL}{\sqrt{L^2+G^2}}\right) + O(1). \end{aligned}$$

This proves (b) under the assumption $g = \arctan \frac{G}{L}$. If $\left|g - \arctan \frac{G}{L}\right| < \frac{C}{|\ell|}$ then we get an additional $O(1)$ error in the above computation which does not change the final result.

Part (c) follows from part (b) since both g and $\arctan\dfrac{G}{L}$ are $O(1/\ell)$. □

A.5. **Second order derivatives.** The following estimates of the second order derivatives are used in integrating the variational equation.

Lemma A.3. *We have the following information for the second order derivatives of Q_4 w.r.t. the Delaunay variables.*

(a) *Under the conditions of Lemma A.2(a) we have*

$$\frac{\partial^2 Q}{\partial g^2}=-Q,\quad \frac{\partial^2 Q}{\partial g\partial G}\perp\frac{\partial Q}{\partial G},\quad \left(\frac{\partial}{\partial G},\frac{\partial}{\partial g}\right)\left(\frac{\partial |Q|^2}{\partial g}\right)=(0,0),$$

$$\frac{\partial^2 Q}{\partial G^2}=O(\ell),\quad \frac{\partial^2 Q}{\partial L^2}=O(\ell).$$

(b) *Under the conditions of Lemma A.2(b) we have we have*

$$\begin{aligned}
\frac{\partial^2 Q}{\partial G^2}&=\frac{L^2}{(L^2+G^2)^{3/2}}(L\cosh u,-2G\sinh u)+O(1),\\
\frac{\partial^2 Q}{\partial g\partial G}&=\left(-\frac{L^2\sinh u}{\sqrt{L^2+G^2}},0\right)+O(1),\\
\frac{\partial^2 Q}{\partial g\partial L}&=\left(\frac{GL\sinh u}{\sqrt{L^2+G^2}},-2\sqrt{L^2+G^2}\cosh u\right)+O(1),\\
\frac{\partial^2 Q}{\partial G\partial L}&=\frac{L}{(L^2+G^2)^{3/2}}\left(-LG\cosh u,(L^2+3G^2)\sinh u\right)+O(1).
\end{aligned}$$

(c) *Under the conditions of Lemma A.2(c) we have*

$$\begin{aligned}
&\frac{\partial^2 Q}{\partial G^2}=-\cosh u(1,0)+O(1),\quad \frac{\partial^2 Q}{\partial g\partial G}=-L\sinh u(1,0)+O(1),\\
&\frac{\partial^2 Q}{\partial g\partial L}=L\sinh u(0,2)+O(1),\quad \frac{\partial^2 Q}{\partial G\partial L}=\cosh u(0,1)+O(1).
\end{aligned}$$

Proof. The estimate $\dfrac{\partial^2 Q}{\partial G^2}=O(\ell)$ follows immediately from Lemma A.2. The estimate $\dfrac{\partial^2 Q}{\partial L^2}=O(\ell)$ follows immediately from (A.5) (or (A.6)).

The estimates of the derivatives involving g are relatively easy since the dependence of Q on g is through a rotation. We consider $\dfrac{\partial^2 Q}{\partial L\partial g}$, for example, the other derivatives are similar. Differentiating (A.10) with respect to g and using (A.7) we get

$$\begin{aligned}\frac{\partial^2 Q}{\partial L \partial g} &= \cosh u f_L(-L^2 \sin g + LG\cos g, L^2\cos g + LG\sin g)\\ &\quad - \sinh u(-2L\sin g + G\cos g, 2L\cos g + G\sin G) + O(1)\\ &= -\sinh u \frac{G^2}{L(L^2+G^2)}\left(\frac{-L^2G + L^2G}{\sqrt{L^2+G^2}}, \frac{L^3+LG^2}{\sqrt{L^2+G^2}}\right)\\ &\quad - \sinh u\left(\frac{-2LG+LG}{\sqrt{L^2+G^2}}, \frac{2L^2+G^2}{\sqrt{L^2+G^2}}\right) + O(1)\\ &= -\sinh u\left(0, \frac{G^2}{\sqrt{L^2+G^2}}\right) - \sinh u\left(-\frac{LG}{\sqrt{L^2+G^2}}, \frac{2L^2+G^2}{\sqrt{L^2+G^2}}\right) + O(1)\\ &= \sinh u\left(\frac{LG}{\sqrt{L^2+G^2}}, -2\sqrt{L^2+G^2}\right) + O(1).\end{aligned}$$

Next, we compute $\frac{\partial^2 Q}{\partial G \partial L}$ and $\frac{\partial^2 Q}{\partial G^2}$. We consider only the case $u > 0$ and take the $+$ sign. The other cases are similar.

As in the proof of Lemma A.2 it suffices to consider the case $g = \arctan \frac{G}{L}$. Differentiating the expression for $\frac{\partial Q}{\partial G}$ and using Lemma A.1, (A.7) and (A.8) we obtain

$$\begin{aligned}\frac{\partial^2 Q}{\partial G^2} &= -L(\sinh u((\ln e)_G)^2 - \cosh u(\ln e)_{GG})(\cos gL + \sin gG, \sin gL - \cos gG)\\ &\quad + 2L\cosh u(\ln e)_G(\sin g, -\cos g) + O(1)\\ &= L\sinh u\left(\frac{L^2-2G^2}{(L^2+G^2)^2}\right)\left(\frac{L^2}{(L^2+G^2)^{1/2}} + \frac{G^2}{(L^2+G^2)^{1/2}}, 0\right),\\ &\quad + 2L\sinh u\frac{G}{L^2+G^2}\left(\frac{G}{(L^2+G^2)^{1/2}}, -\frac{L}{(L^2+G^2)^{1/2}}\right) + O(1)\\ &= \frac{L^2}{(L^2+G^2)^{3/2}}\sinh u(L, -2G) + O(1)\end{aligned}$$

proving the estimate for $\frac{\partial^2 Q}{\partial G^2}$. Next,

$$\frac{\partial^2 Q}{\partial G\partial L} = -(\sinh u)_{LG}(\cos gL^2+\sin gLG, \sin gL^2-\cos gLG)$$
$$-(\sinh u)_L(\sin gL, -\cos gL)-(\sinh u)_G(2\cos gL+\sin gG, 2\sin gL-\cos gG)$$
$$-\sinh u(\sin g, -\cos g)+O(1)$$
$$=-(\sinh u(\ln e)_L(\ln e)_G-\cosh u(\ln e)_{GL})(L(L^2+G^2)^{1/2},0)$$
$$+\cosh u(\ln e)_L\left(\frac{GL}{(L^2+G^2)^{1/2}},-\frac{L^2}{(L^2+G^2)^{1/2}}\right)+\cosh u(\ln e)_G$$
$$\cdot\left(\frac{2L^2+G^2}{(L^2+G^2)^{1/2}},\frac{GL}{(L^2+G^2)^{1/2}}\right)-\sinh u\left(\frac{G}{(L^2+G^2)^{1/2}},-\frac{L}{(L^2+G^2)^{1/2}}\right)+O(1)$$
$$=\frac{L}{(L^2+G^2)^{3/2}}\sinh u\left(-LG,L^2+3G^2\right)+O(1).$$

Part (c) follows from part (b) as in Lemma A.2. □

Appendix B. Gerver's mechanism

B.1. **Gerver's result in** [G2]**.** We summarize the result of [G2] in the following table. Recall that the Gerver scenario deals with the limiting case $\chi\to\infty, \mu\to 0$. Accordingly Q_1 disappears at infinity and there is no interaction between Q_3 and Q_4. Hence both particles perform Kepler motions. The shape of each Kepler orbit is characterized by energy, angular momentum and the argument of periapsis. In Gerver's scenario, the incoming and outgoing asymptotes of the hyperbola are always horizontal and the semimajor of the ellipse is always vertical. So we only need to describe on the energy and angular momentum.

	1st collision	@$(-\varepsilon_0\varepsilon_1, \varepsilon_0+\varepsilon_1)$	2nd collision	@$(\varepsilon_0^2, 0)$
	Q_3	Q_4	Q_3	Q_4
energy	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}\to-\frac{\varepsilon_1^2}{2\varepsilon_0^2}$	$\frac{1}{2}\to\frac{\varepsilon_1^2}{2\varepsilon_0^2}$
angular momentum	$\varepsilon_1\to-\varepsilon_0$	$p_1\to-p_2$	$-\varepsilon_0$	$\sqrt{2}\varepsilon_0$
eccentricity	$\varepsilon_0\to\varepsilon_1$		$\varepsilon_1\to\varepsilon_0$	
semimajor	1	-1	$1\to\left(\frac{\varepsilon_0}{\varepsilon_1}\right)^2$	$1\to-\frac{\varepsilon_1^2}{\varepsilon_0^2}$
semiminor	$\varepsilon_1\to\varepsilon_0$	$p_1\to p_2$	$\varepsilon_0\to\frac{\varepsilon_0^2}{\varepsilon_1}$	$\sqrt{2}\varepsilon_0\to\sqrt{2}\varepsilon_1$

Here
$$p_{1,2}=\frac{-Y\pm\sqrt{Y^2+4(X+R)}}{2},\quad R=\sqrt{X^2+Y^2}.$$
and (X,Y) stands for the point where collision occurs (the parenthesis after @ in the table). We will call the two points the Gerver's collision points.

In the above table ε_0 is a free parameter and $\varepsilon_1=\sqrt{1-\varepsilon_0^2}$.

At the collision points, the velocities of the particles are the following.
For the first collision,

$$v_3^- = \left(\frac{-\varepsilon_1^2}{\varepsilon_0\varepsilon_1+1}, \frac{-\varepsilon_0}{\varepsilon_0\varepsilon_1+1}\right), \quad v_4^- = \left(1-\frac{Y}{Rp_1}, \frac{1}{Rp_1}\right).$$

$$v_3^+ = \left(\frac{\varepsilon_0^2}{\varepsilon_0\varepsilon_1+1}, \frac{\varepsilon_1}{\varepsilon_0\varepsilon_1+1}\right), \quad v_4^+ = \left(-1+\frac{Y}{Rp_2}, -\frac{1}{Rp_2}\right).$$

For the second collision,

$$v_3^- = \left(\frac{-\varepsilon_1}{\varepsilon_0}, \frac{-1}{\varepsilon_0}\right),\ v_4^- = \left(1, \frac{\sqrt{2}}{\varepsilon_0}\right), \quad v_3^+ = \left(1, \frac{-1}{\varepsilon_0}\right),\ v_4^+ = \left(\frac{-\varepsilon_1}{\varepsilon_0}, \frac{\sqrt{2}}{\varepsilon_0}\right).$$

B.2. **Numerical information for a particularly chosen $\varepsilon_0 = 1/2$.** For the first collision $e_3 : \frac{1}{2} \to \frac{\sqrt{3}}{2}$.
We want to figure out the Delaunay coordinates (L, u, G, g) for both Q_3 and Q_4. (Here we replace ℓ by u for convenience.) The first collision point is

$$(X, Y) = (-\varepsilon_0\varepsilon_1, \varepsilon_0+\varepsilon_1) = \left(-\frac{\sqrt{3}}{4}, \frac{1+\sqrt{3}}{2}\right).$$

Before collision

$$(L,u,G,g)_3^- = \left(1, -\frac{5\pi}{6}, \frac{\sqrt{3}}{2}, \pi/2\right), \quad (L,u,G,g)_4^- = (1, 1.40034, p_1, -\arctan p_1),$$

$$v_3^- = \left(\frac{-3}{\sqrt{3}+4}, \frac{-2}{\sqrt{3}+4}\right) \simeq -(0.523, 0.349),$$

$$v_4^- = \left(1-\frac{2(1+\sqrt{3})}{(4+\sqrt{3})p_1}, \frac{4}{(4+\sqrt{3})p_1}\right) \simeq (-0.805, 1.322),$$

where

$$p_1 = \frac{-Y+\sqrt{Y^2+4(X+R)}}{2} = \frac{-(\varepsilon_0+\varepsilon_1)+\sqrt{5+2\varepsilon_0\varepsilon_1}}{2} = 0.52798125.$$

After collision

$$(L,u,G,g)_3^+ = \left(1, \frac{2\pi}{3}, -\frac{1}{2}, \pi/2\right), \quad (L,u,G,g)_4^+ = (1, 0.515747, -p_2, -\arctan p_2),$$

$$v_3^+ = \left(\frac{1}{\sqrt{3}+4}, \frac{2\sqrt{3}}{\sqrt{3}+4}\right) \simeq (0.174, 0.604),$$

$$v_4^+ = \left(-1+\frac{2(1+\sqrt{3})}{(4+\sqrt{3})p_2}, -\frac{4}{(4+\sqrt{3})p_2}\right) \simeq (-1.503, 0.368)$$

where

$$p_2 = \frac{-Y - \sqrt{Y^2 + 4(X+R)}}{2} = \frac{-(\varepsilon_0 + \varepsilon_1) - \sqrt{5 + 2\varepsilon_0\varepsilon_1}}{2} = -1.894006654.$$

For the second collision $e_3 : \frac{\sqrt{3}}{2} \to \frac{1}{2}$.

The collision point is $(X,Y) = (\varepsilon_0^2, 0) = \left(\frac{1}{4}, 0\right)$.

Before collision

$$(L,u,G,g)_3^- = \left(1, -\frac{\pi}{6}, -\frac{1}{2}, \pi/2\right),\ (L,u,G,g)_4^- = \left(1, 0.20273, \sqrt{2}/2, -\arctan\frac{\sqrt{2}}{2}\right),$$

$$v_3^- = \left(-\sqrt{3}, -2\right), \quad v_4^- = \left(1, 2\sqrt{2}\right).$$

After collision

$$(L,u,G,g)_3^+ = \left(\frac{1}{\sqrt{3}}, \frac{\pi}{3}, -\frac{1}{2}, -\frac{\pi}{2}\right),\ (L,u,G,g)_4^+ = \left(\frac{1}{\sqrt{3}}, -0.45815, \frac{\sqrt{2}}{2}, \arctan\frac{\sqrt{6}}{2}\right),$$

$$v_3^+ = (1, -2), \quad v_4^+ = \left(-\sqrt{3}, 2\sqrt{2}\right).$$

B.3. **Control the shape of the ellipse.** As it was mentioned before Lemma 2.1 was stated by Gerver in [G2]. There is a detailed proof of part (a) of our Lemma 2.1 in [G2]. However since no details of the proof of part (b) were given in [G2] we go other main steps here for the reader's convenience even though computations are quite straightforward.

Proof of Lemma 2.1. Recall that Gerver's map depends on a free parameter e_4 (or equivalently G_4). In the computations below however it is more convenient to use the polar angle ψ of the intersection point as the free parameter. It is easy to see that as G_4 changes from large negative to large positive value the point of intersection covers the whole orbit of Q_3 so it can be used as the free parameter. Our goal is to show that by changing the angles ψ_1 and ψ_2 of the first and second collision we can prescribe the values of $\bar{\bar{e}}_3$ and $\bar{\bar{g}}_3$ arbitrarily. Due to the Implicit Function Theorem it suffices to show that

$$\det\begin{bmatrix} \frac{\partial\bar{\bar{e}}_3}{\partial\psi_1} & \frac{\partial\bar{\bar{g}}_3}{\partial\psi_1} \\ \frac{\partial\bar{\bar{e}}_3}{\partial\psi_2} & \frac{\partial\bar{\bar{g}}_3}{\partial\psi_2} \end{bmatrix} \neq 0.$$

To this end we use the following set of equations

$$G_3^+ + G_4^+ = G_3^- + G_4^-, \tag{B.1}$$

$$\frac{e_3^+}{G_3^+}\cos(\psi + g_3^+) + \frac{e_4^+}{G_4^+}\cos(\psi - g_4^-) = \frac{e_3^-}{G_3^-}\cos(\psi + g_3^-) + \frac{e_4^-}{G_4^-}\cos(\psi - g_4^-), \tag{B.2}$$

(B.3) $$\frac{(G_3^+)^2}{1-e_3^+\sin(\psi+g_3^+)}=\frac{(G_3^-)^2}{1-e_3^-\sin(\psi+g_3^-)},$$

(B.4) $$\frac{(G_3^+)^2}{1-e_3^+\sin(\psi+g_3^+)}=\frac{(G_4^+)^2}{1-e_4^+\sin(\psi-g_4^+)},$$

(B.5) $$g_4^+=\arctan\frac{G_4^+}{L_4^+}.$$

Here e_3, e_4 and L_4 are functions of the other variables according to the formulas of Appendix A.

(B.1)–(B.5) are obtained as follows. (B.1) is the angular momentum conservation, (B.3) means that the position of Q_3 does not change during the collision, (B.4) means that Q_3 and Q_4 are at the same point immediately after the collision and (B.5) says that after the collision the outgoing asymptote of Q_4 is horizontal.

It remains to derive (B.2). Represent the position vector as $\vec{r}=r\hat{e}_r$. Then the velocity is $\dot{\vec{r}}=\dot{r}\hat{e}_r+r\dot{\psi}\hat{e}_\psi$. The momentum conservation gives

$$(\dot{\vec{r}}_3)^-+(\dot{\vec{r}}_4)^-=(\dot{\vec{r}}_3)^++(\dot{\vec{r}}_4)^+.$$

Taking the angular component of the velocity we get

(B.6) $$r_3^-\dot{\psi}_3^-+r_4^-\dot{\psi}_4^-=r_3^+\dot{\psi}_3^++r_4^+\dot{\psi}_4^+.$$

In our notation the polar representation of the ellipse takes form $r=\dfrac{G^2}{1-e\sin(\psi+g)}$. Differentiating this equation we obtain the following relation for the radial component of the Kepler motion

$$\dot{r}=\frac{G^2}{(1-e\sin(\psi+g))^2}e\cos(\psi+g)\dot{\psi}=\frac{r^2}{G^2}e\cos(\psi+g)\frac{G}{r^2}=\frac{e}{G}\cos(\psi+g).$$

Plugging this into (B.6) we obtain (B.2).

We can write (B.1)–(B.5) in the form

$$\mathbb{F}(Z^-,\tilde{Z},Z^+)=0$$

where $Z^-=(E_3^-,G_3^-,g_3^-,\psi)$, $Z^+=(E_3^+,G_3^+,g_3^+,G_4^+,g_4^+)$, and $\tilde{Z}=(G_4^-,g_4^-)$ are considered as functions Z^-.

By the Implicit Function Theorem we have

$$\frac{\partial Z^+}{\partial Z^-}=-\left(\frac{\partial\mathbb{F}}{\partial Z^+}\right)^{-1}\left(\frac{\partial\mathbb{F}}{\partial Z^-}+\frac{\partial\mathbb{F}}{\partial\tilde{Z}}\frac{\partial\tilde{Z}}{\partial Z^-}\right).$$

Thus to complete the computation we need to know $\frac{\partial \tilde{Z}}{\partial Z^-}$. In order to compute this expression we use the equations

$$g_4^- = -\arctan \frac{G_4^-}{L_4^-} \tag{B.7}$$

which means that the incoming asymptote of Q_4 is horizontal and

$$\frac{(G_3^-)^2}{1 - e_3^- \sin(\psi + g_3^-)} = \frac{(G_4^-)^2}{1 - e_4^- \sin(\psi - g_4^-)}, \tag{B.8}$$

which means that Q_3 and Q_4 are at the same place immediately before the collision. Writing these equations as $\mathbb{I}(Z^-, \tilde{Z}) = 0$ we get by the Implicit Function Theorem

$$\frac{\partial \tilde{Z}}{\partial Z^-} = -\left(\frac{\partial \mathbb{I}}{\partial \tilde{Z}}\right)^{-1} \frac{\partial \mathbb{I}}{\partial Z^-}$$

so that the required derivative equals to

$$\frac{\partial Z^+}{\partial Z^-} = -\left(\frac{\partial \mathbb{F}}{\partial Z^+}\right)^{-1} \left(\frac{\partial \mathbb{F}}{\partial Z^-} - \frac{\partial \mathbb{F}}{\partial \tilde{Z}} \left(\frac{\partial \mathbb{I}}{\partial \tilde{Z}}\right)^{-1} \frac{\partial \mathbb{I}}{\partial Z^-}\right). \tag{B.9}$$

Combining (B.9) with the formula

$$de_3 = -\frac{2G_3E_3dG_3 + G_3^2 dE_3}{\sqrt{1 - 2G_3^2E_3}}$$

which follows from the relation $e_3 = \sqrt{1 - 2G_3^2E_3}$ we obtain the two entries

$$\frac{\partial \bar{\bar{e}}_3}{\partial \psi_2} = -0.158494 \text{ and } \frac{\partial \bar{\bar{g}}_3}{\partial \psi_2} = 0.369599.$$

The meanings of these two entries are the changes of the eccentricity and argument of periapsis after the second collision if we vary the phase of the *second* collision.

We need more work to figure out the two entries $\frac{\partial \bar{\bar{e}}_3}{\partial \psi_1}$ and $\frac{\partial \bar{\bar{g}}_3}{\partial \psi_1}$, which are the changes of the eccentricity and argument of periapsis after the second collision if we vary the phase of the *first* collision. We describe the computation of the first entry, the second one is similar. We use the relation

$$\frac{\partial \bar{\bar{e}}_3}{\partial \psi_1} = \frac{\partial \bar{\bar{e}}_3}{\partial \bar{E}_3^+} \frac{\partial \bar{E}_3^+}{\partial \psi_1} + \frac{\partial \bar{\bar{e}}_3}{\partial \bar{G}_3^+} \frac{\partial \bar{G}_3^+}{\partial \psi_1} + \frac{\partial \bar{\bar{e}}_3}{\partial \bar{g}_3^+} \frac{\partial \bar{g}_3^+}{\partial \psi_1}.$$

Now $\left(\frac{\partial \bar{E}_3^+}{\partial \psi_1}, \frac{\partial \bar{G}_3^+}{\partial \psi_1}, \frac{\partial \bar{g}_3^+}{\partial \psi_1}\right)$ is computed using (B.9) and the data for the first collision. Noticing that the quantities E_3, G_3, g_3 after the first collision are the same as those before the second collision, we replace $\left(\frac{\partial \bar{\bar{e}}_3}{\partial \bar{E}_3^+}, \frac{\partial \bar{\bar{e}}_3}{\partial \bar{G}_3^+}, \frac{\partial \bar{\bar{e}}_3}{\partial \bar{g}_3^+}\right)$ by $\left(\frac{\partial \bar{\bar{e}}_3}{\partial \bar{\bar{E}}_3^-}, \frac{\partial \bar{\bar{e}}_3}{\partial \bar{\bar{G}}_3^-}, \frac{\partial \bar{\bar{e}}_3}{\partial \bar{\bar{g}}_3^-}\right)$

and compute it using (B.9) and the data for the second collision. It turns out that the resulting matrix is

$$\begin{bmatrix} \dfrac{\partial \bar{\bar{e}}_3}{\partial \psi_1} & \dfrac{\partial \bar{\bar{g}}_3}{\partial \psi_1} \\ \dfrac{\partial \bar{\bar{e}}_3}{\partial \psi_2} & \dfrac{\partial \bar{\bar{g}}_3}{\partial \psi_2} \end{bmatrix} = \begin{bmatrix} 0.620725 & 2.9253 \\ -0.158494 & 0 \end{bmatrix},$$

which is obviously nondegenerate. □

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