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A Local Limit Theorem for sums of independent random vectors *

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Abstract

We prove a local limit theorem for sums of independent random vectors satisfying appropriate tightness assumptions. In particular, the local limit theorem holds in dimension 1 if the summands are uniformly bounded.

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1 Introduction.

1.1 The main result.

A classical Local Limit Theorem says that the distribution of the sum of i.i.d. random variables considered at a small scale is approximately invariant with respect to translations by a large[1] subgroup of $\mathbb{R}^d$. Several authors addressed a generalization of this result for non-identically distributed terms (see e.g. [1, 2, 4, 5, 6, 7, 8, 9, 11] and references therein). Here we show that a reasonable theory can be obtained if we impose appropriate tightness assumptions on individual summands.

Consider a sum $S_N = \sum_{j=1}^N X_j$ where X_j are independent, $\mathbb{R}^d$ valued random variables such that

$$\mathbb{E}(X_j) = 0, \tag{1.1}$$

$$\mathbb{E}(|X_j|^3) \le \mathfrak{m}_3 \tag{1.2}$$

and there exists a constant $\varepsilon_0 > 0$ such that for each $s \in \mathbb{R}^d$

$$\mathbb{E}(\langle X_j, s\rangle^2) \ge \varepsilon_0 |s|^2. \tag{1.3}$$

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[1]in the sense that the quotient of $\mathbb{R}^d$ by that subgroup is a compact group

Note that in the presence of (1.2) condition (1.3) is equivalent to existence of $\varepsilon_1, \varepsilon_2 > 0$ such that for each proper affine subspace $\Pi \subset \mathbb{R}^d$ we have

$$\mathbb{P}(d(X_j, \Pi) \leq \varepsilon_1) \leq 1 - \varepsilon_2. \tag{1.4}$$

Let V_N denote the covariance matrix

$$V_{N,l_1,l_2} = \sum_{j=1}^{N} \mathbb{E}(X_{j,(l_1)} X_{j,(l_2)})$$

(here and below we denote by $X_{(l)}$ the l-th coordinate of vector X).

We call a closed subgroup $H \subset \mathbb{R}^d$ *sufficient* if there is a deterministic sequence a_N such that $S_N - a_N \mod H$ converges almost surely. The *minimal subgroup,* denoted by $\mathcal{H}$, is defined as the intersection of all sufficient subgroups.

Proposition 1.1. *(a) If H is sufficient then $\mathbb{R}^d/H$ is compact.*

(b) The minimal subgroup is sufficient.

If $\mathcal{H}$ is a proper subgroup of $\mathbb{R}^d$ we call the sequence $\{X_N\}$ *arithmetic*, otherwise it is called *nonarithmetic*[2].

Due to Proposition 1.1 there exists a bounded sequence a_N such that $S_N - a_N \mod \mathcal{H}$ converges almost surely. Fix such a sequence and denote the limiting random variable by $\mathbb{S}$.

We refer the reader to Subsection 1.3 for examples of computation of the minimal subgroup for $d = 1$.

Given a random variable Y let $\mathcal{C}_Y$ be the convolution operator

$$\mathcal{C}_Y(g)(x) = \mathbb{E}(g(x + Y)).$$

We denote by $C(\mathbb{R}^d)$ (respectively $C^r(\mathbb{R}^d)$) the space of continuous (respectively r times differentiable) functions on $\mathbb{R}^d$. The subscript 0 indicates that we consider only functions of compact support in the corresponding space.

Theorem 1.2. *For each $g \in C_0(\mathbb{R}^d)$ for each sequence $z_N = \mathcal{O}(\sqrt{N})$ such that $z_N - a_N \in \mathcal{H}$ we have*

$$\lim_{N\to\infty} \left[\frac{\mathbb{E}(g(S_N - z_N))}{u_N(z_N)} \right] = \int_{\mathcal{H}} \mathcal{C}_{\mathbb{S}}(g)(h) d\lambda_{\mathcal{H}}(h)$$

where $\lambda_{\mathcal{H}}$ is the Haar measure on $\mathcal{H}$ and $u_N(z)$ is the density of the normal random variable with zero mean and covariance V_N.

In particular, in the non-arithmetic case for each sequence $z_N = \mathcal{O}(\sqrt{N})$ we have

$$\lim_{N\to\infty} \left[\frac{\mathbb{E}(g(S_N - z_N))}{u_N(z_N)} \right] = \int_{\mathbb{R}^d} g(x) dx.$$

The Haar measure in the above theorem is defined as follows. $\mathcal{H}$ is isomorphic to the product of $\mathbb{Z}^{d_1} \times \mathbb{R}^{d-d_1}$. $\lambda_{\mathcal{H}}$ is the product of the counting measure on the first factor and the Lebesgue measure on the second factor normalized as follows. Choose a set D so that each $x \in \mathbb{R}^d$ can be uniquely written as $x = h + \theta$ where $h \in \mathcal{H}$, $\theta \in D$. $\lambda_{\mathcal{H}}$ is normalized so that

$$\int_{\mathbb{R}^d} g(x) dx = \int_{\mathcal{H}} \int_D g(h + \theta) d\lambda_{\mathcal{H}}(h) d\lambda_D(\theta) \tag{1.5}$$

where λ_D is the Lebesgue measure on D normalized to have total volume 1.

[2] Sometimes in the literature the term *arithmetic* is reserved to the case where $\mathcal{H}$ is a discrete subgroup of $\mathbb{R}^d$ while the case where it has both discrete and continuous parts is called *mixed* but in our presentation we will not distinguish between those two cases.

1.2 One dimensional case.

If $d = 1$ there are several simplifications. Namely V_N is a scalar and $\mathcal{H}$ is either $\mathbb{R}$ or $h\mathbb{Z}$ for some $h \in \mathbb{R}$. So Theorem 1.2 can be restated as follows.

Corollary 1.3. *Either*

(i) for each $g \in C_0(\mathbb{R})$ for each sequence z_N such that $\lim_{N\to\infty} \frac{z_N}{\sqrt{V_N}} = z$

$$\lim_{N\to\infty} \left[\sqrt{V_N}\mathbb{E}(g(S_N - z_N))\right] = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)dx \tag{1.6}$$

or (ii) there exists $h > 0$ and a bounded sequence a_N such that $S_N - a_N \mod h$ converges almost surely to a random variable $\mathbb{S}$ and for each $g \in C_0(\mathbb{R})$ for each sequence z_N such that $z_N = a_N + k_N h$ with $k_N \in \mathbb{Z}$ and $\lim_{N\to\infty} \frac{z_N}{\sqrt{V_N}} = z$

$$\lim_{N\to\infty} \left[\sqrt{V_N}\mathbb{E}(g(S_N - z_N))\right] = \frac{he^{-z^2/2}}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} \mathcal{C}_{\mathbb{S}}(g)(jh).$$

In Section 8 we deduce the following consequence of this result.

Corollary 1.4. *Let X_j be independent random variables of zero mean which are uniformly bounded (that is, there is $\mathcal{K}$ such that $|X_j| \le \mathcal{K}$ with probability one). Then either S_N converges almost surely to some random variable $\mathbb{S}$ in which case*

$$\sqrt{V_N}\mathbb{E}(g(S_N)) \to \sqrt{V(\mathbb{S})}\mathbb{E}(g(\mathbb{S})) \tag{1.7}$$

or S_N satisfies the conclusions of Corollary 1.3.

1.3 Examples.

Here we provide several examples of computing the minimal subgroup, the normalizing sequence a_N and the shape of local distribution $\mathbb{S}$.[3]

They provide a good illustration of versatility of Corollary 1.4, even though the computations in each individual example presented below could be done by hand. Namely, all cases where $\mathcal{H} \neq \mathbb{R}$ follow immediately from Kolmogorov's Three Series Theorem. The cases where $\mathcal{H} = \mathbb{R}$ seem a little more tricky and could be most easily analyzed with the help of Lemma 3.2.

Example 1.5. X_1 has a continuous distribution and X_n for $n \ge 2$ are i.i.d and $\mathbb{P}(X_n \in a + h\mathbb{Z}) = 1$ where h is the maximal number with this property. Then

$$\mathcal{H} = h\mathbb{Z}, \quad a_N = Na \bmod h, \quad \mathbb{S} = X_1.$$

Example 1.6. X_n are integer valued and $|X_n| \le M$ with probability 1. According to Corollary 1.4 there are two cases

(I) $\sum_N (X_N - \mathbb{E}(X_N))$ converges[4]. Let b_N be the closest integer to $\mathbb{E}(X_N)$. Then either $X_N = b_N$ or $|X_N - \mathbb{E}(X_N)| \ge 1/2$. Therefore the case (b1) is characterized by the condition

$$\sum_N \left(1 - \max_k P(X_N = k)\right) < \infty.$$

[3] The reader should keep in mind that the choices of a_N and $\mathbb{S}$ are not unique. Namely, we can replace $(a_N, \mathbb{S})$ by $(a_N + \tilde{a}_N + c, \mathbb{S} - c)$ where c is an arbitrary constant and $\tilde{a}_N$ is a sequence converging to 0. In Examples 1.5–1.8 we give one possible choice.

[4] Note that we do not assume here that X_N have zero mean since $\mathbb{E}(X_N)$ need not be an integer, so we can not reduced the general case to the zero mean case by subtracting the mean.

(II) The minimal subgroup is $h\mathbb{Z}$ for some $h \le 2M$. Note that the same argument as in (b1) shows that $h\mathbb{Z}$ is sufficient iff

$$\sum_N \left(1 - \max_k P(X_N \equiv k \text{ mod } h)\right) \tag{1.8}$$

converges.

We now distinguish to further subcases:

(IIa) The series (1.8) converges only for $h = 1$. In this case $\mathbb{S} = 0$ and we obtain the classical arithmetic local limit theorem

$$\sqrt{V_N}\mathbb{P}(S_N = k_N) \to \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{if} \quad \frac{k_N}{\sqrt{V_N}} \to z.$$

(IIb) The maximal h for which the series (1.8) converges is larger than 1. In this case $\mathcal{H} = h\mathbb{Z}$ with h as above,

$$a_N = \sum_{n=1}^{N} k_n \text{ mod } h, \text{ where } k_n = \arg\max \mathbb{P}(X_n \equiv k \text{ mod } h)$$

and $\mathbb{S} = \sum_{n=1}^{\infty}(X_n - k_n)$ (note that due to Borel-Cantelli Lemma this sum has only finitely many non-zero terms with probability 1).

The LLT in Example 1.6 is proven in [10] (except that our results are slightly more precise in case (IIb). The fact that (1.2) and (1.3) are sufficient for the LLT is noted in [12] which obtains the LLT under slightly weaker conditions than (1.2) and (1.3) (under the assumption that X_N are integer valued!).

Example 1.7. $X_n = \xi_n + \varepsilon_n \eta_n$ where $\{\xi_n\}$ and $\{\eta_n\}$ are i.i.d random variables, ξs and ηs are independent, ξ_n take values ± 1 with probability $\frac{1}{2}$ and η_n have continuous distribution with finite third moment. Then either

(I) $\sum_n \varepsilon_n^2$ converges and

$$\mathcal{H} = 2\mathbb{Z}, \quad a_N = N \text{ mod } 2, \quad \mathbb{S} = \sum_{n=1}^{\infty} \varepsilon_n \eta_n$$

or (II) $\sum_n \varepsilon_n^2$ diverges in which case $\mathcal{H} = \mathbb{R}$ and we are in the non-arithmetic situation.

Example 1.8.

$$\mathbb{P}(X_n = -1) = \frac{1}{2} + p_n, \quad \mathbb{P}(X_n = 1 + \varepsilon_n) = \frac{1}{2} - p_n, \text{ where } \varepsilon_n = \frac{4p_n}{1 - 2p_n}$$

(so that $\mathbb{E}(X_n) = 0$). We assume that $p_n \to 0$. Then either

(I) $\sum_n \varepsilon_n^2$ converges (which is equivalent to the convergence of $\sum_n p_n^2$). Then

$$\mathcal{H} = 2\mathbb{Z}, \quad a_N = \left(N + \frac{1}{2}\sum_{n=1}^{N} \varepsilon_n\right) \text{ mod } 2, \quad \mathbb{S} = \sum_{n=1}^{\infty} \varepsilon_n \left(1_{X_n = 1+\varepsilon_n} - \frac{1}{2}\right)$$

or (II) $\sum_n \varepsilon_n^2$ diverges in which case $\mathcal{H} = \mathbb{R}$ and we are in the non-arithmetic situation.

1.4 Plan of the paper.

In Section 2 we prove Proposition 1.1. In Section 3 we show that the non-arithmetic case is characterized by the condition that the characteristic function of S_N tends to 0 everywhere except for the origin. In Section 4 we show that if the characteristic function is large at some point then it decays rapidly nearby. This estimate is used in Section 5 to prove the Local Limit Theorem for test functions whose Fourier transform is compactly supported. In Section 6 we use an approximation argument to prove the Local Limit Theorem for continuous functions of compact support. The proof relies on an auxiliary estimate saying that a probability to visit a cube of a unit size is $\mathcal{O}(\det(V_N^{-1/2}))$. That estimate is established in Section 7. Finally, in Section 8 we prove Corollary 1.4.

Throughout the paper $\hat{g}$ denotes the Fourier transform of a function g. $\mathcal{U}_\varepsilon(A)$ denotes ε-neighborhood of a set $A \subset \mathbb{R}^d$. B_R is a ball of radius R centered at the origin.

2 Minimal subgroup.

We need the following deterministic fact.

Lemma 2.1. *Let $\tilde{H}, \tilde{\tilde{H}}$ be closed subgroups of $\mathbb{R}^d$ such that $\mathbb{R}^d/H$ is a compact subgroup, where $H = \tilde{H} \cap \tilde{\tilde{H}}$. Let s_N be a sequence such that both $s_N \mod \tilde{H}$ and $s_N \mod \tilde{\tilde{H}}$ converge. Then $s_N \mod H$ converges.*

Proof. Let

$$p : \mathbb{R}^d \to \mathbb{R}^d/H, \quad \tilde{p} : \mathbb{R}^d/H \to \mathbb{R}^d/\tilde{H}, \quad \tilde{\tilde{p}} : \mathbb{R}^d/H \to \mathbb{R}^d/\tilde{\tilde{H}}$$

be natural projections,

$$\tilde{s} = \lim_{N\to\infty} s_N \mod \tilde{H}, \quad \tilde{\tilde{s}} = \lim_{N\to\infty} s_N \mod \tilde{\tilde{H}}, \quad \tilde{S} = \tilde{p}^{-1}\tilde{s}, \quad \tilde{\tilde{S}} = \tilde{\tilde{p}}^{-1}\tilde{\tilde{s}}.$$

Note that $\mathrm{Card}(\tilde{S} \cap \tilde{\tilde{S}}) \leq 1$. On the other hand for each $\varepsilon > 0$

$$p(s_N) \in \mathcal{U}_\varepsilon(\tilde{S}) \cap \mathcal{U}_\varepsilon(\tilde{\tilde{S}})$$

provided that N is large enough. It follows that $\tilde{S}$ and $\tilde{\tilde{S}}$ do indeed intersect and $\lim_{N\to\infty} p(s_N) = \tilde{S} \cap \tilde{\tilde{S}}$. □

Proof of Proposition 1.1. (a) If $\mathbb{R}^d/H$ was not compact then we may assume after an appropriate change of variables that all vectors in H have zero last coordinate. That is, $S_{N,(d)} - a_{N,(d)}$ converges almost surely. By (1.2) and (1.3) we can choose R so large that denoting $\mathcal{X}_N = X_{N,(d)} 1_{|X_{N,(d)}| \leq R}$ we have $V(\mathcal{X}_N) \geq \varepsilon_0/2$. Thus $\sum_N V(\mathcal{X}_N)$ diverges and so $S_{N,(d)} - a_{N,(d)}$ diverges due to Kolmogorov's Three Series Theorem.

To prove (b) let $\tilde{H}, \tilde{\tilde{H}}$ be sufficient subgroups such that $S_N - \tilde{a}_N \mod \tilde{H}$ and $S_N - \tilde{\tilde{a}}_N \mod \tilde{\tilde{H}}$ converge. Let

$$\tilde{b}_N = \tilde{a}_N - \tilde{a}_{N-1}, \quad \tilde{\tilde{b}}_N = \tilde{\tilde{a}}_N - \tilde{\tilde{a}}_{N-1}, \quad H = \tilde{H} \cap \tilde{\tilde{H}}.$$

We claim that $\mathbb{R}^d/H$ is compact. Indeed take R so large that

$$\mathbb{P}(|X_N| \geq R) \leq \varepsilon_2/2$$

where ε_2 is the constant from (1.4). By our assumptions for each δ_1, δ_2

$$\mathbb{P}(X_N \in \tilde{b}_N + \mathcal{U}_{\delta_1}(\tilde{H})) \geq 1 - \delta_2, \quad \mathbb{P}(X_N \in \tilde{\tilde{b}}_N + \mathcal{U}_{\delta_1}(\tilde{\tilde{H}})) \geq 1 - \delta_2$$

provided that N is large enough. Hence if $2\delta_2 + \varepsilon_2/2 < 1$ then

$$\mathbb{P}\left(X_N \in \left[\left(\tilde{b}_N + \mathcal{U}_{\delta_1}(\tilde{H})\right) \cap \left(\tilde{\tilde{b}}_N + \mathcal{U}_{\delta_1}(\tilde{\tilde{H}})\right) \cap B_R\right]\right) > 0.$$

Therefore the set $\left(\tilde{b}_N + \mathcal{U}_{\delta_1}(\tilde{H})\right) \cap \left(\tilde{\tilde{b}}_N + \mathcal{U}_{\delta_1}(\tilde{\tilde{H}})\right) \cap B_R$ is non empty, it contains a point $\hat{b}_N$. Then

$$\mathbb{P}(X_N \in \hat{b}_N + (\mathcal{U}_{2\delta_1}(\tilde{H}) \cap \mathcal{U}_{2\delta_1}(\tilde{\tilde{H}}))) \geq 1 - 2\delta_2. \tag{2.1}$$

Take δ_1 so small that

$$(\mathcal{U}_{2\delta_1}(\tilde{H}) \cap \mathcal{U}_{2\delta_1}(\tilde{\tilde{H}})) \cap B_{2R} \subset \mathcal{U}_{\varepsilon_1}(H). \tag{2.2}$$

Now note that if $\mathbb{R}^d/H$ was not compact there would be a proper subspace $L \supset H$ and so (2.1) and (2.2) would contradict (1.4) with $\Pi = \hat{b}_N + L$.

Our next claim is that H is sufficient. Indeed pick $\bar{\omega}$ so that both $S_N(\bar{\omega}) - \tilde{a}_N \mod \tilde{H}$ and $S_N(\bar{\omega}) - \tilde{\tilde{a}}_N \mod \tilde{\tilde{H}}$ converge. Then for almost every ω both $S_N(\omega) - S_N(\bar{\omega}) \mod \tilde{H}$ and $S_N(\omega) - S_N(\bar{\omega}) \mod \tilde{\tilde{H}}$ converge. Now Lemma 2.1 tells us that $S_N - a_N \mod H$ converges almost surely where $a_N = S_N(\bar{\omega})$. Hence H is sufficient.

Observe that $H_0 = \mathbb{R}^d$ is sufficient. If it is not minimal there is a proper sufficient subgroup $H_1 \subset H_0$. If H_1 is minimal we are done. Otherwise there is $H_1' \not\subset H_1$ which is sufficient and by the foregoing discussion $H_2 = (H_1 \cap H_1')$ is sufficient. Continuing we obtain a chain of proper subgroups

$$H_0 \supset H_1 \supset H_2 \cdots \supset H_k \supset \dots$$

such that H_k is sufficient for each k. Note that either $\dim(H_k) < \dim(H_{k-1})$ or $\mathrm{Vol}(H_{k-1}/H_k)$ is an integer greater than 1. On the other hand the proof of part (a) shows that if R is large enough then H_k has a basis in B_R for each k. Thus the chain can not be continued indefinitely ending at some finite r. Then H_r is minimal and it is sufficient by construction.
□

3 Distinguishing between the arithmetic and non-arithmetic cases.

We start with an auxiliary estimate.

Lemma 3.1. *Each random variable $\mathcal{X}$ can be decomposed as $\mathcal{X} = b + \mathcal{Y} + \mathcal{Z}$ where b is a constant, $\mathcal{Z} \in 2\pi\mathbb{Z}$, $|\mathcal{Y}| \leq 2\pi$, $\mathbb{E}(\mathcal{Y}) = 0$, and*

$$|\mathbb{E}(e^{i\mathcal{X}})| \leq 1 - \frac{\mathbb{E}(\mathcal{Y}^2)}{14}.$$

Proof. Let $\mathbb{E}(e^{i\mathcal{X}}) = \rho e^{i\bar{b}}$ where $\rho, \bar{b} \in \mathbb{R}$. Decompose $\mathcal{X} - \bar{b} = \bar{\mathcal{Y}} + \mathcal{Z}$ where $\mathcal{Z} \in 2\pi\mathbb{Z}$ and $|\bar{\mathcal{Y}}| \leq \pi$. Then

$$\rho = \mathbb{E}(e^{i(\mathcal{X}-\bar{b})}) = \Re(\mathbb{E}(e^{i(\mathcal{X}-\bar{b})})) = \mathbb{E}(\cos((\mathcal{X} - \bar{b}))) = \mathbb{E}(\cos(\bar{\mathcal{Y}})).$$

Using that [5] $\cos(x) \leq 1 - \frac{x^2}{14}$ if $|x| \leq \pi$ we get $\rho < 1 - \frac{E(\bar{\mathcal{Y}}^2)}{14} \leq 1 - \frac{V(\bar{\mathcal{Y}})}{14}$. This proves the result with $\mathcal{Y} = \bar{\mathcal{Y}} - \mathbb{E}(\bar{\mathcal{Y}})$ and $b = \bar{b} + \mathbb{E}(\bar{\mathcal{Y}})$. □

We will refer to the decomposition of Lemma 3.1 as *the useful decomposition of $\mathcal{X}$*.

The next result will help us to distinguish between the arithmetic and non-arithmetic cases.

Lemma 3.2. *Let $\mathcal{X}_N$ be independent random variables with zero mean. Let $\mathcal{S}_N = \sum_{n=1}^N \mathcal{X}_n$. The following are equivalent*

(a) There is a sequence a_N such that $\mathcal{S}_N - a_N$ mod 2π converges;

[5] Indeed

$$\cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} = 1 - \frac{x^2}{2}\left(1 - \frac{x^2}{12}\right) \leq 1 - \frac{x^2}{2}\left(1 - \frac{\pi^2}{12}\right) \leq 1 - \frac{x^2}{2} \times \frac{1}{7}.$$

(b) If $\mathcal{X}_N = \mathfrak{b}_N + \mathcal{Y}_N + \mathcal{Z}_N$ is a useful decomposition of $\mathcal{X}_N$ then $\sum_N V(\mathcal{Y}_N)$ converges;
(c)[6] $\lim\limits_{N_0\to\infty}\lim\limits_{N\to\infty}\left|\mathbb{E}\left(e^{i[\mathcal{S}_N-\mathcal{S}_{N_0}]}\right)\right| = 1.$

Proof. If $\mathcal{S}_N - a_N \bmod 2\pi$ converges then

$$\lim_{N_0\to\infty}\lim_{N\to\infty}\left(([\mathcal{S}_N - a_N] - [\mathcal{S}_{N_0} - a_{N_0}]) \mod 2\pi\right) = 0$$

and hence

$$\lim_{N_0\to\infty}\lim_{N\to\infty}\left|\mathbb{E}\left(e^{i[\mathcal{S}_N-\mathcal{S}_{N_0}]}\right)\right| = \lim_{N_0\to\infty}\lim_{N\to\infty}\left|\mathbb{E}\left(e^{i[(\mathcal{S}_N-a_N)-(S_{N_0}-a_{N_0})]}\right)\right| = 1.$$

Therefore (a) implies (c).

If $\lim\limits_{N_0\to\infty}\lim\limits_{N\to\infty}\left|\mathbb{E}\left(e^{i[\mathcal{S}_N-\mathcal{S}_{N_0}]}\right)\right| = 1$ then for large N_0

$$\lim_{N\to-\infty}\left|\mathbb{E}\left(e^{i[(\mathcal{S}_N-a_N)-(S_{N_0}-a_{N_0})]}\right)\right| > 0.$$

Denote this limit by e^{-A}. Combining Lemma 3.1 with the inequality $1-x \le e^{-x}$ we get

$$\sum_N V(\mathcal{Y}_N) \le 14A. \tag{3.1}$$

Therefore (c) implies (b).

Finally (b) implies (a) by Kolmogorov's Three Series Theorem. □

We now return to considering a sequence of independent random vectors X_n with $S_N = \sum_{n=1}^N X_n$. Denote

$$\phi_n(s) = \mathbb{E}(e^{i\langle s, X_n\rangle}), \quad \Phi_N(s) = \mathbb{E}(e^{i\langle s, S_N\rangle}).$$

Corollary 3.3. *(a) If $\mathcal{H} = \mathbb{R}^d$ then $\lim\limits_{N\to\infty}\Phi_N(s) = 0$ for $s \neq 0$.*

(b) If[7] *$\mathcal{H} = \mathbb{Z}^{d_1} + \mathbb{R}^{d-d_1}$ then $\lim\limits_{N\to\infty}\Phi_N(s) = 0$ unless the last $d-d_1$ coordinates of s are 0 and the first d_1 coordinates belong to $2\pi\mathbb{Z}^{d_1}$.*

Proof. By Lemma 3.2 if $\lim\limits_{N\to\infty}|\Phi_N(s)| > 0$ then the group

$$\{h : \langle h, s\rangle \in 2\pi\mathbb{Z}\}$$

is sufficient and so $\langle h, s\rangle \in 2\pi\mathbb{Z}$ for $h \in \mathcal{H}$. □

4 A local estimate

One of standard proofs of the Central Limit Theorem relies on the following bound (see e.g. [3, Section XVI.6]).

Lemma 4.1. *(a) $\lim\limits_{N\to\infty}\Phi_N\left(V_N^{-1/2}u\right) - e^{-u^2/2} = 0$ uniformly on compact sets.*

(b) There are positive constants c, δ_0 such that if $|s| \le \delta_0$ then

$$|\Phi_N(s)| \le e^{-c\langle V_N s, s\rangle}.$$

In this section we extend this result to a neighborhood of an arbitrary point (rather than 0). So fix an arbitrary $\bar{s} \in \mathbb{R}^d$.

[6] In other words $\mathbb{E}(e^{i\mathcal{X}_N})$ vanishes for at most finitely many N and if $\mathbb{E}(e^{i\mathcal{X}_N}) \neq 0$ for $N > N_0$ then $\lim\limits_{N\to\infty}\left|\mathbb{E}\left(e^{i[\mathcal{S}_N-\mathcal{S}_{N_0}]}\right)\right| > 0$.

[7] Here and below $\mathbb{Z}^{d_1} + \mathbb{R}^{d-d_1}$ denotes the set of vectors whose first d_1 coordinates are integers.

Lemma 4.2. *(a) Suppose that*

$$\langle X_N, \bar{s}\rangle = b_N + \mathcal{Y}_N + \mathcal{Z}_N \tag{4.1}$$

where $\mathcal{Z}_N \in 2\pi\mathbb{Z}$, $\mathcal{Y}_N$ *is bounded,* $E(\mathcal{Y}_N) = 0$ *and*

$$\sum_{j=1}^{N} V(\mathcal{Y}_j) \le \varepsilon. \tag{4.2}$$

Let $a_N = \sum_{j=1}^{N} b_j$. *Then for each* $L > 0$ *there exists a constant* C *such that for* $|u| \le L$ *we have*

$$\left|\Phi_N\left(\bar{s} + V_N^{-1/2}u\right)e^{-ia_N} - e^{-u^2/2}\right| \le C\left[\sqrt{\varepsilon} + \frac{1}{\sqrt{N}}\right].$$

(b) There are positive constants M, c, δ_0 *such that if* $|\Phi_N(\bar{s})| = e^{-A_N}$ *for some* $\bar{s} \in \mathbb{R}^d$ *then for* $|\Delta| \le \delta_0$ *we have*

$$|\Phi_N(\bar{s} + \Delta)| \le e^{MA_N - c\langle V_N\Delta, \Delta\rangle} \tag{4.3}$$

Proof. We start with (b). Let $\langle X_N, \bar{s}\rangle = b_N + \mathcal{Y}_N + \mathcal{Z}_N$ be a useful decomposition of $\langle X_N, \bar{s}\rangle$. Then

$$\phi_j(\bar{s} + \Delta) = e^{ib_j}\mathbb{E}(e^{i(\mathcal{Y}_j + \mathcal{X}_j)})$$

where

$$\mathcal{X}_j = \langle \Delta, X_j\rangle. \tag{4.4}$$

Next,

$$e^{i(\mathcal{Y}_j + \mathcal{X}_j)} = 1 + i(\mathcal{Y}_j + \mathcal{X}_j) - \frac{1}{2}\left[\mathcal{Y}_j^2 + \mathcal{X}_j^2 + 2(\mathcal{X}_j\mathcal{Y}_j)\right] + \mathcal{O}\left(|\mathcal{X}_j + \mathcal{Y}_j|^3\right).$$

Note that

$$|\mathcal{X}_j + \mathcal{Y}_j|^3 \le 8\max\left(|\mathcal{X}_j|^3, |\mathcal{Y}_j|^3\right) = \mathcal{O}\left(|\Delta|^3|X_j|^3 + |\mathcal{Y}_j|^3\right).$$

Thus (1.2) gives

$$\mathbb{E}\left(e^{i(\mathcal{Y}_j + \mathcal{X}_j)}\right) = 1 - \frac{1}{2}\left[\mathbb{E}(\mathcal{X}_j^2) + 2\mathbb{E}(\mathcal{X}_j\mathcal{Y}_j)\right] + \mathcal{O}\left(\Delta^3 + \mathbb{E}(\mathcal{Y}_j^2)\right). \tag{4.5}$$

Denoting $\mathfrak{p}_j = -\frac{1}{2}\left[\mathbb{E}(\mathcal{X}_j^2) + 2\mathbb{E}(\mathcal{X}_j\mathcal{Y}_j)\right]$ and writing the remainder term as $\mathcal{P}_j + i\mathcal{Q}_j$ where $(\mathcal{P}_j, \mathcal{Q}_j) = \mathcal{O}\left(\Delta^3 + \mathbb{E}(\mathcal{Y}_j^2)\right)$ are real we get

$$\left|\mathbb{E}\left(e^{i(\mathcal{Y}_j + \mathcal{X}_j)}\right)\right| = \sqrt{1 + 2\mathfrak{p}_j + 2\mathcal{P}_j + 2\mathfrak{p}_j\mathcal{P}_j + \mathfrak{p}_j^2 + \mathcal{P}_j^2 + \mathcal{Q}_j^2} = 1 + \mathfrak{p}_j + \mathcal{O}(\mathfrak{p}_j^2 + \mathcal{P}_j + \mathcal{Q}_j^2)$$

$$= 1 - \frac{1}{2}\left[\mathbb{E}(\mathcal{X}_j^2) + 2\mathbb{E}(\mathcal{X}_j\mathcal{Y}_j)\right] + \mathcal{O}(\Delta^3 + \mathbb{E}(\mathcal{Y}_j^2)),$$

where the last step uses that $\mathfrak{p}_j^2 = \mathcal{O}(\Delta^3 + \mathbb{E}(\mathcal{Y}_j^2))$.

Next, the inequality

$$\ln(1 + x) \le x \tag{4.6}$$

gives

$$\ln\left|\mathbb{E}\left(e^{i(\mathcal{Y}_j + \mathcal{X}_j)}\right)\right| \le -\frac{1}{2}\left[\mathbb{E}(\mathcal{X}_j^2) + 2\mathbb{E}(\mathcal{X}_j\mathcal{Y}_j)\right] + \mathcal{O}\left(\Delta^3 + \mathbb{E}(\mathcal{Y}_j^2)\right).$$

Therefore

$$\ln|\Phi_N(\bar{s} + \Delta)| \le -\sum_{j=1}^{N}\left[\frac{1}{2}\left[\mathbb{E}(\mathcal{X}_j^2) + 2\mathbb{E}(\mathcal{X}_j\mathcal{Y}_j)\right] + \mathcal{O}\left(\Delta^3 + \mathbb{E}(\mathcal{Y}_j^2)\right)\right].$$

Denoting $\mathcal{V}_N = \sum_{j=1}^N \mathbb{E}(\mathcal{X}_j^2)$, $\mathcal{W}_N = \sum_{j=1}^N \mathbb{E}(\mathcal{Y}_j^2)$ and using Cauchy-Schwartz inequality and the fact that $|\Delta|^2 N = \mathcal{O}(\mathcal{V}_N)$, due to (1.3), we get

$$\ln|\Phi_N(\bar{s}+\Delta)| \le -\frac{\mathcal{V}_N}{2} + \mathcal{O}\left(|\Delta|\mathcal{V}_N + \mathcal{W}_N + \sqrt{\mathcal{W}_N\mathcal{V}_N}\right).$$

Since for each R

$$\sqrt{\mathcal{W}_N\mathcal{V}_N} \le \frac{1}{2}\left[\frac{\mathcal{V}_N}{R} + R\mathcal{W}_N\right]$$

we see that for small Δ we have

$$\ln|\Phi_N(\bar{s}+\Delta)| \le -\frac{\mathcal{V}_N}{4} + \mathcal{O}\left(\mathcal{W}_N\right). \tag{4.7}$$

Next, Lemma 3.1 tells us that

$$\mathcal{W}_N \le 14A_N \tag{4.8}$$

so (4.3) follows from (4.7).

To prove part (a) we use (4.5) where $\mathcal{Y}_N$ is from (4.1) and $\mathcal{X}_N$ is given by (4.4). The fact that $\mathcal{Y}_N$ was a part of a useful decomposition was used in part (b) only to get (4.8). Here we have a stronger bound (4.2) by the assumptions of part (a). In particular, (4.2) implies that $\mathbb{E}(\mathcal{Y}_j^2) \le \varepsilon$ so all terms in (4.5) are small. Accordingly we can use the Taylor expansion of $\ln(1+x)$ to conclude that

$$\ln\phi_j(\bar{s}+\Delta) - ib_j = -\frac{\mathbb{E}(\mathcal{X}_j^2)}{2} + \mathcal{O}\left(\mathbb{E}(\mathcal{X}_j\mathcal{Y}_j) + |\Delta|^3 + \mathbb{E}(\mathcal{Y}_j^2)\right).$$

Hence

$$\ln\Phi_N(s+\Delta) - ia_N + \frac{\mathcal{V}_N}{2} = \mathcal{O}\left(\sum_{j=1}^N \mathbb{E}(\mathcal{X}_j\mathcal{Y}_j)\right) + \mathcal{O}\left(N\Delta^3\right) + \mathcal{O}\left(\sum_{j=1}^N \mathbb{E}(\mathcal{Y}_j^2)\right).$$

Using (4.2) to estimate the third term, Cauchy-Schwartz to estimate the first term and the fact that $|\Delta|^2 N = \mathcal{O}(\mathcal{V}_N)$ to estimate the second term we get

$$\ln\Phi_N(\bar{s}+\Delta) - ia_N = -\frac{\mathcal{V}_N}{2} + \mathcal{O}\left(|\Delta|\mathcal{V}_N + \varepsilon + \sqrt{\varepsilon\mathcal{V}_N}\right)$$

as stated. □

Corollary 4.3. *Suppose that*

$$\langle X_N, \bar{s}\rangle = b_N + \mathcal{Y}_N + \mathcal{Z}_N$$

where $\mathcal{Z}_N \in 2\pi\mathbb{Z}$, $\mathcal{Y}_N$ *is bounded,* $E(\mathcal{Y}_N) = 0$ *and* $\sum_N \mathcal{Y}_N$ *converges to* $\tilde{S}$ *almost surely. Then*

$$\lim_{N\to\infty} \Phi_N\left(\bar{s} + V_N^{-1/2}u\right)e^{-ia_N} = e^{-u^2/2}\mathbb{E}\left(e^{i\tilde{S}}\right)$$

uniformly on compact sets.

Proof. Given $\varepsilon > 0$ let $\bar{N}$ be such that

$$\sum_{N=\bar{N}+1}^{\infty} V(\mathcal{Y}_N) \le \varepsilon \text{ and } \left|\mathbb{E}\left(e^{i\sum_{j=1}^{\bar{N}}\mathcal{Y}_j}\right) - \mathbb{E}\left(e^{i\tilde{S}}\right)\right| \le \varepsilon.$$

Then $\Phi_N\left(\bar{s} + V_N^{-1/2}u\right)e^{-ia_N} =$

$$\left[\Phi_{\bar{N}}\left(\bar{s}+V_N^{-1/2}u\right)e^{-ia_{\bar{N}}}\right]\mathbb{E}\left(e^{i\left[(\bar{s}+V_N^{-1/2}u)(S_N-S_{\bar{N}})-(a_N-a_{\bar{N}})\right]}\right) := \Phi'_{\bar{N},N}(\bar{s},u)\Phi''_{\bar{N},N}(\bar{s},u)$$

Note that $\Phi'_{\bar{N},N}(\bar{s},u)$ depends on N only through the term $V_N^{-1/2}u$ so

$$\lim_{N\to\infty} \Phi'_{\bar{N},N}(\bar{s},u) = \mathbb{E}\left(e^{i\sum_{j=1}^{\bar{N}} \mathcal{Y}_j}\right) = \mathbb{E}\left(e^{i\tilde{S}}\right) + \mathcal{O}(\varepsilon).$$

On the other hand Lemma 4.2(a) (applied to $\sum_{j=\bar{N}+1}^{N} X_j$) gives

$$\left|\Phi''_{\bar{N},N}(\bar{s},u) - e^{-u^2/2}\right| = \mathcal{O}\left(\sqrt{\varepsilon} + \left(N-\bar{N}\right)^{-1/2}\right).$$

Since ε can be chosen arbitrary small the result follows. □

5 Observables with compact Fourier transform.

Here we prove that formulas of Theorem 1.2 are valid if $\hat{g}$ is continuous and has compact support. So we suppose that $\mathrm{supp}(\hat{g}) \in [-K,K]^d$ for some K.

5.1 Non-arithmetic case.

Assume first, that $\lim\limits_{N\to 0} \Phi_N(s) = 0$ for all $s \neq 0$. By Corollary 3.3 this happens, in particular, in the non arithmetic case. Note that since $|\Phi_N|$ is monotone in N the convergence is uniform on $[-K,K]^d \backslash (-\delta_0,\delta_0)^d$ for each $\delta_0 > 0$. We select δ_0 so that the conditions of Lemma 4.1(b) and 4.2(b) are satisfied. Divide $[-K,K]^d$ into boxes $\{I_j\}$ of side δ_1 where $\delta_1 \le \delta_0/2d$ so that I_0 is the box centered at 0. Then

$$\begin{aligned}\mathbb{E}(g(S_N - z_N)) &= \frac{1}{(2\pi)^d}\int_{[-K,K]^d} \hat{g}(-s)e^{-i\langle s,z_N\rangle}\Phi_N(s)ds \\ &= \frac{1}{(2\pi)^d}\sum_j \int_{I_j} \hat{g}(-s)e^{-i\langle s,z_N\rangle}\Phi_N(s)ds.\end{aligned}$$

We claim that the main contribution comes from

$$\int_{I_0} \hat{g}(-s)e^{-i\langle s,z_N\rangle}\Phi_N(s)ds = \bar{J}_{L,N} + \bar{\bar{J}}_{L,N}$$

where $\bar{J}_L$ denotes the integral over the set

$$Q_L := \{s : V_N^{1/2}s \in [-L,L]^d\}$$

and $\bar{\bar{J}}_{L,N}$ denotes the integral over $I_0 - Q_L$. Making the change of variables $V_N^{1/2}s = u$ we get by Lemma 4.1(a)

$$\begin{aligned}\det(V_N^{1/2})\bar{J}_{L,N} &= \int_{[-L,L]^d} \hat{g}\left(-V_N^{-1/2}u\right)e^{-i\langle V_N^{-1/2}u,z_N\rangle}\Phi_N\left(V_N^{-1/2}u\right)du \\ &= \hat{g}(0)\left[\int_{[-L,L]^d} e^{-u^2/2-i\langle u,\bar{z}_N\rangle}du\right](1+o_{N\to\infty}(1)) \\ &= \hat{g}(0)e^{-\bar{z}_N^2/2}\left[(2\pi)^{d/2} + o_{L\to\infty}(1) + o_{N\to\infty}(1)\right]\end{aligned}$$

where

$$\bar{z}_N = V_N^{-1/2}z_N. \tag{5.1}$$

On the other hand, by Lemma 4.1(b)

$$\det(V_N^{1/2})\bar{\bar{J}}_{L,N} \le \mathrm{Const}\int_{\mathbb{R}^d-[-L,L]^d} e^{-cu^2}du = o_{L\to\infty}(1).$$

Since this holds for all L we can let $L \to \infty$ to conclude that

$$\lim_{N\to\infty} e^{\bar{z}_N^2/2} \det(V_N^{1/2}) \int_{I_0} \hat{g}(-s)\Phi_N(s)ds = (2\pi)^{d/2}\hat{g}(0) = (2\pi)^{d/2} \int_{\mathbb{R}^d} g(x)dx.$$

It remains to show that the contributions of I_j with $j \neq 0$ are smaller.

Lemma 5.1. *If $\mathfrak{I}$ be a cube of size δ_1 such that $\Phi_N(s)$ converges to 0 on $\mathfrak{I}$. Then*

$$\lim_{N\to\infty} \det\left(V_N^{1/2}\right) \int_{\mathfrak{I}} |\Phi_N(s)|ds = 0.$$

Proof. Let

$$e^{-\mathfrak{A}_N} = \max_{\mathfrak{I}} |\Phi_N(s)| \text{ and } \bar{s}_N = \arg\max_{\mathfrak{I}} |\Phi_N(s)|.$$

Split $\displaystyle\int_{\mathfrak{I}} |\Phi_N(s)|ds = \bar{J}_N + \bar{\bar{J}}_N$ where $\bar{J}_N$ denotes the integral over the set

$$\mathfrak{Q}_N := \{c\langle V_N\Delta, \Delta\rangle < 2M\mathfrak{A}_N\} \text{ where } \Delta = s - \bar{s}_N.$$

and $\bar{\bar{J}}_N$ denotes the integral over $\mathfrak{I} - \mathfrak{Q}_N$. Since $\mathfrak{Q}_N$ is contained in a ball or radius $\mathcal{O}\left(\sqrt{\mathfrak{A}_N/N}\right)$ we have

$$\det(V_N^{1/2})\bar{J}_N = \mathcal{O}\left((\mathfrak{A}_N)^{d/2} e^{-\mathfrak{A}_N}\right) \to 0$$

since $\mathfrak{A}_N \to \infty$ as $N \to \infty$. On the other hand, by Lemma 4.2(b)

$$\begin{aligned} |\bar{\bar{J}}_N| &\leq \text{Const} \int_{c\langle V_N\Delta,\Delta\rangle \geq 2MA_{N,j}} e^{-c\langle V_N\Delta,\Delta\rangle} d\Delta \\ &\leq \frac{\text{Const}}{N^{d/2}} \int_{|u|>\bar{c}\sqrt{\mathfrak{A}_N}}^{\infty} e^{-cu^2} du = \mathcal{O}\left(\frac{\mathfrak{A}_N^{d-1/2}}{N^{d/2}} e^{-c\mathfrak{A}_N}\right). \end{aligned}$$

Combining the estimates for $\bar{J}_N$ and $\bar{\bar{J}}_N$ we obtain the lemma. □

Lemma 5.1 shows that the main contribution to $\mathbb{E}(g(S_N))$ comes from I_0 so that

$$e^{\bar{z}_N^2/2} \det\left(V_N^{1/2}\right) \mathbb{E}(g(S_N - z_N)) \to \left(\frac{\sqrt{2\pi}}{2\pi}\right)^d \hat{g}(0) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(x)dx$$

as claimed.

5.2 Arithmetic case.

Next, we consider the arithmetic case. Let $\mathcal{H}$ be the minimal subgroup. After a linear change of variables we can assume that[8] $\mathcal{H} = \mathbb{Z}^{d_1} + \mathbb{R}^{d-d_1}$. Let $X_N = b_N + Y_N + Z_N$ be the decomposition of X_N such that $X_{N,(l)} = b_{N,(l)} + Y_{N,(l)} + Z_{N,(l)}$ is a useful decomposition for $l \leq d_1$ and $b_{N,(l)} = Y_{N,(l)} = 0$ for $l > d_1$. Let $\tilde{S}_N = (S_N - a_N) \mod \mathcal{H}$. Due to Lemma 3.2 we may (and will) assume that a_N is chosen so that

$$\tilde{S}_N = \sum_{j=1}^{N} Y_j \mod \mathcal{H}.$$

[8]Here $\mathbb{Z}^{d_1}$ is the set of vectors whose first d_1 coordinates are integers and the last $d - d_1$ coordinates are zero and $\mathbb{R}^{d-d_1}$ is the set of vectors whose first d_1 coordinates are zero.

Lemma 5.1 shows that the main contribution to

$$\det\left(V_N^{1/2}\right)\mathbb{E}(g(S_N-z_N))$$

comes from small cubes $I(s_m)$ centered at points s_m where

$$\lim_{N\to\infty}|\Phi_N(s_m)|>0.$$

By Corollary 3.3 these points have form $s_m=2\pi m$ with $m\in\mathbb{Z}^{d_1}$. The contribution of $m=0$ is $\frac{e^{-\bar{z}_N^2/2}}{(2\pi)^{d/2}}\hat{g}(0)$ as before.

For $m\neq 0$ note that $e^{i\langle s_m, z_N-a_N\rangle}=1$. Let $\Delta=s-s_m$. Then

$$\frac{1}{(2\pi)^d}\int_{I(s_m)}\hat{g}(-s)e^{-i\langle s,z_N\rangle}\mathbb{E}(e^{i\langle s,S_N\rangle})ds$$

$$=\frac{1}{(2\pi)^d}\int_{I(s_m)}\hat{g}(-s)e^{-i\langle\Delta,(z_N-a_N)\rangle}\mathbb{E}(e^{i\langle s,(S_N-a_N)\rangle})ds.$$

Denoting

$$Q_{m,L,N}=\{s:V_N^{1/2}\Delta\in[-L,L]^d\}$$

we decompose the last integral as $\bar{J}_{m,L,N}+\bar{\bar{J}}_{m,L,N}$ where $\bar{J}_{m,L,N}$ is the integral over $Q_{m,L,N}$ and $\bar{\bar{J}}_{m,L,N}$ is the integral over $I(s_m)-Q_{m,L,N}$. By Corollary 4.3

$$\frac{\det\left(V_N^{1/2}\right)\bar{J}_{j,L,N}}{(2\pi)^d}=\frac{\hat{g}(-s_m)\mathbb{E}\left(e^{i\langle s_m,\mathbb{S}\rangle}\right)+o_{N\to\infty}(1)}{(2\pi)^d}\int_{[-L,L]^d}e^{-u^2/2-i\langle\bar{z}_N,u\rangle}du$$

$$=e^{-\bar{z}_N^2/2}\frac{\hat{g}(-s_m)\mathbb{E}\left(e^{i\langle s_m,\mathbb{S}\rangle}\right)}{(2\pi)^{d/2}}+o_{N\to\infty,L\to\infty}(1)$$

where $\bar{z}_N$ is defined by (5.1). On the other hand by Lemma 4.2(b)

$$\det\left(V_N^{1/2}\right)|\bar{\bar{J}}_{m,L,N}|\leq\text{Const}\int_{\mathbb{R}^d-[-L,L]^d}e^{-cu^2}du=o_{L\to\infty}(1).$$

Since this holds for all L we can let $L\to\infty$ to conclude that

$$\lim_{N\to\infty}e^{\bar{z}_N^2/2}\frac{\det\left(V_N^{1/2}\right)}{(2\pi)^d}\int_{U(s_j)}\hat{g}(-s)e^{-isz_N}\mathbb{E}(e^{isS_N})ds \tag{5.2}$$

$$=\frac{\hat{g}(-s_m)\mathbb{E}(e^{i\langle s_m,\mathbb{S}\rangle})}{(2\pi)^{d/2}}=\frac{\widehat{\mathcal{C}_{\mathbb{S}}g}(-s_m)}{(2\pi)^{d/2}}.$$

Note that the argument above relies on Corollary 4.3, so it only works under the assumption that $|\Phi_N(s_m)|\not\to 0$. However if $\Phi_N(s_m)\to 0$ then the limit in (5.2) is zero due to Lemma 5.1. Hence

$$\lim_{N\to\infty}e^{\bar{z}_N^2/2}\det\left(V_N^{1/2}\right)\mathbb{E}(g(S_N-z_N))=\sum_{m\in\mathbb{Z}^{d_1}}\frac{\widehat{\mathcal{C}_{\mathbb{S}}g}\,(2\pi m)}{(2\pi)^{d/2}}.$$

Define the following function on $\mathbb{R}^{d_1}$

$$\mathcal{G}(x')=\int_{\mathbb{R}^{d-d_1}}(\mathcal{C}_{\mathbb{S}}g)(x',x'')dx''. \tag{5.3}$$

Then

$$\sum_{m\in\mathbb{Z}^{d_1}}\widehat{\mathcal{C}_{\mathbb{S}}g}\,(2\pi m)=\sum_{m\in\mathbb{Z}^{d_1}}\hat{\mathcal{G}}(2\pi m)=\sum_{m\in\mathbb{Z}^{d_1}}\mathcal{G}(m)=\int_{\mathcal{H}}\mathcal{C}_{\mathbb{S}}(g)(h)d\lambda_{\mathcal{H}}.$$

Here the first equality holds since we have identified $m\in\mathbb{Z}^{d_1}$ with $(m,0)\in\mathbb{R}^d$, the second equality follows by the Poisson Summation Formula and the third equality follows by (5.3) and (1.5). This proves Theorem 1.2 for the functions with compactly supported Fourier transform.

6 Proof of the Local Limit Theorem.

Here we finish the proof of Theorem 1.2.

We need the following *a priori* estimate proven in Section 7.

Lemma 6.1. *There is a constant D such that for any cube Q of unit size*

$$\mathbb{P}(S_N \in Q) \leq \frac{D}{N^{d/2}}.$$

To fix the notation we consider a non-arithmetic case, the argument in the arithmetic case is similar.

We note that it is sufficient to prove Theorem 1.2 for $g \in C_0^{d+1}(\mathbb{R}^d)$. Indeed if $g \in C_0(\mathbb{R}^d)$ and $\mathrm{supp}(g) \in [-K,K]^d$ then for each $\varepsilon > 0$ we can find $\tilde{g} \in C_0^{d+1}(\mathbb{R}^d)$ with $\mathrm{supp}(\tilde{g}) \in [-(K+1),(K+1)]^d$ and $||g-\tilde{g}||_{L^\infty} \leq \varepsilon$. Then

$$\det\left(V_N^{1/2}\right)\mathbb{E}(g(S_N - z_N)) \tag{6.1}$$

$$= \det\left(V_N^{1/2}\right)\mathbb{E}(\tilde{g}(S_N - z_N)) + \det\left(V_N^{1/2}\right)\mathcal{O}(\varepsilon)\mathbb{P}\left(S_N \in z_N + [-(K+1),K+1]^d\right).$$

The second term is $\mathcal{O}(\varepsilon)$ by Lemma 6.1. So if Theorem 1.2 is valid for C_0^{d+1} functions then

$$\det\left(V_N^{1/2}\right)\mathbb{E}(g(S_N) - z_N) = e^{-\bar{z}_N^2/2}\int_{[-(K+1),K+1]^d}\tilde{g}(x)dx + o_{N\to\infty}(1) + \mathcal{O}(\varepsilon).$$

Since

$$\left|\int_{[-(K+1),K+1]^d}\tilde{g}(x)dx - \int_{[-(K+1),K+1]^d}g(x)dx\right| \leq \varepsilon(2(K+1))^d$$

the theorem holds for all continuous functions.

So let $g \in C_0^{d+1}(\mathbb{R}^d)$. Then for each ε there is $\bar{g}$ such that $\widehat{\bar{g}}$ has compact support and $|g(x) - \bar{g}(x)| \leq \frac{\varepsilon}{1+|x|^{d+1}}$. Denoting by Q_m the unit cube centered at m we get

$$\det\left(V_N^{1/2}\right)|\mathbb{E}(g(S_N - z_N)) - \mathbb{E}(\bar{g}(S_N - z_N))|$$

$$\leq \sum_{m\in\mathbb{Z}^d}\frac{\varepsilon\det\left(V_N^{1/2}\right)}{1+|m|^{d+1}}\mathbb{P}(S_N - z_N \in Q_m) = \mathcal{O}\left(\sum_{m\in\mathbb{Z}^d}\frac{\varepsilon}{1+|m|^{d+1}}\right) = \mathcal{O}(\varepsilon)$$

where the penultimate step uses Lemma 6.1. Also

$$\int_{\mathbb{R}^d}|g(x) - \bar{g}(x)|dx \leq \varepsilon\int_{\mathbb{R}^d}\frac{dx}{1+|x|^{d+1}} = \mathcal{O}(\varepsilon).$$

Since

$$\frac{\mathbb{E}(\bar{g}(S_N - z_N))}{u(z_N)} \to \int_{\mathbb{R}^d}\bar{g}(x)dx$$

due to the results of Section 5, Theorem 1.2 holds on $C_0^{d+1}(\mathbb{R}^d)$ and, hence, on $C_0(\mathbb{R}^d)$.

7 Concentration Inequality.

The proof of Lemma 6.1 in arbitrary dimension is the same as the proof for $d = 1$ given in [9, Section III.1] but we reproduce the proof here for completeness.

Proof of Lemma 6.1. It is enough to prove the claim for cubes of any fixed size ρ since the unit cube can be covered by a finite number of cubes of size ρ. Let

$$g(x) = \prod_{l=1}^{d} \left(\frac{1 - \cos(\hat{\delta} x_{(l)})}{\hat{\delta}^2 x_{(l)}^2} \right)$$

where $\hat{\delta} = \delta_0 / d$ and δ_0 is the constant of Lemma 4.1(b). Then

$$\hat{g}(s) = (\pi \hat{\delta})^d \prod_{l=1}^{d} \left(\left(1 - \frac{|s_{(l)}|}{\hat{\delta}} \right) 1_{|s_{(l)}| \leq \hat{\delta}} \right).$$

Hence for each a

$$\mathbb{E}(g(S_N - a)) = \int_{\mathbb{R}^d} \hat{g}(-s) e^{i\langle s, a \rangle} \Phi_N(s) ds \leq \int_{\max_l |s_{(l)}| < \delta_0} \hat{g}(s) |\Phi_N(s)| ds$$

since $\hat{g}$ is real and supported inside the cube of size $2\delta_0$. Thus (1.3) and Lemma 4.1(b) imply that there is a constant $\hat{D}$ such that

$$\mathbb{E}(g(S_N - a)) \leq \frac{\hat{D}}{N^{d/2}}$$

On the other hand $g(0) = \frac{1}{2^d}$ so there is a constant ρ such that $g(x) > \frac{1}{4^d}$ on the cube of size ρ centered at 0. Hence if $\mathcal{Q}$ is a cube of size ρ centered at a then

$$\mathbb{E}(g(S_N - a)) \geq \frac{\mathbb{P}(S_N \in \mathcal{Q})}{4^d}.$$

Combining the last two displays we obtain the result. $\square$

8 Bounded random variables.

Proof of Corollary 1.4. If $\sum_j V(X_j)$ converges then S_N converges almost surely by Kolmogorov's Three Series Theorem and so (1.7) holds.

Therefore we assume that $\sum_j V(X_j)$ diverges. Fix a large A and let k_n be a sequence such that denoting $\mathcal{X}_n = \sum_{j=k_{n-1}+1}^{k_n} X_j$ we have

$$\frac{1}{A} \leq V(\mathcal{X}_n) \leq A.$$

Since

$$\mathbb{E}(\mathcal{X}_n^4) = (\mathbb{E}(\mathcal{X}_n^2))^2 + \sum_{j=k_{n-1}+1}^{k_n} V(X_j^2) \leq A^2 + \sum_{j=k_{n-1}+1}^{k_n} \mathbb{E}(X_j^4) \leq A^2 + \mathcal{K}^2 A$$

$\{\mathcal{X}_n\}$ satisfies (1.1), (1.2) and (1.3). Accordingly $\sum_{j=1}^{k_n} X_j$ satisfy the conclusions of Corollary 1.3. Note that this holds for *any* sequence k_N such that

$$\frac{1}{A} \leq \sum_{j=k_{n-1}+1}^{k_n} \mathbb{E}(X_j^2) \leq A \tag{8.1}$$

for some A and all n. We claim that, in fact, the conclusions of Corollary 1.3 are satisfied for our original sum S_N. Indeed, take an arbitrary sequence satisfying (8.1). Suppose, to fix our notation, that S_{k_n} satisfies a non-arithmetic Local Limit Theorem, the arithmetic case is similar. We claim that (1.6) holds. Otherwise there exist sequences

$\{N_l\}$ $\{z_l\}$ such that $z_l/\sqrt{V_{N_l}} \to z$ and a continuous function g of compact support such that $\lim_{l\to\infty}\left[\sqrt{V_{N_l}}\mathbb{E}(g(S_{N_l}-z_l))\right]$ does not converge to $\dfrac{e^{-z^2/2}}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} g(x)dx$. By taking a subsequence we can assume that

$$\sum_{j=N_{l-1}+1}^{N_l} \mathbb{E}(X_j^2) \geq 100A.$$

Let n_l be such that $k_{n_l} \leq N_l < k_{n_{l+1}}$. Replacing k_{n_l} by N_l we obtain a new sequence $\tilde{k}_n$ satisfying (8.1) with A replaced by $2A$. Also, let $\tilde{z}_n = z_l$ if $\tilde{k}_n = N_l$ for some l and $\tilde{z}_n = z\sqrt{V_{\tilde{k}_n}}$ otherwise. Then

$$\lim_{l\to\infty}\left[\sqrt{V_{\tilde{k}_n}}\mathbb{E}(g(S_{\tilde{k}_n}-\tilde{z}_n))\right]$$

fails to exist giving a contradiction with the assumption that (1.6) fails.

Hence (1.6) holds as claimed. □

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