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# Infinite measure renewal theorem and related results

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Abstract

We present abstract conditions under which a special flow over a probability preserving map with a non-integrable roof function is Krickeberg mixing. Our main condition is a version of the local central limit theorem for the underlying map. We check our assumptions for iid random variables (renewal theorem with infinite mean) and for suspensions over Pomeau-Manneville maps.

## 1. *Introduction*

Mixing plays a central role in studying statistical properties of transformations preserving a probability measure. For transformations preserving an infinite measure, mixing is much less understood. In fact, there are several different generalizations of mixing to infinite measure setting [**21**]. One natural definition is to require that for a large collection of (nice) sets of finite measure, the probability that the orbit is in this set at a given time $t$ is asymptotically independent of the initial distribution. This type of mixing is sometimes called *Krickeberg mixing* since it has been studied for Markov chains in [**20**] (other early works on this subject include [**13**, **18**, **29**]). This notion of mixing is related to classical renewal theory ([**14**]) and to limit distributions of ergodic sums of infinite measure preserving transformations [**10**]. Recently, there was a considerable interest in studying mixing properties of hyperbolic transformations preserving an infinite measure in both discrete and continuous time settings (see [**2**, **5**, **15**, **23**, **24**, **26**, **27**, **28**, **31**] and references wherein).

The goal of this note is to describe a method of deducing mixing for flows from local limit results for the first return map to an appropriate section. This approach goes back to [**14**] in the independent setting, and in dynamical setting it was pursued in [**2**, **8**, **9**]. The plan of the paper is the following. In Section 2, we explain how to obtain mixing for flows from the local limit theorem and appropriate large deviation bounds for a section. Section 3 contains tools which are helpful in verifying the abstract conditions of Section 2 in specific examples. In particular, in Theorem 3.3 we obtain sharp large deviation bounds for quasi-independent random variables. The last two sections contain specific examples where our assumptions hold. Section 4 is devoted to independent random variables. The results of this section are not new but we included this example since it allows us to illustrate our approach in the simplest possible setting. In particular, it is known since the work of Garcia-Lamperti ([**14**]) that in the independent case the regular variation of the return time with index $\alpha$ is sufficient for mixing if $\alpha > \frac{1}{2}$ but extra assumptions are needed if $\alpha \leq \frac{1}{2}$. We will present in Section 4 a simple argument to verify our key assumptions (2.4), (2.5) for $\alpha > \frac{1}{2}$, and we will see that a more delicate estimate (4.2) is required in the general case. In Section 5, we show how to verify our assumptions for suspension flows over the Liverani-Saussol-Vaienti map studied in [**22**].

While there is a number of papers dealing with mixing of infinite measure preserving flows (see the references at the beginning of the introduction), we use more elementary tools than

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most of the previous works. In particular, we pay a special attention to isolate the key geometric (quasi-independence) and probabilistic (anticoncetration, exchangebility) ingredients needed in our method. This could make our method useful also for studying more complicated systems.

## 2. *Abstract setting*

### 2.1. *Results.*

Recall that a function $\mathfrak{L}:\mathbb{R}^+\to\mathbb{R}^+$ is called *slowly varying* if for each $h>0$, $\lim_{t\to\infty}\frac{\mathfrak{L}(ht)}{\mathfrak{L}(t)}=1$. A function $\mathfrak{R}:\mathbb{R}^+\to\mathbb{R}^+$ which can be represented in the form $\mathfrak{R}(t)=t^\gamma\mathfrak{L}(t)$ with $\mathfrak{L}$ slowly varying is called *regularly varying of index* $\gamma$. Equivalently, for each $h>0$, $\lim_{t\to\infty}\frac{\mathfrak{R}(ht)}{\mathfrak{R}(t)}=h^\gamma$. We refer the reader to [**4**] for a comprehensive discussion of regularly and slowly varying functions. The properties of these functions needed in this paper will be recalled in a due time.

Let $f:X\to X$ be a map preserving a probability measure $\mu$. Let $\tau:X\to[\tau_{\min},\infty)$, where $\tau_{\min}>0$, be a function whose tail is regularly varying with index $-\alpha$ for some $\alpha\in(0,1)$. That is, we assume that there is a slowly varying function $\mathcal{L}(t)$ such that

$$\mu(\tau>t)\sim\frac{\mathcal{L}(t)}{t^\alpha}. \tag{2.1}$$

In particular $\mu(\tau)=\infty$. Consider the suspension flow of $f$ under the roof function $\tau$. Recall that the phase space of the suspension flow is

$$\Omega=\{(x,s):x\in X,s\geq 0\}/\sim,\ \text{ where } (x,s+\tau(x))\sim(f(x),s) \text{ for any } x\in X \text{ and } s\geq 0.$$

The suspension flow $g_t$ acts on $\Omega$ by $g_t(x,s)=(x,s+t)$ (with a slight abuse of notation, we write $(x,s)$ instead of the equivalence class $[(x,s)]$. We also note that as $T$ is not assumed to be invertible, $g_t$ is sometimes called semiflow in the literature.) Note that $g_t$ preserves the infinite measure $\nu$ defined by

$$d\nu(x,s)=d\mu(x)ds.$$

Let $\tau_k(x)=\sum_{i=0}^{k-1}\tau(f^i(x))$. We are interested in the asymptotics of $\nu(\mathcal{A}\cap g_{-t}\mathcal{B})$ for suitable sets $\mathcal{A},\mathcal{B}$.

Before formulating our first main result we need some background information on regularly varying functions. Applying [**4**, Theorem 1.5.12] to the regularly varying function $t\mapsto t^\alpha/\mathcal{L}(t)$, we find that there is a regularly varying function $\mathcal{R}(t)$ of index $\frac{1}{\alpha}$ such that

$$\lim_{t\to\infty}\frac{t\mathcal{L}(\mathcal{R}(t))}{\mathcal{R}^\alpha(t)}=1. \tag{2.2}$$

We say that two regularly varying functions $\mathcal{R}_1$ and $\mathcal{R}_2$ are *asymptotically equivalent* if $\lim_{t\to\infty}\frac{\mathcal{R}_1(t)}{\mathcal{R}_2(t)}=1$. Clearly, the function $\mathcal{R}$ satisfying (2.2) is uniquely defined up to asymptotic equivalence. Furthermore, [**4**, Theorem 1.5.12] implies that $\mathcal{R}$ can be assumed to be monotonically increasing.

Now we are ready to formulate our results.

Proposition 2.1. *Suppose that there is an algebra $\mathcal{X}$ on $X$, a bounded continuous probability density $\rho$ on $[0,\infty)$ and a constant $\bar{c}$, such that for any $A,B\in\mathcal{X}$, for any $\varepsilon>0$*

*and for any $\mathcal{I}$ compact subset of $\mathbb{R}_+$,*

$$\lim_{k\to\infty} \sup_{t\leq \frac{\mathcal{R}(k)}{\varepsilon}} \sup_{l\in\mathcal{I}} \left| \mathcal{R}(k)\mu\left(x\in A, f^k x\in B, \tau_k(x)\in[t-l,t]\right) - \bar{c}\rho\left(\frac{t}{\mathcal{R}(k)}\right) l\mu(A)\mu(B)\right| = 0 \tag{2.3}$$

*Suppose furthermore that there are constants $\beta_1, \beta_2, \beta_3$ such that*

$$\beta_2 + \frac{\beta_3}{\alpha} < 1, \quad \beta_1 + \beta_2\alpha + \beta_3 = 1 \tag{2.4}$$

*and there is some $C$ so that the for any $l \geq 1$ and for any $k \in \mathbb{Z}_+$,*

$$\mu\left(\tau_k \in [t, t+l]\right) \leq \frac{Cl}{\mathcal{L}^{\beta_2}(t)t^{\beta_1}k^{\beta_2}\mathcal{R}^{\beta_3}(k)}. \tag{2.5}$$

*Then for any $\mathcal{A}, \mathcal{B} \subset \Omega$, $\mathcal{A} = A \times [a_1, a_2]$, $\mathcal{B} = B \times [b_1, b_2]$ we have*

$$\lim_{t\to\infty} \nu(\mathcal{A}\cap g_{-t}\mathcal{B})\mathcal{L}(t)t^{1-\alpha} = \hat{c}\mu(A)\mu(B)(a_2-a_1)(b_2-b_1) = \hat{c}\nu(\mathcal{A})\nu(\mathcal{B}), \tag{2.6}$$

*where*

$$\hat{c} = \bar{c}\alpha\int_0^\infty \frac{\rho(z)}{z^\alpha}dz. \tag{2.7}$$

Remark 2.2. *Let $\Omega_{\leq M} = \{(x,s)\in\Omega : s \leq M\}$. Assume that $X$ is a topological space, $\mathcal{X}$ generates the Borel $\sigma$-algebra and every set in $\mathcal{X}$ has boundary of $\mu$-measure zero. Then one can prove by standard argument (cf. [**3**]) that (2.6) is equivalent to*

$$\nu(\Phi\Psi\circ g_t)\mathcal{L}(t)t^{1-\alpha} \to \hat{c}\nu(\Phi)\nu(\Psi)$$

*with either of the following two classes of functions:*

- *for any continuous functions $\Phi, \Psi : \Omega \to \mathbb{R}$ supported on $\Omega_{\leq M}$ for some $M < \infty$;*
- *for any $\Phi = 1_{\mathcal{A}}$, $\Psi = 1_{\mathcal{B}}$, where $\mathcal{A}, \mathcal{B} \subset \Omega_{\leq M}$ for some $M < \infty$ and $\nu(\partial\mathcal{A}) = \nu(\partial\mathcal{B}) = 0$.*

Assumption (2.3) amounts to the *non-lattice* (mixing) local limit theorem. In fact, the non-lattice assumption is not necessary for mixing of the flow. To clarify the situation, we need some definitions.

Definition 2.3. *Let $(Y, \lambda, T)$ be a dynamical system. We say that an observable $\varphi : Y \to \mathbb{R}$ is* **rational** *if there is a real number $h$ and two measurable functions $\psi : Y \to \mathbb{Z}$, $\mathfrak{h} : Y \to \mathbb{R}$ so that*

$$\varphi = h\psi + \mathfrak{h} - \mathfrak{h}\circ T. \tag{2.8}$$

*A function, which is not rational, is called* **irrational**.
*We say that $\varphi$ is* **periodic** *if there exist real numbers $\mathbf{a}, h$ and two measurable functions $\psi : Y \to \mathbb{Z}$, $\mathfrak{h} : Y \to \mathbb{R}$ so that*

$$\varphi = \mathbf{a} + h\psi + \mathfrak{h} - \mathfrak{h}\circ T. \tag{2.9}$$

*A function, which is not periodic, is called* **aperiodic**.

A rational function is clearly periodic with $\mathbf{a} = 0$. Conversely, suppose that (2.9) holds *and* $\frac{\mathbf{a}}{h}$ is rational, that is $\mathbf{a} = \frac{ph}{q}$. In this case, $\varphi = \bar{h}\bar{\psi} + \mathfrak{h} - \mathfrak{h}\circ T$ with $\bar{\psi} = \frac{\mathbf{a}+h\psi}{\bar{h}}$ and $\bar{h} = \frac{h}{q}$, where $\bar{\psi}$ is integer valued and so $\varphi$ is rational.

Thus we have three cases: $\varphi$ can be either aperiodic, periodic irrational or rational.

Proposition 2.1 addresses mixing in the case $\tau$ is aperiodic. Next, we consider the simplest case, i.e. when $\tau$ is rational.

PROPOSITION 2.4. *If $\tau$ is rational, then $g_t$ is not mixing.*

*Proof.* Assume that (2.8) holds for $\varphi = \tau$. Note that $\mathfrak{h}$ is defined up to an additive constant. Let us choose this constant in such a way that $\mu(A) > 0$, where $A = \{x \in X : 0 < \mathfrak{h}(x) \leq \tau(x)\}$. Next, we consider the set $\mathcal{A} = \{(x, \mathfrak{h}(x)), x \in A\} \subset \Omega$. For $x \in A$, let $\varsigma(x) = \min\{s > 0 : g_s(x, \mathfrak{h}(x)) \in \mathcal{A}\}$ (for almost every $x \in A$, $\varsigma(x)$ is finite by the Poincaré recurrence theorem). Furthermore, let $n(x)$ denote the number of times the orbit $\{g_t(x, \mathfrak{h}(x)), t \in (0, \varsigma(x))\}$ hits the roof. Then we find that by (2.8),

$$\varsigma(x) = -\mathfrak{h}(x) + \mathfrak{h}(f^{n(x)}x) + \sum_{i=0}^{n(x)-1} \tau(f^i(x)) = \sum_{i=0}^{n(x)-1} h\psi(f^i(x)) \in h\mathbb{Z}.$$

Thus with a sufficiently small $\varepsilon$ and $\mathcal{A}_\varepsilon = \{(x, s) \in \Omega : x \in A, |s - \mathfrak{h}(x)| < \varepsilon\}$, we have that $\mathcal{A}_\varepsilon \cap g_{-t}\mathcal{A}_\varepsilon = \emptyset$ whenever $t \in h\mathbb{Z} + h/2$. This shows that $g_t$ is not mixing. □

It remains to address the periodic irrational case. This is done in Proposition 2.5 below.

Given a function $\mathfrak{h} : X \to \mathbb{R}$ and numbers $k \in \mathbb{Z}_+, w_k \in \mathbb{R}$, let $\mathcal{F}_{k,\mathfrak{h},w_k} : X \to X \times X \times \mathbb{R}$ be defined by

$$\mathcal{F}_{k,\mathfrak{h},w_k} : x \mapsto (x, f^k x, \tau_k(x) - \mathfrak{h}(x) + \mathfrak{h}(f^k(x)) - w_k).$$

PROPOSITION 2.5. *Assume that $X$ is a topological space.* (2.6) *remains valid for any $A$, $B$ with $\mu(\partial A) = \mu(\partial B) = 0$ if* (2.3) *is replaced by the following assumption*

*"There is a bounded and continuous function $\mathfrak{h} : X \to \mathbb{R}$ and constants $\bar{c}, \mathbf{a}, h$ such that $\frac{\mathbf{a}}{h}$ is irrational and such that for any $\varepsilon > 0$, for any $C < \infty$, for any $\phi \in \mathcal{C}(X \times X \times \mathbb{R})$, compactly supported in the last coordinate, we have*

$$\lim_{k\to\infty} \sup_{w\in[0,1/\varepsilon]} \sup_{w_k\in\mathbf{a}k+h\mathbb{Z}:|w_k/\mathcal{R}(k)-w|<C} \left| \mathcal{R}(k) \int \phi d(\mathcal{F}_{k,\mathfrak{h},w_k})_*\mu - \bar{c}\rho(w) \int \phi d(\mu \times \mu \times u) \right| = 0 \tag{2.10}$$

*where $u$ is $h$ times the counting measure on $h\mathbb{Z}$."*

## 2.2. *Proofs.*

*Proof of Proposition 2.1.*

$$\nu(\mathcal{A} \cap g_{-t}\mathcal{B}) = \int_{a_1}^{a_2} \sum_k \mu\left(x \in A, f^k x \in B, \tau_k(x) + b_1 \leq a + t \leq \tau_k(x) + b_2\right) da. \tag{2.11}$$

The last condition can be rewritten as

$$\tau_k(x) \in [t + a - b_2, t + a - b_1].$$

Define

$$\mathcal{N}(t) = \inf\{s : \mathcal{R}(s) > t\} \tag{2.12}$$

where $\mathcal{R}$ is defined by (2.2). [**4**, Theorem 1.5.12] implies that $\mathcal{N}(t)$ is asymptotically equivalent to $t \mapsto t^\alpha/\mathcal{L}(t)$. Clearly $\mathcal{N}(t)$ is monotonic.

Fix a small constant $\varepsilon$ and decompose the sum (2.11) as $I + II + III$ where $I$ includes the terms with $k < \varepsilon\mathcal{N}(t)$, $III$ includes the terms with $k \geq \mathcal{N}(t)/\varepsilon$ and $II$ comprises the remaining terms. By (2.5) and the Karamata Theorem ([**4**, §1.5.6])

$$I \leq \frac{\text{Const}}{\mathcal{L}^{\beta_2}(t)t^{\beta_1}} \sum_{k=1}^{\varepsilon\mathcal{N}(t)} \frac{1}{k^{\beta_2}\mathcal{R}^{\beta_3}(k)} \leq \frac{\text{Const}}{\mathcal{L}^{\beta_2}(t)t^{\beta_1}} \frac{(\varepsilon\mathcal{N}(t))^{1-\beta_2}}{\mathcal{R}^{\beta_3}(\varepsilon\mathcal{N}(t))}. \tag{2.13}$$

Since $\mathcal{R}$ is regularly varying we have (see [**4**, §1.5.7]) that

$$\lim_{t\to\infty} \frac{\mathcal{R}(\mathcal{N}(t))}{t} = 1. \tag{2.14}$$

Hence

$$\lim_{t\to\infty} \frac{\mathcal{R}(\varepsilon\mathcal{N}(t))}{t} = \varepsilon^{1/\alpha}.$$

Thus

$$I \leq \text{Const}\varepsilon^{1-\beta_2-\beta_3/\alpha} \frac{\mathcal{N}^{1-\beta_2}(t)}{\mathcal{L}^{\beta_2}(t)t^{\beta_1+\beta_3}}$$

Comparing (2.2) and (2.14) we obtain

$$\lim_{t\to\infty} \frac{\mathcal{L}(t)\mathcal{N}(t)}{t^\alpha} = 1. \tag{2.15}$$

Therefore

$$I \leq \text{Const}\frac{\varepsilon^{1-\beta_2-\beta_3/\alpha}}{\mathcal{L}(t)t^{\beta_1+(\beta_2-1)\alpha+\beta_3}} = \text{Const}\ \frac{\varepsilon^{1-\beta_2-\beta_3/\alpha}}{\mathcal{L}(t)t^{1-\alpha}} \tag{2.16}$$

and so $I$ is negligible in the sense that we can make it as small as we wish by choosing a sufficiently small $\varepsilon$.

Next, by (2.3) and the Karamata Theorem

$$\mathit{III} \leq \text{Const} \sum_{k>\mathcal{N}(t)/\varepsilon} \frac{1}{\mathcal{R}(k)} \leq \text{Const}\varepsilon^{(1/\alpha)-1} \frac{\mathcal{N}(t)}{\mathcal{R}(\mathcal{N}(t))}.$$

Using (2.14) and (2.15) we see that

$$\mathit{III} \leq \text{Const}\varepsilon^{(1/\alpha)-1} \frac{t^\alpha}{\mathcal{L}(t)t}$$

which is also negligible.

On the other hand by (2.3) we have

$$\mathit{II} \sim \bar{c}(a_2-a_1)(b_2-b_1)\mu(A)\mu(B) \sum_{k=\varepsilon\mathcal{N}(t)}^{\mathcal{N}(t)/\varepsilon} \rho\left(\frac{t}{\mathcal{R}(k)}\right)\frac{1}{\mathcal{R}(k)}. \tag{2.17}$$

By regular variation

$$\frac{t}{\mathcal{R}(k)} = \frac{\mathcal{R}(\mathcal{N}(t))}{\mathcal{R}(k)}(1+o(1)) = \left(\frac{\mathcal{N}(t)}{k}\right)^{1/\alpha}(1+o(1))$$

so the sum in (2.17) is asymptotic to

$$\frac{1}{t}\sum_{k=\varepsilon\mathcal{N}(t)}^{\mathcal{N}(t)/\varepsilon} \rho\left(\left(\frac{\mathcal{N}(t)}{k}\right)^{1/\alpha}\right)\left(\frac{\mathcal{N}(t)}{k}\right)^{1/\alpha}.$$

Let $z_k = \left(\frac{\mathcal{N}(t)}{k}\right)^{1/\alpha}$. Then $z_k - z_{k+1} \sim \frac{\mathcal{N}^{1/\alpha}(t)}{\alpha k^{1+1/\alpha}}$. Writing

$$\rho\left(\left(\frac{\mathcal{N}(t)}{k}\right)^{1/\alpha}\right)\left(\frac{\mathcal{N}(t)}{k}\right)^{1/\alpha} \sim \rho(z_k)\alpha(z_k-z_{k+1})k = \alpha\mathcal{N}(t)\rho(z_k)\frac{z_k-z_{k+1}}{z_k^\alpha}$$

we see that the sum in (2.17) is asymptotic to

$$\frac{\alpha\mathcal{N}(t)}{t}\int_{L_1(\varepsilon)}^{L_2(\varepsilon)}\frac{\rho(z)}{z^\alpha}\,dz \tag{2.18}$$

where $L_1(\varepsilon)\to 0$ and $L_2(\varepsilon)\to\infty$ as $\varepsilon\to 0$. Combining the estimates for $I$, $I\!I$, and $I\!I\!I$ and using (2.15) to eliminate $\mathcal{N}(t)$ from (2.18) we obtain the result. □

*Proof of Proposition 2.5.* Let

$$C(a)=\{(x,y,z)\in X\times X\times\mathbb{R}: x\in A, y\in B, z-\mathfrak{h}(x)+\mathfrak{h}(y)\in[a-b_2,a-b_1]\}$$

Then

$$\nu(\mathcal{A}\cap g_{-t}\mathcal{B})=\int_{a_1}^{a_2}\sum_k(\mathcal{F}_{k,\mathfrak{h},t})_*\mu(C(a))da$$

We decompose this sum into $I+I\!I+I\!I\!I$ as before, and use the same estimates for $I+I\!I\!I$. To revisit the computation of $I\!I$, first observe that $\mu(\partial A)=\mu(\partial B)=0$ implies $(\mu\times\mu\times u)(\partial C(a))=0$ and thus by approximating $1_C$ with continuous functions, we find that (2.10) implies

$$I\!I\sim\sum_{k=\varepsilon\mathcal{N}(t)}^{\mathcal{N}(t)/\varepsilon}\frac{\bar{c}}{\mathcal{R}(k)}\rho\left(\frac{t}{\mathcal{R}(k)}\right)\int_{a_1}^{a_2}(\mu\times\mu\times u)(\{(x,y,z):(x,y,z-k\mathbf{a})\in C(a)\})da.$$

Fixing some $Q$ large positive integer and writing

$$\sum_{k=\varepsilon\mathcal{N}(t)}^{\mathcal{N}(t)/\varepsilon}=\sum_{i=0}^{(\frac{1}{\varepsilon}-\varepsilon-1)\frac{\mathcal{N}(t)}{Q}}\ \sum_{k=\varepsilon t^\alpha+iQ}^{\varepsilon t^\alpha+(i+1)Q-1},$$

we find that

$$I\!I\sim\bar{c}Q\sum_{i=0}^{t^\alpha(\frac{1}{\varepsilon}-\varepsilon-1)\frac{1}{Q}}\rho\left(\frac{t}{\mathcal{R}(\varepsilon\mathcal{N}(t)+iQ)}\right)\frac{1}{\mathcal{R}(\varepsilon\mathcal{N}(t)+iQ)}$$
$$\int_{a_1}^{a_2}\frac{1}{Q}\sum_{k=\varepsilon t^\alpha+iQ}^{\varepsilon t^\alpha+(i+1)Q-1}(\mu\times\mu\times u)(\{(x,y,z):(x,y,z-k\mathbf{a})\in C(a)\})da$$

Since $\frac{\mathbf{a}}{h}$ is irrational, Weyl's theorem implies that the integrand in the second line of the last display is $\mu(A)\mu(B)[b_2-b_1](1+o_{Q\to\infty}(1))$, uniformly in $a,t,\varepsilon$ and $i$. Thus $I\!I$ is asymptotic to a Riemann sum and the proof can be completed as in the case of Proposition 2.1. □

REMARK 2.6. *The conclusions of Propositions 2.1 and 2.5 remain valid if (2.5) is replaced by the assumption that for some $r\in\mathbb{Z}_+$ there exists constants $C,\beta_{1,1},\beta_{2,1},\beta_{3,1},\dots,\beta_{1,r},\beta_{2,r},\beta_{3,r}$ such that for each $l\geq 1$*

$$\mu\left(\tau_k\in[t,t+l]\right)\leq\sum_{j=1}^r\frac{Cl}{\mathcal{L}^{\beta_{2,j}}(t)t^{\beta_{1,j}}k^{\beta_{2,j}}\mathcal{R}^{\beta_{3,j}}(k)}, \tag{2.19}$$

*where $\beta_{1,j}$ $\beta_{2,j}$ and $\beta_{3,j}$ satisfy (2.4) for each $j$. Indeed, (2.5) was only used to derive (2.16). Under (2.19) we can replace (2.16) by*

$$I\leq\text{Const}\sum_{j=1}^r\sum_{k=1}^{\varepsilon\mathcal{N}(t)}\frac{1}{\mathcal{L}^{\beta_{2,j}}(t)t^{\beta_{1,j}}k^{\beta_{2,j}}\mathcal{R}^{\beta_{3,j}}(k)}\leq\text{Const}\varepsilon^{1-\max_j\left(\beta_{2,j}+\frac{\beta_{3,j}}{\alpha}\right)}\frac{t^{\alpha-1}}{\mathcal{L}(t)}$$

*which shows that $I$ remains negligible.*

According to a common terminology, $\tau$ *satisfies a mixing local limit theorem* if either $\tau$ is aperiodic and (2.3) holds or $\tau$ is periodic (either rational or irrational) and (2.10) holds. The results of this section could be summarized as follows.

THEOREM 2.7. *If $\tau$ is irrational, satisfies a mixing local limit theorem and* (2.19), *then* (2.6) *holds.*

In other words, if the appropriate local limit theorem and large deviation bounds hold for the base map, then the special flow is mixing in both the aperiodic and the periodic irrational cases but not in the rational case. A similar result holds in the finite measure case (see [**9**, Section 2]).

In Sections 4 and 5 we provide examples of systems satisfying the conditions of Theorem 2.7.

2.3. *Power tail.*

Here we consider an important special case where the function $\mathcal{L}$ is asymptotically constant. Thus we assume that there is a constant $\mathbf{c}$ such that

$$\mu(\tau > t) \sim \frac{\mathbf{c}}{t^\alpha} \quad \text{for} \quad 0 < \alpha < 1. \tag{2.20}$$

In this case one can take $\mathcal{R}(k) = (\mathbf{c}k)^{1/\alpha}$ and the statements of Propositions 2.1 and 2.5 can be simplified as follows.

PROPOSITION 2.8. *Suppose that* (2.20) *holds and there is a bounded continuous density $\rho$ on $[0,\infty)$ such that either*

*(i) $\tau$ is aperiodic and there is a constant $\bar{c}$, such that for any $A, B \in \mathcal{X}$, for any $\varepsilon > 0$ and for any $\mathcal{I}$ compact subset of $\mathbb{R}_+$,*

$$\lim_{k\to\infty} \sup_{t \le \frac{(\mathbf{c}k)^{1/\alpha}}{\varepsilon}} \sup_{l\in\mathcal{I}} \left| (\mathbf{c}k)^{1/\alpha} \mu\left(x \in A, f^k x \in B, \tau_k(x) \in [t-l, t]\right) - \bar{c}\rho\left(\frac{t}{(\mathbf{c}k)^{1/\alpha}}\right) l\mu(A)\mu(B) \right| = 0, \tag{2.21}$$

*or*

*(ii) $\tau$ is periodic irrational and there is a bounded and continuous function $\mathfrak{h} : X \to \mathbb{R}$ and constants $\bar{c}, \mathbf{a}, h$ such that $\dfrac{\mathbf{a}}{h}$ is irrational and such that for any $\varepsilon > 0$, for any $C < \infty$, for any $\phi \in \mathcal{C}(X \times X \times \mathbb{R})$, compactly supported in the last coordinate, we have*

$$\lim_{k\to\infty} \sup_{w\in[0,1/\varepsilon]} \sup_{w_k \in \mathbf{a}k + h\mathbb{Z} : |w_k/(\mathbf{c}k)^{1/\alpha} - w| < C} \left| (\mathbf{c}k)^{1/\alpha} \int \phi \, d(\mathcal{F}_{k,\mathfrak{h},w_k})_* \mu - \bar{c}\rho(w) \int \phi \, d(\mu \times \mu \times u) \right| = 0 \tag{2.22}$$

*where $u$ is $h$ times the counting measure on $h\mathbb{Z}$.*

*Assume in addition that for some $r \in \mathbb{Z}_+$ there exist constants $C, \gamma_{1,1}, \gamma_{2,1}, \dots \gamma_{1,r}, \gamma_{2,r}$ such that for all $l \ge 1$*

$$\mu\left(\tau_k \in [t, t+l]\right) \le \sum_{j=1}^r \frac{Cl}{t^{\gamma_{1,j}} k^{\gamma_{2,j}}}. \tag{2.23}$$

*where for each $j = 1 \dots r$*

$$\gamma_{2,j} < 1, \quad \gamma_{1,j} + \gamma_{2,j}\alpha = 1. \tag{2.24}$$

*Then for $\mathcal{A}, \mathcal{B}$ satisfying the assumptions of Proposition 2.1 we have*

$$\lim_{t\to\infty} \nu(\mathcal{A} \cap g_{-t}\mathcal{B})t^{1-\alpha} = \hat{c}\nu(\mathcal{A})\nu(\mathcal{B}), \tag{2.25}$$

*where $\hat{c}$ is defined in* (2.7).

## 3. *Tools.*

### 3.1. *Anticoncentration inequlity.*

Here we obtain a useful *a priori* bound.

PROPOSITION 3.1. *Suppose that there exists some $\delta > 0$ so that for $|s| \le \delta$*

$$\left|\mu\left(e^{is\tau_k}\right)\right| \le \left(1 - c\mathcal{L}\left(\frac{1}{|s|}\right)|s|^\alpha\right)^k. \tag{3.1}$$

*Then there exists a constant $D > 0$ so that for any interval $I$ of unit size*

$$\mu\left(\tau_k \in I\right) \le \frac{D}{\mathcal{R}(k)}. \tag{3.2}$$

*Proof.* Without loss of generality we may assume that $\delta \le 1$. Denote $Z(s) = \mathcal{L}(1/|s|)|s|^\alpha$. Note that by (2.2), $Z\left(\frac{1}{\mathcal{R}(k)}\right) = \frac{1 + o_{k\to\infty}(1)}{k}$. We have

$$\frac{Z(1/\mathcal{R}(k))}{Z(s)} = (|s|\mathcal{R}(k))^{-\alpha}\frac{\mathcal{L}(\mathcal{R}(k))}{\mathcal{L}(1/|s|)}. \tag{3.3}$$

Note that by the Potter bounds ([**4**, §1.5.4]), for any fixed $\beta < \alpha$, there is a constant $C_1(\beta)$ such that for $\frac{1}{\mathcal{R}(k)} \le |s| \le \delta$, we have

$$\frac{\mathcal{L}(\mathcal{R}(k))}{\mathcal{L}(1/|s|)} \le C_1(\beta)(\mathcal{R}(k))^{\alpha-\beta}|s|^{\alpha-\beta}. \tag{3.4}$$

Combining (3.3) and (3.4) we conclude that

$$Z(s) \ge \frac{1}{C_1}(\beta)Z\left(\frac{1}{\mathcal{R}(k)}\right)(|s|\mathcal{R}(k))^\beta \ge C_2(\beta)\frac{(|s|\mathcal{R}(k))^\beta}{k}$$

for a suitable $C_2(\beta)$.

Thus (3.1) implies that for $s$ with $|s| \in [1/\mathcal{R}(k), \delta]$

$$\left|\mu\left(e^{is\tau_k}\right)\right| \le e^{-C_3(\beta)(s\mathcal{R}(k))^\beta}. \tag{3.5}$$

Clearly, (3.5) holds for $s$ with $|s| \le 1/\mathcal{R}(k)$ as well since the left hand side is bounded by 1. Let $H(x) = \frac{1-\cos\delta x}{\pi\delta^2 x^2}$. Then $\hat{H}(s) = 1_{|s|<\delta}\left(\frac{1}{\delta} - \frac{|s|}{\delta^2}\right)$. Therefore for each $a$

$$\mu(H(\tau_k - a)) \le \frac{1}{2\pi}\int_{-\delta}^{\delta}\left|\mu\left(e^{-is\tau_k}\right)\right|\hat{H}(s)ds \le \frac{1}{2\pi\delta}\int_{-\delta}^{\delta}\left|\mu\left(e^{-is\tau_k}\right)\right|ds \le \frac{C_4}{\mathcal{R}(k)},$$

where the last step uses (3.5).

On the other hand $H(x) \ge \frac{47}{96\pi}$ if $|x| \le \delta/2$. Therefore

$$\mu\left(\tau_k - a \in \left[-\frac{\delta}{2}, \frac{\delta}{2}\right)\right) \le \frac{96\pi}{47}\mu(H(\tau_k - a)) \le \frac{C_5}{\mathcal{R}(k)}.$$

This proves our claim for intervals of size $\delta$. Since any interval of unit size can be covered by a bounded number of intervals of size $\delta$, the result follows. □

3.2. *Large deviations.*

PROPOSITION 3.2. *Suppose* (2.1). *Then there is a constant $D > 0$ so that for any $k \in \mathbb{Z}_+$,*

$$\mu(\tau_k(x) > t) \leq \frac{Dk\mathcal{L}(t)}{t^\alpha}. \tag{3.6}$$

*Proof.* Let us write

$$\tau^- = \tau 1_{\tau \leq t}, \quad \tau^+ = \tau 1_{\tau > t}, \quad \tau_k^\pm(x) = \sum_{j=0}^{k-1} \tau^\pm(f^j x).$$

Then

$$\mu(\tau_k > t) \leq \mu(\tau_k^- > t/2) + \mu(\tau_k^+ > t/2) \leq \mu(\tau_k^- > t/2) + \mu(\tau_k^+ > 0)$$

where the last step uses that $\tau_k^+ > 0$ implies $\tau_k^+ > t$. By (2.1), we have the following estimate for sufficiently large $t$ and for any $k$

$$\mu(\tau_k^+ > 0) = \mu(\exists j \in 0, \dots k-1 : \tau(f^j x) > t) \leq k\mu(\tau^+ > 0) \leq 2k\mathcal{L}(t)t^{-\alpha}.$$

Next, by the Karamata theory (see [**4**, Theorem 1.6.4]), there is some $C$ so that

$$\mu(\tau_k^-) = k\mu(\tau^-) \leq Ckt^{1-\alpha}\mathcal{L}(t).$$

Hence by the Markov inequality, we have

$$\mu\left(\tau_k^- \geq t/2\right) \leq C\frac{kt^{1-\alpha}\mathcal{L}(t)}{t/2} = 2Ck\mathcal{L}(t)t^{-\alpha}.$$

The proposition follows. □

As we will see, the following *quasi-independence* property holds in several interesting applications. The sets $\{(\tau \circ f^j) > t\}$ are called quasi-independent if there is some $K \in \mathbb{R}$ so that for any $j_1, j_2 \in \mathbb{Z}_+$,

$$\mu(\tau(f^{j_1}x) > t, \tau(f^{j_2}x) > t) \leq K\mu(\tau(f^{j_1}x) > t)\mu(\tau(f^{j_2}x) > t). \tag{3.7}$$

The next theorem shows that (3.6) is asymptotically sharp under the quasi-independence assumption. In particular, we can recover a large deviation estimate for sums of independent random variables (see [**7**, **32**]). Theorem 3.3 is not used in other parts of this paper, however we include this result since it is of independent interest (e.g., see related results for Young towers in [**17**, **25**]) and also because similar ideas will be used in Sections 4 and 5 to check (2.19).

Recall (2.12).

THEOREM 3.3. *Suppose that in addition to* (2.1) *the sets $\{(\tau \circ f^j) > t\}$ are quasi-independent. Then for every $\varepsilon > 0$ there is some $\delta = \delta(\varepsilon) > 0$ and $T = T(\varepsilon) < \infty$ such that for any $t > T$ and for every $k = 1, 2, \dots \lfloor \delta\mathcal{N}(t) \rfloor$,*

$$\left| \frac{t^\alpha}{k\mathcal{L}(t)} \mu(\tau_k > t) - 1 \right| \leq \varepsilon.$$

*Proof.* It is sufficient to prove the theorem with $\mathcal{N}(t)$ being replaced by $t^\alpha\mathcal{L}^{-1}(t)$. Indeed, recall that $\mathcal{N}(t)$ is asymptotically equivalent to $t^\alpha\mathcal{L}^{-1}(t)$. Thus if we prove the desired estimate for

$$k \leq \delta t^\alpha \mathcal{L}^{-1}(t), \tag{3.8}$$

it immediately follows for $k \leq (\delta/2)\mathcal{N}(t)$ assuming that $t > T_0$.

First, we prove the lower bound, i.e.

$$\frac{t^\alpha}{k\mathcal{L}(t)} \mu(\tau_k > t) \geq 1 - \varepsilon. \tag{3.9}$$

Observe that

$$\mu(\tau_k > t) \geq \mu\left(\max_j \tau(f^j x) > t\right) = \mu(\exists j \in 0, \dots, k-1 : \tau(f^j x) > t).$$

By the Bonferroni inequality this probability is bounded from below by

$$\sum_{j=0}^{k-1} \mu(\tau(f^j x) > t) - \sum_{j_1=0}^{k-2} \sum_{j_2=j_1+1}^{k-1} \mu(\tau(f^{j_1} x) > t, \tau(f^{j_2} x) > t).$$

Using the quasi-independence, (2.1), and the $f$-invariance of $\mu$ we conclude that there exists a constant $T_1(\varepsilon)$ such that

$$\mu(\tau_k > t) \geq \left(1 - \frac{\varepsilon}{2}\right) \frac{k\mathcal{L}(t)}{t^\alpha} - \frac{Kk^2\mathcal{L}^2(t)}{t^{2\alpha}},$$

for all $t \geq T_1(\varepsilon)$. This implies (3.9) for any $\delta < \frac{\varepsilon}{2K}$.

Next, we prove the upper bound, i.e.

$$\frac{t^\alpha}{k\mathcal{L}(t)} \mu(\tau_k > t) \leq 1 + \varepsilon. \tag{3.10}$$

The proof is similar to that of Proposition 3.2. Namely, we choose

$$\tau^- = \tau 1_{\tau \leq H}, \quad \tau^+ = \tau 1_{\tau > H}, \quad \tau_k^\pm = \sum_{j=0}^{k-1} \tau^\pm(f^j x)$$

this time with some $H < t$ (to be specified later). Then by the Karamata theory (see [**4**, Theorem 1.6.4]), there is a constant $\bar{C}$ so that

$$\mu(\tau_k^-) = k\mu(\tau^-) \leq \bar{C} k H^{1-\alpha} \mathcal{L}(H).$$

Hence by Markov inequality for each $\bar{\varepsilon} > 0$ we have

$$\mu\left(\tau_k^- \geq \bar{\varepsilon} t\right) \leq \bar{C} \frac{kH^{1-\alpha}\mathcal{L}(H)}{\bar{\varepsilon} t}. \tag{3.11}$$

Next

$$\mu\left(\tau_k^+ \geq (1-\bar{\varepsilon})t\right) \leq \mu\left(\tau_k^+ 1_{A_1} \geq (1-\bar{\varepsilon})t\right) + \mu\left(\tau_k^+ 1_{A_2} \geq (1-\bar{\varepsilon})t\right)$$

where $A_1$ is the set where $\tau(f^j x) > H$ for exactly one index $j \in [0, k-1]$ and $A_2$ is the set where $\tau(f^j x) > H$ for at least two indices $j \in [0, k-1]$. On $A_1$ we should have $\tau(f^j x) > (1-\bar{\varepsilon})t$ so

$$\mu\left(\tau_k^+ 1_{A_1} \geq (1-\bar{\varepsilon})t\right) \leq \sum_{j=0}^{k-1} \mu\left(\tau(f^j x) > (1-\bar{\varepsilon})t\right) \leq \frac{k\mathcal{L}((1-\bar{\varepsilon})t)}{[(1-\bar{\varepsilon})t]^\alpha} =: U.$$

Now we choose $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon)$ so that $(1-\bar{\varepsilon})^{-\alpha} < 1 + \varepsilon/5$ and then $T_2 = T_2(\varepsilon, \bar{\varepsilon})$ so that $\mathcal{L}((1-\bar{\varepsilon})t) < (1+\varepsilon/5)\mathcal{L}(t)$ for all $t > T_2$. Then

$$U \leq (1+\varepsilon/2) k\mathcal{L}(t) t^{-\alpha} \quad \text{for all} \quad t > T_2. \tag{3.12}$$

On the other hand the probability that there are two indices where $\tau \circ f^j$ is large can be estimated using (3.7) as

$$\sum_{j_1=0}^{k-2} \sum_{j_2=j_1+1}^{k-1} \mu(\tau(f^{j_1}x) > H, \tau(f^{j_2}x) > H) \le K k^2 \frac{\mathcal{L}^2(H)}{H^{2\alpha}}. \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), we see that there is a constant $C(\varepsilon)$ such that for $t \ge \max(T_1(\varepsilon), T_2(\varepsilon))$ we have

$$\mu(\tau_k > t) \le C(\varepsilon) \left[ \frac{kH^{1-\alpha}\mathcal{L}(H)}{t} + \frac{k^2\mathcal{L}^2(H)}{H^{2\alpha}} \right] + (1 + \varepsilon/2)k\mathcal{L}(t)t^{-\alpha}.$$

In order to verify (3.10), it remains to prove that for a suitable choice of $H$ we have

$$\frac{kH^{1-\alpha}\mathcal{L}(H)}{t} + \frac{k^2\mathcal{L}^2(H)}{H^{2\alpha}} \le \varepsilon' k\mathcal{L}(t)t^{-\alpha}, \tag{3.14}$$

where $\varepsilon' = \frac{\varepsilon}{2C(\varepsilon)}$. Indeed, set

$$\eta = (\varepsilon'/4)^{1/(1-\alpha)}, \quad H = \eta t. \tag{3.15}$$

Then $H^{1-\alpha}/t \le (\varepsilon'/4)t^{-\alpha}$. Since $\mathcal{L}$ is slowly varying, we have

$$\mathcal{L}(\eta t) < 2\mathcal{L}(t) \quad \text{for} \quad t > T_3(\varepsilon). \tag{3.16}$$

Multiplying the last two inequalities, we obtain

$$\frac{H^{1-\alpha}\mathcal{L}(H)}{t} \le \frac{\varepsilon'}{2}\mathcal{L}(t)t^{-\alpha}. \tag{3.17}$$

Next, we show that with $H$ given by (3.15) and $\delta$ sufficiently small, we have

$$\frac{k\mathcal{L}^2(H)}{H^{2\alpha}} \le \frac{\varepsilon'}{2}\mathcal{L}(t)t^{-\alpha}. \tag{3.18}$$

Clearly, (3.17) and (3.18) imply (3.14).

To prove (3.18), set $\delta = \eta^{2\alpha}\varepsilon'/8$. Recalling (3.8), we obtain $k\mathcal{L}(t)t^{-\alpha}\eta^{-2\alpha} < \varepsilon'/8$. Using (3.16) we find that $k\mathcal{L}^2(\eta t)t^{-\alpha}\eta^{-2\alpha} < \mathcal{L}(t)\varepsilon'/2$, which proves (3.18). □

## 4. *Independent Random Variables.*

Here we consider the case where $t_j = \tau \circ f^{j-1}$ are i.i.d. random variables having non-lattice distribution. We will recover a result of [**12**]. We note that the optimal results for the infinite measure renewal theorem for independent random variables are obtained in [**6**]. However, we include the section on independent random variables in order to illustrate our approach in the simplest possible setting.

To see how the i.i.d. case fits into our abstract setup, set $X = \mathbb{R}_+^\infty$. Assume that a Borel probability measure $\mathbb{P}$ is given on $[c, \infty)$ with $c > 0$ so that $\mathbb{P}([t, \infty)) \sim \mathcal{L}(t)t^{-\alpha}$. Furthermore, we assume that $\mathbb{P}$ is not supported on any discrete subgroup of $\mathbb{R}$. Let $f : X \to X$ be the left shift, that is for $(t_1, t_2, ...) \in X$, $f((t_1, t_2, ...)) = (t_2, t_3, ...)$. Let $\mu = \mathbb{P}^\infty$ (more precisely, we consider the $\sigma$-algebra generated by finite dimensional cylinder sets on $X$ and the unique measure $\mu$ whose projection on the first $N$ coordinates is equal to $\mathbb{P}^N$ for any finite $N$). Finally, let $\tau((t_1, t_2, ...)) = t_1$.

We need to check (2.3) and (2.5). Let us first note that (3.1) is satisfied in our case (see e.g. [**19**, Eq. (2.6.38)]) and hence (3.2) holds.

4.1. *Local Limit Theorem.*

In the special case $A = B = X$, (2.3) is proven in [**30**]. Now let $A$ and $B$ be finite dimensional cylinder sets, i.e.

$$A = \{(t_1, t_2 \dots) : (t_1, t_2 \dots t_n) \in \mathbb{A}\}, \quad B = \{(t_1, t_2, \dots t_n) \in \mathbb{B}\}.$$

for some Borel sets $\mathbb{A}, \mathbb{B} \subset \mathbb{R}_+^n$. Let us write $\mathbb{A}_R = \mathbb{A} \cap [0, R]^n$. We have

$$\mu(x \in A, f^k x \in B, \tau_k \in I) = \int_{\mathbb{A}} \mu\left(\tau_{k-n} \in I - \sum_{j=1}^n u_j\right) d\mathbb{P}(u_1) \otimes \dots \otimes d\mathbb{P}(u_n) \mathbb{P}^n(\mathbb{B}).$$

Decompose the above integral as

$$\int_{\mathbb{A}_R} \dots + \int_{\mathbb{A} \setminus \mathbb{A}_R} \dots =: I_1 + I_2.$$

Using the Local Limit Theorem of [**30**] for $I_1$, the anticoncentration inequality (3.2) for $I_2$ and finally letting $R \to \infty$, we obtain (2.3).

4.2. *Local Large Deviations: $\alpha > \frac{1}{2}$.*

Next we prove (2.5) if $\alpha > \frac{1}{2}$.

PROPOSITION 4.1. *There is a constant $\bar{C}$ such that for any $k$ and for any $t, u$ with $0 < t \le u$, we have*

$$\mu(\tau_k \in [u, u+1]) \le \frac{\bar{C} k \mathcal{L}(t)}{t^\alpha \mathcal{R}(k)}.$$

This gives (2.5) with $\beta_1 = \alpha$, $\beta_2 = -1$, $\beta_3 = 1$. Thus $\frac{\beta_3}{\alpha} + \beta_2 = \frac{1}{\alpha} - 1 < 1$ iff $\alpha > \frac{1}{2}$.

*Proof.* Denote $I = [u, u+1]$. Then,

$$\mu(\tau_k \in I) \le \mu(\tau_{k/2} > t/2, \tau_k \in I) + \mu(\tau_k - \tau_{k/2} > t/2, \tau_k \in I).$$

By symmetry is suffices to consider the first term

$$\mu(\tau_{k/2} > t/2, \tau_k \in I) = \mu(\tau_{k/2} > t/2) \mu(\tau_k - \tau_{k/2} \in I - \tau_{k/2} | \tau_{k/2} > t/2).$$

The first term is bounded by $\frac{Ck\mathcal{L}(t)}{t^\alpha}$ due to (3.6) and the second term is bounded by $\frac{C}{\mathcal{R}(k)}$ by (3.2). □

4.3. *Local Large Deviations: $\alpha \le \frac{1}{2}$.*

In case $\alpha \le \frac{1}{2}$, an example of [**14**] shows that the condition (2.1) alone is insufficient to obtain the conclusion of Theorem 2.7. Since we verified all the other assumptions of that theorem, it follows that (2.19) does not hold in general. Here we obtain (2.19) assuming an additional control over the tail of the roof function.

PROPOSITION 4.2. *Suppose that for some (and hence all) $K$ there is a constant $C(K)$ such that for any $t > 0$*

$$\mu(\tau \in [t, t+K]) \le \frac{C(K)\mathcal{L}(t)}{t^{1+\alpha}}. \tag{4.1}$$

*Then there are constants $C_1, C_2$ such that for any $k$ and for any $t, u$ with $0 < t \le u$,*

$$\mu(\tau_k \in [u, u+1]) \le \frac{C_1 \mathcal{L}(t) k}{t^{1+\alpha}} + \frac{C_2}{t}. \tag{4.2}$$

This gives (2.19) with

$$\beta_{1,1} = 1 + \alpha, \quad \beta_{1,2} = -1, \quad \beta_{1,3} = 0, \quad \beta_{2,1} = 1, \quad \beta_{2,2} = \beta_{3,2} = 0.$$

We note that in case $\tau$ is integer-valued, a stronger result, namely a precise asymptotics, in the style of Theorem 3.3, is proven in [**11**]. It is likely that in case (4.1) holds with asymptotic equality, a similar result holds in the present setting as well. However, the one-sided bound established here is sufficient for our purposes.

*Proof.* Choosing $C_2 > 2$, (4.2) follows for $t < 2$. Thus we can assume $t \ge 2$.

We proceed as in the proof of Proposition 4.1. Denote $I = [u, u+1]$. Fix large constants $L$ and $r$. Namely we will take $L = 5$ and will impose finitely many lower bounds on $r$ (see equations (4.5), (4.10), (4.11), (4.14) and (4.15) for the precise conditions on $r$ we require). One can then take the biggest of these lower bounds. Set

$$\tau_k^+ = \sum_{j=0}^{k-1} \tau \circ f^j 1_{\tau \circ f^j > t}, \quad \tau_k^- = \sum_{j=0}^{k-1} \tau \circ f^j 1_{\tau \circ f^j < \mathcal{R}(k) L^r}, \quad \tau_{k,l} = \sum_{j=0}^{k-1} \tau \circ f^j 1_{\tau \circ f^j \in [t/L^{l+1}, t/L^l]}.$$

Let $\bar{l}_{k,t}$ be the smallest integer $l$ that satisfies $tL^{-l-1} \le \mathcal{R}(k) L^r$. Then

$$\tau_k \le \tau_k^+ + \left( \sum_{l=0}^{\bar{l}_{k,t}} \tau_{k,l} \right) + \tau_k^-.$$

Since $t/4 + (\sum_{l=0}^{\bar{l}_{k,t}} t/(4 \times 2^l)) + t/4 \le t \le u$, we have

$$\{\tau_k \in I\} \subset A_+ \cup \bigcup_{l=0}^{\bar{l}_{k,t}} A_l \cup A_-,$$

where

$$A^{\pm} = \{\tau_k^{\pm} > t/4, \tau_k \in I\}, \quad A_l = \{\tau_{k,l} > t/(4 \times 2^l), \tau_k \in I\}.$$

(In fact, we have $A^+ = \{\tau_k^+ > s, \tau_k \in I\}$ for any $s \in (0, t]$ as $\tau_k^+ > 0$ is equivalent to $\tau_k^+ > t$.) Therefore it suffices to show that $\mu(A_-) + \mu(\bigcup_{l=0}^{\bar{l}_{k,t}} A_l) + \mu(A_+)$ is bounded by the right hand side of (4.2). We thus divide the proof into three steps.

**Step 1: Estimating $\mu(A^+)$.** To estimate $\mu(A^+)$ let $\mathbf{j}$ be the first index $j \le k-1$ when $\tau \circ f^j > t$. Conditioning on the values of $\tau \circ f^j$ for $j \ne \mathbf{j}$ we get

$$\mu(A^+ 1_{\mathbf{j}=j_0}) = \tag{4.3}$$

$$\int_{\mathcal{A}_{j_0}} \mathbb{P}\left( \tau > t \text{ and } \tau \in I - \sum_{\substack{j \in [0,k-1] \\ j \ne j_0}} v_j \right) d\mathbb{P}(v_1) \otimes \cdots \otimes \mathbb{P}(v_{j_0-1}) \otimes \mathbb{P}(v_{j_0+1}) \otimes \cdots \otimes d\mathbb{P}(u_n)$$

where

$$\mathcal{A}_{j_0} = \left\{ (v_0, ..., v_{j_0-1}, v_{j_0+1}, ..., v_{k-1}) : v_l \le t \text{ for all } l < j_0 \text{ and } \sum_{\substack{j \in [0,k-1] \\ j \ne j_0}} v_j + t \le u+1 \right\}.$$

Now (4.1) shows that the integrand in (4.3) is at most

$$\frac{C(1)\mathcal{L}(s)}{s^{1+\alpha}} \text{ where } s = s(v_1, \dots, v_k) = u - \sum_{\substack{j\in[0,k-1]\\ j\neq j_0}} v_j \geq t-1.$$

By the Potter bounds [**4**, §1.5.4], there is a constant $\bar{C}$ such that for all $s \geq t-1 \geq 1$, we have $\frac{\mathcal{L}(s)}{s^{1+\alpha}} \leq \bar{C}\frac{\mathcal{L}(t)}{t^{1+\alpha}}$. Hence for each $j_0$, we have

$$\mu(A^+ 1_{\mathbf{j}=j_0}) \leq \frac{C\mathcal{L}(t)}{t^{1+\alpha}}\mathbb{P}^{k-1}(\mathcal{A}_{j_0}) \leq \frac{C\mathcal{L}(t)}{t^{1+\alpha}},$$

where $C = C(1)\bar{C}$. Summing over $j_0$ we obtain

$$\mu(A^+) \leq \frac{C\mathcal{L}(t)k}{t^{1+\alpha}}. \tag{4.4}$$

**Step 2: Estimating $\sum_{l=0}^{\bar{l}_{k,t}} \mu(A_l)$.**

Note that on $A_l$, there are at least $\lceil \frac{t}{4\times 2^l}\frac{L^l}{t}\rceil = \lceil \frac{1}{4}\left(\frac{L}{2}\right)^l \rceil =: m(l)$ indices $j$ such that $\tau \circ f^j \in [t/L^{l+1}, t/L^l]$. Let $\mathbf{j}$ denote the first such index $j$. Proceeding as in Step 1, we define

$$\mathcal{A}_{l,j_0} = \Bigg\{(v_0, ..., v_{j_0-1}, v_{j_0+1}, ..., u_{k-1}) : u_l \notin [t/L^{l+1}, t/L^l] \text{ for } l < j_0$$
$$\text{and } \left(\left(\sum_{\substack{j\in[0,k-1]\\ j\neq j_0}} v_j\right) + [t/L^{l+1}, t/L^l]\right) \cap I \neq \emptyset\Bigg\}.$$

By (4.1), we have

$$\mu\left(\tau + \sum_{\substack{j\in[0,k-1]\\ j\neq j_0}} v_j \in I\right) \leq C(1)\frac{\mathcal{L}(s)}{s^{1+\alpha}}, \quad \text{where} \quad s = s(v_1, \dots, v_k) = u - \sum_{\substack{j\in[0,k-1]\\ j\neq j_0}} v_j.$$

From the definition of $\mathcal{A}_{l,j_0}$ it follows that

$$\frac{t}{L^{l+1}} \leq s(v_1, \dots v_k) \leq \frac{t}{L^l} + 1$$

for any $(v_1, \dots v_k) \in \mathcal{A}_{j_0,l}$. By the choice of $\bar{l}_{k,t}$, we have $t/L^{\bar{l}_{k,t}+1} > L^{r-1}\inf_k \mathcal{R}(k)$ (which is equal to $L^{r-1}\mathcal{R}(1)$ as we assumed that $\mathcal{R}$ is monotone). Thus choosing $r$ sufficiently large, we can assume

$$\sup_{s\in[t/L^{l+1}, t/L^l+1]} \mathcal{L}(s) \leq 2\mathcal{L}\left(\frac{t}{L^{l+1}}\right) \tag{4.5}$$

for all $l = 0, 1, ..., \bar{l}_{k,t}$. Consequently, for any $(v_1, \dots v_k) \in \mathcal{A}_{j_0,l}$, we have

$$\mu\Big(\tau + \sum_{\substack{j\in[0,k-1]\\ j\neq j_0}} v_j \in I\Big) \leq C^*\mathcal{L}\left(\frac{t}{L^{l+1}}\right)\left(\frac{L^{l+1}}{t}\right)^{\alpha+1}, \tag{4.6}$$

where $C^* = 2C(1)$. Now arguing as in Step 1, we obtain

$$\mu(A_l 1_{\mathbf{j}=j_0}) \leq C^*\mathcal{L}\left(\frac{t}{L^{l+1}}\right)\left(\frac{L^{l+1}}{t}\right)^{\alpha+1}\mathbb{P}^{k-1}(\mathcal{A}_{l,j_0}). \tag{4.7}$$

Now in contrast with Step 1, it is not sufficient to estimate the last factor by 1, we need a more precise control. To this end we observe that if $Y$ is a binomially distributed random variable with parameters $\bar{k} \in \mathbb{Z}_+$ and $p \in (0,1)$, then for every non-negative integer $a$,

$$P(Y \geq a) \leq \bar{k}^a p^a. \tag{4.8}$$

Indeed, letting $(i_1, i_2, \ldots, i_a)$ be the first $a$ trials which result in a success, we have

$$P(Y \leq a) \leq \sum_{1 \leq i_1 < i_2 < \cdots < i_a \leq \bar{k}} P(\text{ trials } i_1, \ldots, i_a \text{ are successful }) \leq \bar{k}^a p^a.$$

Applying (4.8) with $\bar{k} = k - j_0 - 1$ and $p_{t,l} := \mathbb{P}([t/L^{l+1}, t/L^l])$ and $a = m(l) - 1$, we obtain

$$\mathbb{P}^{k-1}(\mathcal{A}_{l,j_0}) \leq \mu(\#\{j > j_0 : \tau \circ f^j \in [t/L^{l+1}, t/L^l]\} \geq m(l) - 1) \leq \bar{k}^a p_{t,l}^a \tag{4.9}$$

Now we simply use (2.1) instead of (4.1) to estimate $p_{t,l}$. Namely if $r$ is sufficiently large so that $t/L^l$ is large for all $l \leq \bar{l}_{k,t}$, then

$$p_{t,l} \leq \mu\left(\tau \geq \frac{t}{L^{l+1}}\right) \leq 2\left(\frac{L^{l+1}}{t}\right)^\alpha \mathcal{L}\left(\frac{t}{L^{l+1}}\right). \tag{4.10}$$

Combining (4.9) and (4.10), obtain

$$\mathbb{P}^{k-1}(\mathcal{A}_{l,j_0}) \leq \left[2k\left(\frac{L^{l+1}}{t}\right)^\alpha \mathcal{L}\left(\frac{t}{L^{l+1}}\right)\right]^{m(l)-1}.$$

Substituting to (4.7) and summing over $j_0$, we obtain

$$\mu(A_l) \leq C^* \left[2k\left(\frac{L^{l+1}}{t}\right)^\alpha \mathcal{L}\left(\frac{t}{L^{l+1}}\right)\right]^{m(l)} \frac{L^{l+1}}{t} =: q_l.$$

Since, $m(0) = 1$, it follows that there is a constant $C_L$ such that

$$q_0 \leq \frac{C_L \mathcal{L}(t) k}{t^{1+\alpha}}.$$

Thus in order to complete Step 2, it suffices to prove that for $L$ and $r$ sufficiently large,

$$q_{l+1}/q_l \leq 1/2 \tag{4.11}$$

holds for all $l \leq \bar{l}_{k,l} - 1$.

First observe that by (4.5), we have $\mathcal{L}\left(\frac{t}{L^{l+2}}\right) \leq 2\mathcal{L}\left(\frac{t}{L^{l+1}}\right)$. Consequently,

$$\frac{q_{l+1}}{q_l} \leq \left(\frac{\left(4k\mathcal{L}\left(\frac{t}{L^{l+1}}\right)\right)^{1/\alpha} L^{l+1}}{t}\right)^{\alpha(m(l+1)-m(l))} L^{\alpha m(l+1)+1}. \tag{4.12}$$

Recall that by the choice of $\bar{l}_{k,t}$, we have

$$\mathcal{R}(k) L^r < t/L^{\bar{l}_{k,t}} \leq \mathcal{R}(k) L^{r+1}. \tag{4.13}$$

By the Potter bounds ([**4**, §1.5.4]) if $r$ is large and hence $t/L^l$ is large for all $l \leq \bar{l}_{k,t}$, then

$$\mathcal{L}(t/L^{l+1}) \leq 2L^{\alpha(\bar{l}_{k,t}-l-1)} \mathcal{L}(t/L^{\bar{l}_{k,t}}). \tag{4.14}$$

Next, we compute

$$\frac{\left(4k\mathcal{L}\left(\frac{t}{L^{l+1}}\right)\right)^{1/\alpha} L^{l+1}}{t} \overset{(4.14)}{\leq} \frac{L^{\bar{l}_{k,t}-l-1}\left(8k\mathcal{L}\left(\frac{t}{L^{\bar{l}_{k,t}}}\right)\right)^{1/\alpha} L^{l+1}}{t} = \frac{L^{\bar{l}_{k,t}}\left(8k\mathcal{L}\left(\frac{t}{L^{\bar{l}_{k,t}}}\right)\right)^{1/\alpha}}{t}$$
$$\overset{(4.13)}{\leq} \frac{L^{\bar{l}_{k,t}}\left(16k\mathcal{L}\left(\mathcal{R}(k)\right)\right)^{1/\alpha}}{t} \leq \frac{L^{\bar{l}_{k,t}}\bar{C}^{1/\alpha}\mathcal{R}(k)}{t} \overset{(4.13)}{\leq} \bar{C}^{1/\alpha} L^{-r},$$

where $\bar{C}$ is such that $k\mathcal{L}(\mathcal{R}(k)) \leq \frac{\bar{C}}{16}k^\alpha$ holds for all $k$ (such $\bar{C}$ exists by (2.2)). Substituting into (4.12), we find

$$\frac{q_{l+1}}{q_l} \leq (\bar{C}^{1/\alpha}L^{-r})^{\alpha\frac{1}{4}(\frac{L}{2}-1)(\frac{L}{2})^l} L^{\alpha\frac{1}{4}(\frac{L}{2})^{l+1}+1}$$

$$= (\bar{C}^{1/\alpha}L^{-r+\frac{L}{L-2}})^{\alpha\frac{1}{4}(\frac{L}{2}-1)(\frac{L}{2})^l} L. \tag{4.15}$$

Now choosing $L = 5$ and $r$ sufficiently large, we find that $q_{l+1}/q_l \leq 1/2$. We have verified (4.11) and thus completed Step 2.

**Step 3: Estimating $\mu(A^-)$.**

We have

$$\mu(A^-) \leq \mu\left(\sum_{j=0}^{k/2}\tau\circ f^j 1_{\tau\circ f^j<\mathcal{R}(k)L^r} > \frac{t}{8}, \tau_k \in I\right) + \mu\left(\sum_{j=k/2+1}^{k}\tau\circ f^j 1_{\tau\circ f^j<\mathcal{R}(k)L^r} > \frac{t}{8}, \tau_k \in I\right).$$

We estimate the first term, the second one is similar. By the Markov inequality

$$\mu\left(\sum_{j=0}^{k/2}\tau\circ f^j 1_{\tau\circ f^j<\mathcal{R}(k)L^r} > \frac{t}{8}\right) \leq \frac{8}{t}\mu\left(\sum_{j=0}^{k/2}\tau\circ f^j 1_{\tau\circ f^j<\mathcal{R}(k)L^r}\right) \leq C\frac{k\left(\mathcal{R}(k)\right)^{1-\alpha}\mathcal{L}(\mathcal{R}(k))}{t}$$

where the last step relies on Karamata theory ([**4**, Theorem 1.6.4]).

On the other hand by (3.2)

$$\mu\left(\tau_k \in I\Big|\sum_{j=0}^{k/2}\tau\circ f^j 1_{\tau\circ f^j<k^{1/\alpha}L^r} > \frac{t}{8}\right) \leq \frac{D}{\mathcal{R}(k)}.$$

Combining the last two displays we obtain

$$\mu\left(\sum_{j=0}^{k/2}\tau\circ f^j 1_{\tau\circ f^j<k^{1/\alpha}L^r} > \frac{t}{8}, \tau_k \in I\right) \leq \frac{\bar{D}}{t}\times\frac{k\mathcal{L}(\mathcal{R}(k))}{\mathcal{R}^\alpha(k)}.$$

and hence

$$\mu(A^-) \leq \frac{\tilde{D}}{t}\times\frac{k\mathcal{L}(\mathcal{R}(k))}{\mathcal{R}^\alpha(k)}.$$

Now (2.2) gives

$$\mu(A^-) \leq \frac{\hat{D}}{t}. \tag{4.16}$$

This completes Step 3. □

## 5. *LSV map.*

### 5.1. *The result.*

Let $\tilde{X} = [0,1]$ and $\tilde{f} : \tilde{X} \to \tilde{X}$ be the map

$$\tilde{f}(x) = \begin{cases} x(1+(2x)^r) & \text{if } x \notin X; \\ 2x-1 & \text{if } x \in X \end{cases}$$

where $X = [1/2, 1]$. Consider the special flow $\tilde{g}_t$ of $\tilde{f}$ under a roof function $\tilde{\tau}$ which is positive and piecewise Hölder, in the sense, that its restrictions on both $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ are Hölder. Let $\tilde{\Omega}$ be the phase space of this flow. That is,

$$\tilde{\Omega} = \{(x,s) : x \in X, s \geq 0\}/\approx, \text{ where } (x, s+\tau(x)) \approx (\tilde{f}(x), s) \text{ for any } x \in \tilde{X} \text{ and } s \geq 0.$$

By [**22**], there is a unique (up to scaling) ergodic absolutely continuous $\tilde{f}$-invariant measure $\tilde{\mu}$ on $\tilde{X}$. We assume $r > 1$. Then the invariant measure is infinite. Let us normalize it so that $\tilde{\mu}([1/2, 1]) = 1$. Then $\tilde{\nu}$, defined by $d\tilde{\nu}(x, s) = d\tilde{\mu}(x)ds$ is an infinite invariant measure of $\tilde{g}_t$.

THEOREM 5.1. *Assume that $\tilde{\tau}$ is irrational. Then for any $\varepsilon > 0$ and for any $\tilde{A}, \tilde{B} \subset [\varepsilon, 1]$ with $\tilde{\mu}(\partial\tilde{A}) = \tilde{\mu}(\partial\tilde{B}) = 0$, for any*

$$0 < \tilde{a}_1 < \tilde{a}_2 < \inf_{x\in\tilde{A}}\{\tilde{\tau}(x)\}, \quad 0 < \tilde{b}_1 < \tilde{b}_2 < \inf_{x\in\tilde{B}}\{\tilde{\tau}(x)\}, \quad \tilde{\mathcal{A}} = \tilde{A} \times [\tilde{a}_1, \tilde{a}_2], \quad \tilde{\mathcal{B}} = \tilde{B} \times [\tilde{b}_1, \tilde{b}_2]$$

*we have*

$$\lim_{t\to\infty} \tilde{\nu}(\tilde{\mathcal{A}} \cap \tilde{g}_{-t}\tilde{\mathcal{B}})t^{1-1/r} = \hat{c}\tilde{\nu}(\tilde{\mathcal{A}})\tilde{\nu}(\tilde{\mathcal{B}}). \tag{5.1}$$

Recall from Section 2 that the irrationality condition is necessary for (5.1). We also note that irrationality holds for typical roof functions $\tilde{\tau}$. In particular, a sufficient condition for the irrationality of $\tilde{\tau}$ is that there are two periodic orbits for the flow $\tilde{g}$, the ratio of whose periods is irrational, see, for example, the discussion in [**16**, page 394].

To reduce Theorem 5.1 to our setting we note that $\tilde{g}$ can be represented as a special flow over the first return map $f : X \to X$. Specifically, let $R(x) = \min\{n \geq 1 : \tilde{f}^n(x) \in X\}$ be the first return time to $X$ and let $f : X \to X$, $f(x) = \tilde{f}^{R(x)}(x)$ be the first return map. Let us also extend the definition of $R$ to $\tilde{X} \setminus X$ with the same formula (first hitting time). For a function $\phi : \tilde{X} \to \mathbb{R}$, let $\phi_X : X \to \mathbb{R}$ be defined by $\phi_X(x) = \sum_{i=0}^{R(x)-1} \phi(\tilde{f}^i(x))$. Define the roof function $\tau : X \to \mathbb{R}_+$ by $\tau = (\tilde{\tau})_X$. As before, $g_t$ is the special flow under roof function $\tau$, $\Omega$ is its phase space and $\nu$ with $d\nu(x, s) = d\mu(x)ds$ is a $g_t$-invariant measure. There is a natural surjection $\iota : \Omega \to \tilde{\Omega}$ which maps the class of $(x, s)$ w.r.t. the equivalence relation $\sim$ to the class with respect to the equivalence relation $\approx$ (note that by definition $(x, s) \sim (x', s')$ implies $(x, s) \approx (x', s')$). Clearly, $\iota$ is not invertible. Thus $(\tilde{\Omega}, \tilde{\nu}, \tilde{g}_t)$ is a factor of $(\Omega, \nu, g_t)$.

A cylinder of length $n$ (or shortly, $n$-cylinder) is a set

$$\{x \in X : R(x) = m_1, R(fx) = m_2 \ldots R(f^{n-1}x) = m_n\}.$$

Let us consider the topology on $X$ generated by the cylinder sets. Let us also fix a metric $d(x, y) = \theta^{s(x,y)}$, where $s(x, y)$ is the smallest $n$ so that $x$ and $y$ belong to different cylinders of length $n$ and $\theta < 1$ is sufficiently close to 1.

LEMMA 5.2. *$\tau$ is rational if and only if $\tilde{\tau}$ is rational.*

*Proof.* Assume that $\tilde{\tau} = b\tilde{\psi} + \mathfrak{h} - \mathfrak{h} \circ \tilde{f}$ holds on $\tilde{X}$ with $\tilde{g} : \tilde{X} \to \mathbb{Z}$. Then by definition, $\tau = (\tilde{\tau})_X = b(\tilde{\psi})_X + \mathfrak{h} - \mathfrak{h} \circ f$ on $X$. Thus $\tau$ is rational if $\tilde{\tau}$ is rational.

Next, assume that $\tau = b\psi + \mathfrak{h} - \mathfrak{h} \circ f$ on $X$. Define the functions $\psi', \mathfrak{h}', \tau' : \tilde{X} \to \mathbb{R}$ by

$$\psi'(x) = \psi(x)1_{\{x\in X\}}, \quad \mathfrak{h}'(x) = \mathfrak{h}(x)1_{\{x\in X\}}, \quad \tau' = b\psi' + \mathfrak{h}' - \mathfrak{h}' \circ \tilde{f}.$$

Observe that by construction, $\tau'_X = \tau$ on $X$. In general $\tau'$ may not be equal to $\tilde{\tau}$ on $\tilde{X}$. However $\tilde{\tau} - \tau' = \mathfrak{h}'' - \mathfrak{h}'' \circ \tilde{f}$ on $\tilde{X}$, where $\mathfrak{h}'' : \tilde{X} \to \mathbb{R}$ satisfies $\mathfrak{h}''(x) = \sum_{i=0}^{R(x)-1} \tilde{\tau}(\tilde{f}^i(x)) - \tau'(\tilde{f}^i(x))$. We conclude that $\tilde{\tau} = b\psi' + \mathfrak{h}' + \mathfrak{h}'' - \mathfrak{h}' \circ \tilde{f} - \mathfrak{h}'' \circ \tilde{f}$. Thus $\tilde{\tau}$ is rational if $\tau$ is rational. $\square$

PROPOSITION 5.3. *Assume that $\tau$ in irrational. Then for any $A, B \subset X$ with $\mu(\partial A) = \mu(\partial B) = 0$, for any $\mathcal{A} = A \times [a_1, a_2]$, $\mathcal{B} = B \times [b_1, b_2]$ we have*

$$\lim_{t\to\infty} \nu(\mathcal{A} \cap g_{-t}\mathcal{B})t^{1-1/r} = \hat{c}\nu(\mathcal{A})\nu(\mathcal{B}). \tag{5.2}$$

First, we prove Proposition 5.3 and then derive Theorem 5.1 from Proposition 5.3 and Lemma 5.2.

5.2. *Special flow over the induced system.*

*Proof of Proposition 5.3.* The proof of Proposition 5.3 is divided into two steps. In Step 1, we check that either (2.21) or (2.22) holds. In Step 2, we check that (2.23) holds. By the results of Section 1 (with $\alpha = 1/r$), these will imply Proposition 5.3.

**Step 1: Checking (2.21) and (2.22).**

First, we note that by [**1**, Theorem 6.3], (2.21) holds if $\tau : X \to \mathbb{R}$ is aperiodic. According to [**1**], the function $\tau : X \to \mathbb{R}$ is aperiodic if there is no $\lambda \in \mathcal{S}^1$ (here $\mathcal{S}^1$ is the complex unit circle) and measurable function $\mathfrak{g} : X \to \mathcal{S}^1$ (other than the trivial $\lambda = 1$, $\mathfrak{g} = 1$) satisfying

$$e^{it\tau(x)} = \lambda\mathfrak{g}(x)/\mathfrak{g}(fx). \tag{5.3}$$

Observe that this definition coincides with definition of periodicity. Indeed, if $\tau(x) = \mathbf{a} + \mathfrak{h}(x) - \mathfrak{h}(f(x)) + \frac{2\pi}{t}\psi(x)$ with $\psi : X \to \mathbb{Z}$, then (5.3) holds with $\mathfrak{g} = e^{it\mathfrak{h}}$. Conversely, assume that (5.3) holds. Then, by Corollary 2.2 of [**1**], $\mathfrak{g}$ is Hölder. Next, we define a Hölder function $\mathfrak{h}$ which satisfies $e^{i\mathfrak{h}} = \mathfrak{g}$. By the Hölder property of $\mathfrak{g}$, there is some $K$ such that the oscillation of $\mathfrak{g}$ on $K$-cylinders is less than $\sqrt{2}$. For any $K$-cylinder $\xi$, fix some $x_\xi \in \xi$ and define $\mathfrak{h}(x_\xi)$ as the only number in $[0, 2\pi)$ that satisfies $e^{i\mathfrak{h}(x_\xi)} = \mathfrak{g}(x_\xi)$. Then for any $y \in \xi$, we choose the unique $\mathfrak{h}(y)$ which satisfies $|\mathfrak{h}(y) - \mathfrak{h}(x_\xi)| < \pi$ and $e^{i\mathfrak{h}(y)} = \mathfrak{g}(y)$. By construction, $\mathfrak{h}$ is Hölder. We have now $\tau(x) = \mathbf{a} + \mathfrak{h}(x) - \mathfrak{h}(f(x)) + \psi(x)$, where $\mathbf{a} = -\frac{i}{t}\log\lambda$ and $\psi : X \to \frac{2\pi}{t}\mathbb{Z}$. Hence (2.9) holds, so the definition of [**1**] is equivalent to ours. It follows that (2.21) holds in the aperiodic case.

Let us now assume that $\tau$ is periodic irrational. By the previous paragraph, we can assume that $\mathfrak{h}$ is Hölder. In order to verify (2.22), it is enough to consider test functions of the form $\phi(x, y, z) = 1_{x\in\mathcal{C}}1_{y\in\mathcal{D}}\phi(z)$, where $\mathcal{C}$ and $\mathcal{D}$ are cylinders and $\phi(z)$ is compactly supported. Then (2.22) follows from [**1**, Theorem 6.5], applied to $\psi$, and the continuous mapping theorem.

**Step 2: Checking (2.23).**

We note that (3.1) is verified in [**1**]. In particular (3.2) holds.

The Gibbs-Markov property of $f$ implies that there is a constant $K$ such that if $\mathcal{C}_1, \mathcal{C}_2$ are cylinders and the length of $\mathcal{C}_1$ is less than $j$

$$\mu(\mathcal{C}_1 \cap f^{-j}\mathcal{C}_2) \le K\mu(\mathcal{C}_1)\mu(\mathcal{C}_2). \tag{5.4}$$

Also applying (5.4) inductively we see that if $\mathcal{C}_1, \mathcal{C}_2 \dots \mathcal{C}_l$ are cylinders and $j_1, j_2 \dots j_{l-1}$ are numbers with length$(\mathcal{C}_m) \le j_m$ then

$$\mu(\mathcal{C}_1 \cap f^{-j_1}\mathcal{C}_2 \dots \cap f^{-j_1-\dots-j_{l-1}}\mathcal{C}_l) \le K^{l-1}\mu(\mathcal{C}_1)\mu(\mathcal{C}_2)\dots\mu(\mathcal{C}_l) \tag{5.5}$$

In particular (3.7) holds and so (3.6) is satisfied. This allows to check (2.5) in case $r < 2$ and so $\alpha > \frac{1}{2}$. In the general case we verify (2.23) which is the consequence of the Proposition 5.4 below. □

5.3. *Local Large Deviations*

PROPOSITION 5.4. $\mu(\tau_k > t, \tau_k \in I) \le \dfrac{C_1 k}{t^{1+\alpha}} + \dfrac{C_2}{t}.$

*Proof.* Note that (4.1) holds with $\alpha = \frac{1}{r}$ ([**22**]). We follow the approach of Proposition 4.2. In particular, we shall use the notation of Proposition 4.2. We need to adapt Steps 1-3 of our proof of Proposition 4.2 to the present case. There are two differences in the proof: now $\mathcal{L}(t)$ is asymptotically constant (a simplification) and the independence is replaced by quasi-independence (a complication).

We will say that a cylinder $\mathcal{D}$ of length 1 is *high* if $\tau > t/3$ on $\mathcal{D}$. We say that the cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ of lengths $m_1$ and $m_2$ respectively are compatible with $\mathcal{D}$ if $m_1 + m_2 = k - 1$ and there is a point $x \in \mathcal{C}_1 \bigcap f^{-m_1}\mathcal{D} \bigcap f^{-m_1-1}\mathcal{C}_2$ such that $\tau_k(x) \in I$. Thus

$$\mu(A^+) \leq \sum_{\mathcal{C}_1,\mathcal{D},\mathcal{C}_2} \mu\left(\mathcal{C}_1 \bigcap f^{-m_1}\mathcal{D} \bigcap f^{-m_1-1}\mathcal{C}_2\right)$$

where the sum is over compatible cylinders. By Gibbs-Markov property

$$\mu(A^+) \leq K^2 \sum_{\mathcal{C}_1,\mathcal{D},\mathcal{C}_2} \mu(\mathcal{C}_1)\mu(\mathcal{D})\mu(\mathcal{C}_2).$$

Next given $\mathcal{C}_1, \mathcal{C}_2, I$ there is an interval $\hat{I}$ of bounded size such that if $\mathcal{C}_1, \mathcal{D}$, and $\mathcal{C}_2$ are compatible, then $\tau(x) \in \hat{I}$ for each $x \in \mathcal{D}$. ($\hat{I}$ maybe empty if there are no high cylinders compatible with $\mathcal{C}_1$ and $\mathcal{C}_2$). Therefore for each $\mathcal{C}_1, \mathcal{C}_2$

$$\sum_{\mathcal{D}:\mathcal{C}_1,\mathcal{D},\mathcal{C}_2 \text{ are compatible}} \mu(\mathcal{D}) \leq \frac{C}{t^{1+\alpha}}.$$

On the other hand for each $m_1, m_2$, $\sum_{\mathcal{C}_j:\text{length}(\mathcal{C}_j)=m_j} \mu(\mathcal{C}_j) = 1$. Since $m_1$ can take $k-1$ possible values we get

$$\mu(A^+) \leq \frac{Ck}{t^{1+\alpha}}$$

completing Step 1. Step 2 is similar except instead of using three cylinders to describe the itinerary of $x$ we use $2m(l)+1$ cylinders, where $\mathcal{D}_1, \mathcal{D}_2, \dots \mathcal{D}_{m(l)}$ are cylinders of length 1 such that $\tau \in \left[\frac{t}{L^{l+1}}, \frac{t}{L^l}\right]$ on $\mathcal{D}_j$ and $\mathcal{C}_1 \dots \mathcal{C}_{m(l)+1}$ are complementary cylinders. Note in particular that for the proof of (4.8) the quasi-independence of trials is sufficient.

To complete Step 3, we need to establish (4.16). We have

$$\mu(A^-) \leq \sum_{\mathcal{C}_1,\mathcal{C}_2} K\mu(\mathcal{C}_1)\mu(\mathcal{C}_2),$$

where the sum is over all cylinders of length $k/2$ such that $\tau(f^j x) \leq 2L^r k^{1/\alpha}$ for all $x$ in $\mathcal{C}_1 \bigcap f^{-k/2}\mathcal{C}_2$ and all $j < k$ and either $\tau_{k/2}(x) > t/100$ for all $x \in \mathcal{C}_1$ or $\tau_{k/2}(x) > t/100$ for all $x \in \mathcal{C}_2$. To estimate this sum we note that

$$\sum_{\mathcal{C}_1:\, [\sum_{j=0}^{k/2-1}(\tau\circ f^j)1_{\tau\circ f^j \in [0,2L^r k^{1/\alpha}]}]>t/100 \text{ on } \mathcal{C}_1} \mu(\mathcal{C}_1) \leq \frac{C(L,r)k^{1/\alpha}}{t}$$

by the Markov inequality. On the other hand (3.2) shows thar for each $\mathcal{C}_1$

$$\sum_{\mathcal{C}_2:\tau_k(x)\in I \text{ for some } x\in\mathcal{C}_1\cap f^{-k/2}\mathcal{C}_2} \mu(\mathcal{C}_2) \leq \frac{C}{k^{1/\alpha}}.$$

This shows that the contribution of terms where $\tau_{k/2} > t/100$ on $\mathcal{C}_1$ is $O\left(t^{-1}\right)$. Likewise the contribution of terms where $\tau_{k/2} > t/100$ on $\mathcal{C}_2$ is $O\left(t^{-1}\right)$. This proves (4.16). □

5.4. *Mixing away from the origin.*

Here we deduce Theorem 5.1 from Proposition 5.3. Define $\mathcal{A} = \iota^{-1}(\tilde{\mathcal{A}}) \subset \Omega$ and $\mathcal{B} = \iota^{-1}(\tilde{\mathcal{B}}) \subset \Omega$. Since $\iota$ is a homomorphism, we have

$$\tilde{\nu}(\tilde{\mathcal{A}} \cap \tilde{g}_{-t}\tilde{\mathcal{B}}) = \nu(\mathcal{A} \cap g_{-t}\mathcal{B}) \quad \text{and} \quad \tilde{\nu}(\tilde{\mathcal{A}}) = \nu(\mathcal{A}), \quad \tilde{\nu}(\tilde{\mathcal{B}}) = \nu(\mathcal{B}).$$

It is easy to check that for any $\tilde{\mathcal{E}} \subset \tilde{\Omega}$ with $\tilde{\nu}(\partial\tilde{\mathcal{E}}) = 0$ (w.r.t. the usual product topology on $\tilde{\Omega}$) and for $\mathcal{E} = \iota^{-1}\tilde{\mathcal{E}}$, we have $\mu(\partial\mathcal{E}) = 0$ (w.r.t. the product topology on $\Omega$ where the topology in the base is defined by $d$). Unfortunately, $\mathcal{A}$ and $\mathcal{B}$ are not subsets of $\Omega_{\leq M}$ in general. Indeed $\Omega_{\leq M}$ is defined by the requirement that the *backward* return time to the base $X$ is bounded, while the condition $\tilde{A}, \tilde{B} \subset [\varepsilon, 1]$ in Theorem 5.1 allows us to bound *forward* return time to $X$. Since our system is non-invertible, the forward and backward directions play different roles. Thus we cannot apply Proposition 5.3 directly and an additional analysis is required.

*Proof of Theorem 5.1.*

By Lemma 5.2, $\tau$ is irrational.

Let $y_0 = 1$, and $y_{n+1}$ be the preimage of $y_n$ in $[0, 1/2]$. Let $x_{n+1}$ be the preimage of $y_n$ in $(1/2, 1]$. The intervals $X_n = (x_{n+1}, x_n]$ form a partition of $X$. In fact, $X_n$ coincides with the 1-cylinder $\{x \in X : R(x) = n\}$. Furthermore, the intervals $Y_n = (y_{n+1}, y_n]$, $n \geq 1$ form a partition of $(0, 1/2]$. Note that $Y_0 := (1/2, 1] = X$ (up to measure zero). For $n \geq 0$ let

$$\tilde{\Omega}^n = \{(x, s) : x \in Y_n, 0 \leq s \leq \tilde{\tau}(x)\}, \quad \hat{\Omega}^N = \bigcup_{n=0}^{N} \tilde{\Omega}^n.$$

Since $\tilde{A}$ and $\tilde{B}$ are disjoint from $[0, \varepsilon)$, there is a finite $N = N(\varepsilon)$ so that $\tilde{\mathcal{A}}, \tilde{\mathcal{B}} \subset \hat{\Omega}^N$. So it is sufficient to prove (5.1) for $\tilde{\mathcal{A}}, \tilde{\mathcal{B}} \subset \hat{\Omega}^N$ with $\tilde{\nu}(\tilde{\mathcal{A}}) = 0$, $\tilde{\nu}(\tilde{\mathcal{B}}) = 0$. This will be done in three steps.

**Step 1:** (5.1) **holds for $\tilde{\mathcal{A}}, \tilde{\mathcal{B}} \subset \hat{\Omega}^0$.**

Indeed, in this case $\mathcal{A}, \mathcal{B} \in \Omega_{\leq M}$ with $M = \|\tilde{\tau}\|_\infty$, so the result follows from Proposition 5.3.

**Step 2:** (5.1) **holds if $\tilde{\mathcal{A}} \subset \hat{\Omega}^0$, and $\tilde{\mathcal{B}} \subset \hat{\Omega}^N$ for some** $N$.

The proof is by induction on $N$. The base of induction was done at Step 1. So let us assume that the result holds for $\hat{\Omega}^{N-1}$ and prove it for $\hat{\Omega}^N$. Let $\tilde{\mathcal{B}}_N = \tilde{\mathcal{B}} \cap \tilde{\Omega}^N$. Since $\tilde{\mathcal{B}} - \tilde{\mathcal{B}}_N \subset \hat{\Omega}^{N-1}$ it is enough to show that the pair $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}_N)$ satisfies (5.1).

Partition $\tilde{\mathcal{B}}_N$ into subsets $\tilde{\mathcal{B}}_{N,l}$ of small diameter $\delta$. It suffices to check that for each $l$, the pair $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}_{N,l})$ satisfies (5.1). Let

$$t_l^- = \sup_{(x,s) \in \tilde{\mathcal{B}}_{N,l}} \min\{t > 0 : \tilde{g}_t(x, s) \in \tilde{\Omega}^{N-1}\}, \quad \tilde{\mathcal{B}}^*_{N,l} = \tilde{g}_{t_l^-}\tilde{\mathcal{B}}_{N,l}.$$

If $\delta$ is sufficiently small, then $\tilde{\mathcal{B}}^*_{N,l} \subset \tilde{\Omega}^{N-1}$ and the diameter of $\tilde{\mathcal{B}}^*_{N,l}$ is less than $\tilde{\tau}_{min}/2$. Consequently, the preimage of $\tilde{\mathcal{B}}^*_{N,l}$ under $\tilde{g}_{\mathfrak{s}}$ is the disjoint union of two sets: $\tilde{\mathcal{B}}'_{N,l} \subset \tilde{\Omega}^N$ and $\tilde{\mathcal{B}}''_{N,l} \subset \tilde{\Omega}^0$, where

$$\mathfrak{s} = \sup\{s < \tilde{\tau}_{min}/2 : \exists x \in Y_{N-1} : (x, s) \in \tilde{\mathcal{B}}^*_{N,l}\}.$$

Furthermore, the preimage of $\tilde{\mathcal{B}}'_{N,l}$ under $\tilde{g}_{t_l^- - \mathfrak{s}}$ is $\tilde{\mathcal{B}}_{N,l}$.

Thus

$$\{(x, s) \in \tilde{\mathcal{A}} : \tilde{g}_t(x, s) \in \tilde{\mathcal{B}}_{N,l}\} =$$

$$\{(x, s) \in \tilde{\mathcal{A}} : \tilde{g}_{t+t_l^-}(x, s) \in \tilde{\mathcal{B}}^*_{N,l}\} \setminus \{(x, s) \in \tilde{\mathcal{A}} : \tilde{g}_{t+t_l^- - \mathfrak{s}}(x, s) \in \tilde{\mathcal{B}}''_{N,l}\}.$$

By the inductive hypothesis, the RHS is asymptotic to

$$\hat{c}t^{1/r-1}\tilde{\nu}(\tilde{\mathcal{A}}) \left[\tilde{\nu}(\tilde{\mathcal{B}}^*_{N,l}) - \tilde{\nu}(\mathcal{B}''_{N,l})\right].$$

Since $\tilde{g}$ is measure preserving, we have

$$\tilde{\nu}(\tilde{\mathcal{B}}^*_{N,l}) - \tilde{\nu}(\mathcal{B}''_{N,l}) = \tilde{\nu}(\tilde{\mathcal{B}}'_{N,l}) = \tilde{\nu}(\tilde{\mathcal{B}}_{N,l}),$$

which proves (5.1).

**Step 3: (5.1) holds for $\tilde{\mathcal{A}}, \tilde{\mathcal{B}} \subset \hat{\Omega}^N$ for arbitrary $N$.** It suffices to show that for any fixed $\xi > 0$, we have

$$\hat{c}t^{1/r-1}\tilde{\nu}(\tilde{\mathcal{A}})\tilde{\nu}(\tilde{\mathcal{B}})(1-\xi) \leq \tilde{\nu}(\tilde{\mathcal{A}} \cap \tilde{g}_{-t}\tilde{\mathcal{B}}) \leq \hat{c}t^{1/r-1}\tilde{\nu}(\tilde{\mathcal{A}})\tilde{\nu}(\tilde{\mathcal{B}})(1+\xi) \tag{5.6}$$

provided that $t$ is large enough.

To establish (5.6), we partition $\tilde{\mathcal{A}}$ into sets $\tilde{\mathcal{A}}_l$ of small diameter $\delta$. Let

$$t_l^+ = \sup_{(x,s)\in\tilde{\mathcal{A}}_l} \min\{t > 0 : \tilde{g}_t(x,s) \in \tilde{\Omega}^0\}, \quad \tilde{\mathcal{A}}_l^* = \tilde{g}_{t_l^+}\tilde{\mathcal{A}}_l.$$

By bounded distortion, given $\xi$, we can find $\delta_0(\xi)$ such that if $\delta < \delta_0(\xi)$, then the Jacobian J(x,s) of $\tilde{g}_{t_l^+}$ satisfies

$$\left(1 - \frac{\xi}{2}\right)\frac{\tilde{\nu}(\tilde{\mathcal{A}}_l^*)}{\tilde{\nu}(\tilde{\mathcal{A}}_l)} \leq J(x,s) \leq \left(1 + \frac{\xi}{2}\right)\frac{\tilde{\nu}(\tilde{\mathcal{A}}_l^*)}{\tilde{\nu}(\tilde{\mathcal{A}}_l)}.$$

Consequently,

$$\frac{\tilde{\nu}(\tilde{\mathcal{A}}_l)}{\left(1+\frac{\xi}{2}\right)\tilde{\nu}(\tilde{\mathcal{A}}_l^*)}\tilde{\nu}(\tilde{\mathcal{A}}_l^* \cap \tilde{g}_{-(t-t_l^+)}\tilde{\mathcal{B}}) \leq \tilde{\nu}(\tilde{\mathcal{A}}_l \cap \tilde{g}_{-t}\tilde{\mathcal{B}}) \leq \frac{\tilde{\nu}(\tilde{\mathcal{A}}_l)}{\left(1-\frac{\xi}{2}\right)\tilde{\nu}(\tilde{\mathcal{A}}_l^*)}\tilde{\nu}(\tilde{\mathcal{A}}_l^* \cap \tilde{g}_{-(t-t_l^+)}\tilde{\mathcal{B}}).$$

By Step 2,

$$\lim_{t\to\infty} t^{1-1/r}\tilde{\nu}(\tilde{\mathcal{A}}_l^* \cap \tilde{g}_{-(t-t_l^+)}\tilde{\mathcal{B}}) = \hat{c}\tilde{\nu}(\tilde{\mathcal{A}}_l^*)\tilde{\nu}(\tilde{\mathcal{B}}).$$

Therefore for large $t$

$$\hat{c}t^{1/r-1}\tilde{\nu}(\tilde{\mathcal{A}}_l)\tilde{\nu}(\tilde{\mathcal{B}})(1-\xi) \leq \tilde{\nu}(\tilde{\mathcal{A}}_l \cap \tilde{g}_{-t}\tilde{\mathcal{B}}) \leq \hat{c}t^{1/r-1}\tilde{\nu}(\tilde{\mathcal{A}}_l)\tilde{\nu}(\tilde{\mathcal{B}})(1+\xi).$$

Summing over $l$ we obtain (5.6) completing the proof of the theorem. □

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