TOTALLY ASYMMETRIC DYNAMICAL WALKS IN RANDOM ENVIRONMENT

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Abstract. In this note we study dynamical random walks with internal states. We consider a one dimensional model where a particle which moves to the right ballistically, and moreover out of every three steps it moves to the right at least twice. We provide sufficient conditions for z_n to satisfy the central limit theorem.

1. The result.

Dynamical random walk (DRW) is a map F defined on $M \times \mathbb{Z}^d$ where M is an internal state of the walker. Namely, for each $z \in \mathbb{Z}^d$ we have a map $T_z : M \to M$ and a partition $M = \bigcup_{v \in \mathcal{V}} W_{v,z}$ (*gate partition*) where $\mathcal{V}$ is a finite subset of $\mathbb{Z}^d$. Let

$$F(x, z) = \left(T_z x, z + \sum_v 1_{W_v} v \right).$$

In other words, the internal state of the particle changes by the local dynamics and then it moves to a neighboring site as prescribed by the gate. In the case the local dynamics is random, that is, the pairs $(f_z, \{W_{\cdot,z}\})$ are independent for different $z \in \mathbb{Z}^d$ we have *Dynamical Walk in Random Environment*.

We note that the DRW introduced above contain several classical examples of random motion. As an example, consider the following system: let $d = 1$, $M = \mathbb{T}^1$, $T_z(x) = 2x \pmod 1$ and $W_{z,-1} = [0, \frac{1}{2})$, $W_{z,1} = [\frac{1}{2}, 1)$ for all $z \in \mathbb{Z}$. One can see that the DRW defined in this way is equivalent to the one dimensional random walk with the transition probabilities $(1/2, 1/2)$. More generally it shown in [1] that DRW with expanding local dynamics and Markov gates can model random walks in random environment (RWRE). In particular, all types of behavior observed in RWRE, appear also in DRW, so the particle can be transient with zero speed ([14]) or it can exhibit Sinai behavior ([13]) where after n steps the particle is located at the distance of order $\ln^2 n$ from the origin. However, Markov condition on the gates is pretty restrictive and so it is of interest to develop tools to handle non Markovian dynamics. This is the goal of the present article.

In this note we consider a simple model of DRW where $d = 1$, $M = \mathbb{T}$ and $T_z : \mathbb{T} \to \mathbb{T}$ are smooth uniformly expanding maps. We will also assume that the particle's coordinate changes every time, thus $\mathbb{T} = W_{n,-1} \cup W_{n,1}$ and moreover that the forward gate is much larger, so that that among every three moves at least two are to the right.

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Thus the particle moves to the right ballistically. Namely $z_n \geq n/3$ and $z_n \geq z_m - 1$ for $n > m$. We are interested in the central limit theorem for the particle motion.

We consider the following models.

Model A. There are constants[1] $3 < \gamma \leq K$ and $K_1 > 0$, such that for all n and all $x, y \in \mathbb{T}$ we have

$$\gamma \leq |T_n'(x)| \leq K, \quad \sup_{x \neq y \in \mathbb{T}} \frac{|T_n'(x) - T_n'(y)|}{|x-y|} \leq K_1. \tag{1.1}$$

There exists $c > 0, \delta_0$, so that for all $n \in \mathbb{Z}$

$$c\delta_0 \leq |W_{n,-1}| \leq \delta_0, \tag{1.2}$$

and

$$T_n(W_{n,-1}) \cap W_{n-1,-1} = \emptyset, \quad T_{n-1}T_n((W_{n,-1})) \cap W_{n,-1} = \emptyset. \tag{1.3}$$

Model A′. In addition to properties (1.1)–(1.3), we assume that for some large [2] $N > 0$ we have that for all $n \in \mathbb{Z}$ and $1 \leq k \leq N$

$$(T_{n+k-1} \circ \cdots \circ T_n \circ T_{n-1} \circ T_n(W_{n,-1})) \cap W_{n+k,-1} = \emptyset. \tag{1.3_N}$$

To assure the reader that all the conditions (1.1)-(1.3_N) above can coexist, consider a sequence of expanding maps $\{T_n\}_{n\in\mathbb{Z}}$ satisfying (1.1) such that there is $x_0 \in \mathbb{T}$, so that $T_n(x_0) = x_0$, for all $n \in \mathbb{Z}$. Then take x' such that $x' > x_0$ and $|x - x'|$ is so small that the points

$$\{x', T_{n,n}(x'), \ldots, T_{n,n+N}(x')\},$$

are disjoint for all $n \in \mathbb{Z}$ (where $T_{n,n+k} = T_{n+k}(T_{n+k-1} \cdots T_n(x)))$. Also note that $|T_{n,n+k}(x') - T_{n,n+k+1}(x')| \geq \gamma|x' - x_0|$, for all k with $k + 1 \leq N$. Hence, for δ_0 sufficiently small and for gates satisfying $W_n \subset [x' - \delta_0/2, x' + \delta_0/2]$, $\forall n \in \mathbb{Z}$, the conditions (1.3) and (1.3_N) will be satisfied.

Model B. We fix a map $\bar{T}$ and a segment $\bar{W}$ such that $\bar{T}^p\bar{W} \cap W = \emptyset$ for $p = 1, 2$. We now suppose that for a sufficiently small δ_0 we have that for all n $|T_n - \bar{T}|_{C^{1+\mathrm{Lip}}(\mathbb{T})} \leq \delta_0$ and the Hausdorff distance between $W_{n,-1}$ and $\bar{W}$ is smaller than δ_0.

Our first result is the CLT for the hitting time. Namely let $\tau_n(z, k)$ be the smallest time t such that $F^t(z, k) \in \mathbb{T} \times \{n\}$. Define the maps $G_n : \mathbb{T} \to \mathbb{T}$ by

$$G_n(x) = \pi_{\mathbb{T}} F^{r_n(x)}(x, n) \text{ where } r_n(x) = \tau_{n+1}(x, n). \tag{1.4}$$

We shall also write $\tau_n(x) := \tau_n(x, 0)$. Note that

$$\tau_n(x) = \sum_{k=0}^{n-1} r_k(G_{k-1} \circ \cdots \circ G_0 x). \tag{1.5}$$

We say that the DRW satisfies the CLT for hitting times if

$$\frac{\tau_n - \mathbb{E}(\tau_n)}{\sqrt{\mathrm{Var}(\tau_n)}} \Rightarrow \mathcal{N}(0, 1) \text{ as } n \to \infty \tag{1.6}$$

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[1] One could replace the condition $\gamma > 3$ by $\gamma > 1$ at the expense of increasing N in the condition (1.3_N). We prefer to keep the present assumption $\gamma > 3$ in order to simplify the argument.

[2] The precise conditions on N will be given later as they depend on the mixing rate of the family $\{T_n\}$, see (4.5).

where $\mathcal{N}(a, \sigma^2)$ denotes the normal distribution with mean a and standard deviation σ. Here and elsewhere in this article we assume (unless it is explicitly stated otherwise) that x is uniformly distributed on $\mathbb{T}$.

Theorem 1.1. *(a) Given γ, K, K_1, c there exist $\bar{\delta}_0$ and N such that if* (1.1)–(1.3_N) *hold with $\delta_0 \le \bar{\delta}_0$ then the DRW from Model A′ satisfies the CLT for hitting times.*

(a′) Given γ, K, K_1, c there exist $\bar{\delta}_0$ such that if (1.1)–(1.3) *hold with $\delta_0 \le \bar{\delta}_0$ and the variance of τ_N tends to infinity then the DRW from Model A satisfies the CLT for hitting times.*

(b) Given $\bar{T}$ there exist $\bar{\delta}_0$ such that if $\delta_0 \le \bar{\delta}_0$ then the DRW from Model B satisfies the CLT for hitting times.

The random setting. One way to verify the conditions of Theorem 1.1(a′) is to assume that the environment is stationary. Let $E = \{(\mathcal{T}_1, W_1), \dots, (\mathcal{T}_m, W_m)\}$ be a finite collection of pairs of maps and gates, so that any sequence $(T_n, W_n)_{n\in\mathbb{Z}}$, with $(T_n, W_n) \in E$, satisfies the conditions of Model a A or Model B. We make an iid selection from this collection. The next Corollary follows from Theorem 1.1(a′) and Lemma 7.1.

Corollary 1.2. *Assume that any realization $(T_n, W_n)_{n\in\mathbb{Z}}$ satisfies the conditions of Model A or Model B with δ_0 sufficiently small. Then for almost every realizations $(T_n, W_n)_{n\in\mathbb{Z}}$, τ_n satisfies* (1.6).

To deduce an information about the distribution of z coordinate from this result we need some additional terminology. The scale function is defined by $\mathcal{S}(z) = \mathbb{E}(\tau_z)$. This function is obviously monotone so we can consider an inverse function $\mathcal{Z}(s) = \max(x : \mathcal{S}(x) \le s)$. Let $\hat{\sigma}_n = \sqrt{\mathrm{Var}(\tau_{\mathcal{Z}(n)})}$.

Theorem 1.3. *Under the conditions of Theorem 1.1 we have*

$$\frac{\mathcal{S}(z_n) - n}{\hat{\sigma}_n} \Rightarrow \mathcal{N}(0, 1) \text{ as } n \to \infty.$$

Theorem 1.4. *Consider DWRE of Model A and Model B and take δ_0 so small that every realization of $\{(T_n, W_n)\}_{n\in\mathbb{Z}}$ satisfies the conditions of Corollary 1.2.*

(a) (Quenched CLT) *There is a constant $\boldsymbol{\sigma}$ such that for almost all realizations of the pairs (T_n, W_n) there are constants $b_n = b_n(\omega)$ such that if x is uniformly distributed on $\mathbb{T}$ then*

$$\frac{z_n - b_n}{\boldsymbol{\sigma}\sqrt{n}} \Rightarrow \mathcal{N}(0, 1) \text{ as } n \to \infty.$$

(b) (Annealed CLT) *There are constants v, σ such that if x and $\{(T_n, W_n)\}$ are independent, x is uniformly distributed on $\mathbb{T}$ and (T_n, W_n) are chosen from E in an iid fashion, then*

$$\frac{z_n - vn}{\sigma\sqrt{n}} \Rightarrow \mathcal{N}(0, 1) \text{ as } n \to \infty.$$

Remark 1.5. The ballisticity condition (1.3) ensures that the time needed to move to the right is in BV as a function of the initial condition x. This allows us to apply existing results about the central limit theorem for non-autonomous dynamical systems, such as [2, 3, 4, 6, 12]. In case the return time is unbounded one needs to extend the existing result allowing much less regular functions. This will be done in a separate paper, while here we will present the method in a simplest possible setting. We hope that our approach will be useful for other models of motions is random media, including the random Lorenz gas studied in [1] and the systems with deterministic dynamics and random jumps studied in [10].

The rest of the paper is organized as follows. Section 2 contains necessary background about piecewise expanding maps of the circle. In particular, we recall a CLT for non stationary compositions of piecewise expanding maps from [3]: see Theorem 2.1 below. To apply this theorem we need to establish good mixing properties of the maps as well as a growth of variance for non-stationary sums. In Sections 3 and 4 we verify mixing and variance growth respectively for Model A', and in Sections 5 and 6 we do the same for Model B. Section 7 establishes the linear growth of variance for DRW in random environment. The CLT for the particle position on the same setting is obtained in Section 9. The proof consists in two parts. In Section 8 we establish the CLT for the scale function (Theorem 1.3). This theorem is valid for any fixed environment. In Section 9 we show that in iid environment the scale function is close to linear completing the proof of Theorem 1.4. Finally in Section 10 we give an example showing that without stationarity assumption the scale function does not need to be linear, and as a result the CLT for particle position may fail.

2. Main tools

2.1. **Central Limit Theorem.** Let $n, m \in \mathbb{Z}$, with $n < m$. For the sequence of maps $\{G_n\}_{n\geq 1}$, we define maps $G_{n,m}$ as follows

$$G_{n,m}(x) = (G_m \circ \cdots \circ G_n)(x).$$

We denote by $\mathcal{V}$ the space of all functions with bounded variation and denote by $V(f)$ the variation of the function $f \in \mathcal{V}$. The space $\mathcal{V}$ is equipped with the norm

$$|f|_v := V(f) + ||f||_1,$$

where L^1 norm in $||f||_1$ is taken relative to the Lebesgue measure. For $f \in \mathcal{V}$, we have: $||f||_\infty \leq |f|_v$. Define also $\mathcal{V}_0 = \{f \in \mathcal{V} : \int_{\mathbb{T}} f dx = 0\}$.

We are interested in maps satisfying the following hypothesis.

Hypothesis 1. *$T : \mathbb{T} \to \mathbb{T}$ is such that there exists a finite or countable partition (I_j) of $\mathbb{T}$ such that the restriction of the map to each interval I_j is strictly monotone and $T|_{I_j} \in C^1(I_j)$. We also assume that*

$$\gamma(T) \quad := \inf_j \inf_{x\in I_j} |T'(x)| > 2$$

$$K(T) := \sup_j \sup_{x\neq y\in I_j} \left| \frac{T'(x) - T'(y)}{x - y} \right| < \infty.$$

It is well known that the transfer operator of a map satisfying Hypothesis 1 is given by the following formula

$$P_T f(x) = \sum_j f(\sigma_j x) \frac{1}{|\tau'(\sigma_j x)|} 1_{\tau(I_j)}(x)$$

where σ_j is the inverse function of the restriction of T on I_j. It is clear that the mappings $\{G_n\}$ and $\{T_n\}$ satisfy the Hypothesis 1. Recall that

$$\int (P_T f)\, g dx = \int f(x) g(Tx) dx, \quad \forall f \in L^1, g \in L^\infty.$$

Let $\mathcal{P}$ be a set of contractions on L^1 (a set of linear operators satisfying $\|Pf\|_1 \leq \|f\|_1$, for every $P \in \mathcal{P}$). We say $\mathcal{P}$ *is exponentially mixing* on $\mathcal{V}_0$ if there exist $\theta < 1$ and $D > 0$ such that, for all integers $l \geq 1$, all l-tuples of operators $P_1, \ldots, P_l$ in $\mathcal{P}$ we have

$$(\mathbf{Dec}) \qquad \forall f \in \mathcal{V}_0, \quad |P_l \cdots P_1 f|_v \leq D\theta^l |f|_v.$$

We say that the sequence of operators $\{P_n\}_{n\geq 1}$ satisfies the condition (**Min**), if there exists $\sigma > 0$ with

$$(\mathbf{Min}) \qquad P_n P_{n-1} \ldots P_1 \mathbf{1}(x) \geq \sigma, \quad \forall x \in \mathbb{T}, \quad \forall n \in \mathbb{N}.$$

In sequel we will denote

$$\mathcal{P}^n = P_n P_{n-1} \ldots P_1. \tag{2.1}$$

The relevance of (**Dec**) and (**Min**) comes from the following result.

Theorem 2.1. [3, Theorem 5.1] *Let (f_n) be a sequence of observables so that* $\sup_n |f_n|_v < \infty$. *Assume, that for a sequence of transformations $\{T_n\}_{n\geq 1}$ the corresponding set of transfer operators $\{P_{T_n}\}_{n\geq 1}$ satisfies the conditions* (**Min**) *and* (**Dec**). *Let*

$$S_n(x) = \sum_{k=0}^{n-1} f_n(T_n \cdots T_1 x) - \int_0^1 f_n(T_n \cdots T_1) dx.$$

If the norms $\|S_n\|_2$ are unbounded, as $n \to \infty$, then

$$\frac{S_n}{\|S_n\|_2} \Rightarrow \mathcal{N}(0, 1).$$

2.2. **Sufficient conditions for mixing.** We recall several definitions from [3]. For $p \geq 1$, we define the following metric between two contractions $R, R' \in \mathcal{P}$

$$d_p(R, R') = \sup_{\{f \in \mathcal{V} : |f|_v \leq 1\}} \|Rf - R'f\|_p .$$

For $P \in \mathcal{P}$ we denote

$$B(P, \delta) := \{R \in \mathcal{P} : d_1(R, P) < \delta\}.$$

We say that the sequence $\{P_n\}_{n\geq 1}$ is *exact* if

$$(\mathbf{Exa}) \qquad \forall f \in \mathcal{V}_0, \quad \lim_n \|P_n \cdots P_1 f\|_1 = 0.$$

If $P_n = P_{T_n}$, where the maps T_n are defined on a probability space $(X, \mathcal{A}, m)$, define

$$\mathcal{A}_k = T_1^{-1} \cdots T_k^{-1}(\mathcal{A}). \tag{2.2}$$

The exactness condition is equivalent to the triviality of the algebra $\mathcal{A}_\infty$, where

$$\mathcal{A}_\infty = \bigcap_{k\geq 1} \mathcal{A}_k. \tag{2.3}$$

We say that the collection of contractions $\mathcal{P}$ satisfies *Doeblin-Fortet-Lasota-Yorke inequality*, or (**LY**) for short, if there exists $\rho \in (0,1)$ and $C > 0$, so that for any $P \in \mathcal{P}$ we have

$$\forall f \in \mathcal{V}, \quad |Pf|_v \leq \rho |f|_v + C\|f\|_1. \tag{LY}$$

It follows from (**LY**) that for any $P_n, \dots, P_1 \in \mathcal{P}$, there exists $M > 0$, so that ([3, Lemma 2.4])

$$|P_n \cdots P_1 f|_v \leq M|f|_v, \ \ n \geq 1. \tag{2.4}$$

We say that $\mathcal{P}$ is compact if for any sequence $(R_n) \subset \mathcal{P}$ there is a subsequence (R_{n_j}) and an operator $R \in \mathcal{P}$, such that

$$\forall f \in \mathcal{V}_0, \quad \lim_j \left\|R_{n_j} f - Rf\right\|_1 = 0. \tag{C}$$

Proposition 2.2 (Proposition 2.11, [3])**.** *Let $\mathcal{P}$ be a collection of contractions verifying the condition* (**C**)*. Assume also $\mathcal{P}$ satisfies* (**LY**) *and that any sequence $\{P_n\} \subset \mathcal{P}$ is exact, i.e.* (**Exa**) *holds. Then any sequence in $\mathcal{P}$ satisfies* (**Dec**)*.*

We also need an extension of Proposition 2.10 in [3].

Corollary 2.3. *Let $\mathcal{P}$ be a collection satisfying* (**LY**) *and $\{P_n\}_{n\geq 1} \subset \mathcal{P}$ a collection for which* (**Dec**) *holds. Then there exists $\delta > 0$, so that any sequence $\{R_n\}_{n\geq 1} \subset \mathcal{P}$, with $R_n \in B(P_n, \delta)$, for all $n \geq 1$, satisfies* (**Dec**)*.*

Proof. The proof follows from Lemma 2.4 below and Proposition 2.7 of [3]. □

Lemma 2.4. *Let $\mathcal{P}$ be a collection satisfying* (**LY**) *and $\{P_n\}_{n\geq 1} \subset \mathcal{P}$ a sub-collection satisfying* (**Dec**)*. Then for every $\epsilon > 0$ there are integers $q(\epsilon) \geq 1$ and a real number $\delta(\epsilon) > 0$, such that for all q operators $R_1, \dots, R_q$ in $\cup_{n\geq 1} B(P_n, \delta(\epsilon)) \cap \mathcal{P}$, we have*

$$\forall f \in \mathcal{V}_0, \quad \|R_q \cdots R_1 f\|_1 \leq \varepsilon |f|_v.$$

Proof. Without loos of generality we can assume $R_k \in B(P_k, \delta(\epsilon)) \cap \mathcal{P}$, for all $k \leq n$. By (**Dec**) we can find $q \geq 1$, so that for every $f \in \mathcal{V}_0$ we have

$$\|P_q \cdots P_1 f\| \leq L\theta^q |f|_v \leq \frac{\varepsilon}{2}|f|_v.$$

Next, observe that

$$\begin{aligned}
\|R_n \cdots R_1 f - Q_n \cdots Q_1 f\|_1 &\leq \|R_n(R_{n-1} \cdots R_1)f - Q_n(R_{n-1} \cdots R_1)f\|_1 \\
&\quad + \|Q_n(R_{n-1} \cdots R_1)f - Q_n(Q_{n-1} \cdots Q_1)f\|_1 \\
&\leq d(R_n, Q_n)|R_{n-1} \cdots R_1 f|_v \\
&\quad + \|Q_n(R_{n-1} \cdots R_1 - Q_{n-1} \cdots Q_1)f\|_1 \\
&\leq d(R_n, Q_n)M|f|_v + \|R_{n-1} \cdots R_1 f - Q_{n-1} \cdots Q_1 f\|_1.
\end{aligned}$$

In the last line we used the estimate (2.4). Iterating the last estimate we get

$$\|R_n \cdots R_1 f - Q_n \cdots Q_1 f\|_1 \leq M \sum_{k=1}^{n} d(R_n, Q_n)|f|_v.$$

Hence for any $f \in \mathcal{V}_0$

$$\begin{aligned}\|R_q \cdots R_1 f\|_1 &\le \|R_q \cdots R_1 f - P_q \cdots P_1 f\|_1 + \|P_q \cdots P_1 f\|_1 \\ &\le qM\delta(\varepsilon)|f|_v + \frac{\varepsilon}{2}|f|_v \le \varepsilon |f|_v\end{aligned}$$

if $\delta(\varepsilon)$ is such that $qM\delta(\varepsilon) \le \varepsilon/2$. □

2.3. **Lasota-Yorke inequality.** We need the following standard fact those proof could be found in [3] or [7].

Lemma 2.5. *(a) Let $[u,v] \subset [c,d] \subset [0,1]$, and f be of bounded variation. Then*

$$|\varphi(u)| + |\varphi(v)| \le V_{[c,d]}(\varphi) + \frac{2}{(d-c)} \int_c^d |\varphi(t)| dt. \tag{2.5}$$

(b) In particular

$$|\varphi(u)| + |\varphi(v)| \le V_{[u,v]}(\varphi) + \frac{2}{(v-u)} \int_u^v |\varphi(t)| dt.$$

Lemma 2.6. *Let T satisfy* (1.1) *and suppose that there is an interval $W \subset \mathbb{T}$ such that T is smooth everywhere except, possibly, at the endpoints of W. Then there exists $\rho \in (0,1)$ and $C = C(K) > 0$ such that for all $n \in \mathbb{N}$*

$$V(P_T f) \le \frac{3}{\gamma} \mathrm{V}(f) + C\|f\|_1. \tag{2.6}$$

Proof. Recall that

$$P_T f(x) = \sum_j f(\sigma_j x) \frac{1}{|T'(\sigma_j x)|} 1_{T(I_j)}(x),$$

where σ_j is the inverse function of T on its intervals of monotonicity (I_j). We can assume that $|W| \le \frac{1}{2}$. Otherwise, we will switch W and W^c. Since T has only two discontinuity points, the partition (I_j) can be chosen in such a way that there will be at most one interval $I \in (I_j)$, with $|I| < \frac{1}{2[K]+1}$. Indeed, we can define the partition (I_j) on W^c so that $|I_j| = \frac{|W^c|}{[K]+1}$, $j = 1, \ldots, [K]+1$. Since $\frac{1}{2([K]+1)} < |I_j| < \frac{1}{K}$, then $T|_{I_j}$ will be one-to-one on each one of these intervals. If now $|W| < \frac{1}{2([K]+1)}$, then we will take W to be one of the partition intervals, otherwise we divide W into intervals of size $\frac{1}{2([K]+1)}$ and a reminder interval I_i, so that $|I_i| < \frac{1}{(2[K]+1)}$.

Note that

$$V(P_T f) \le \sum_j V\Big(f(\sigma_j x) \frac{1}{|T'(\sigma_j x)|} 1_{T(I_j)} \Big) \le$$

$$\sum_j V_{T(I_j)} \left[\left(\frac{f}{T'} \right) \circ \sigma_j \right] + \sum_j \left[\left| \frac{f}{T'} \right| (\sigma_j \alpha_j) + \left| \frac{f}{T'} \right| (\sigma_j \beta_j) \right] =: I + \mathit{II}$$

where $T(I_j) = [\alpha_j, \beta_j]$. By an inequality in [3], page 106, we have

$$V_{T(I_j)} \left[\left(\frac{f}{T'} \right) \circ \sigma_j \right] = V_{I_j} \left[\left(\frac{f}{T'} \right) \right] \le \frac{V_{I_j}(f)}{\gamma} + \frac{K_1}{\gamma^2} \int_{I_j} |f(t)| dt.$$

Summing over j we get

$$I \leq \frac{V(f)}{\gamma} + \frac{K_1}{\gamma^2}\|f\|_1. \tag{2.7}$$

Next for all monotonicity intervals except the shortest one we use Lemma 2.5(b) obtaining

$$\left|\frac{f}{T'}\right|(\sigma_j\alpha_j) + \left|\frac{f}{T'}\right|(\sigma_j\beta_j) \leq \frac{1}{\gamma}\left[|f|(\sigma_j\alpha_j) + |f|(\sigma_j\beta_j)\right]$$

$$\leq \frac{V_{I_j}(f)}{\gamma} + \frac{2}{|I_j|}\int_{I_j}|f(x)|dx \leq \frac{V_{I_j}(f)}{\gamma} + \frac{2}{2[K]+1}\int_{I_j}|f(x)|dx. \tag{2.8}$$

It remains to handle the shortest interval I_i. Let I_{i+1} be a partition element adjacent to I_i and set $I = I_i \cup I_{i+1}$. Then $|I| > \frac{1}{2([K]+1)}$ and by (2.5), applied to $I_i \subset I = [c, d]$, we have

$$\left|\left(\frac{f}{T'}\right)(\sigma_i\alpha_i)\right| + \left|\left(\frac{f}{T'}\right)(\sigma_i\beta_i)\right| \leq$$

$$\frac{1}{\gamma}(|f(\sigma_i\alpha_i) + |f|(\sigma_i\beta_i)) \leq \frac{1}{\gamma}V_I(f) + \frac{2}{\gamma|I|}\int_I |f|dx. \tag{2.9}$$

Summing the above estimates we obtain

$$V(P_T f) \leq \frac{3}{\gamma}V(f) + C\|f\|_1$$

where the factor $\frac{3}{\gamma}$ is the sum of three terms of size $\frac{1}{\gamma}$ coming from (2.7), (2.8), and (2.9) respecively. This completes the proof. □

2.4. **Positivity of density.**

Lemma 2.7. *Let $\{T_n\}_{n\geq 1}$ be a sequence of maps satisfying* (1.1). *Assume that the sequence of intervals $\{W_n\}_{n\geq 1}$ is such that for each $n \geq 1$ T_n is continuous everywhere on $\mathbb{T}$, except possibly at the endpoints of W_n. Then there exists σ such that*

$$(P_n \cdots P_1 1)(x) \geq \sigma, \quad \forall x \in \mathbb{T}, n \geq 1,$$

where P_n is the transfer operator for T_n.

Proof. According to Proposition 2 in [2], it is sufficient to show that for each $I \subset \mathbb{T}$, there exists $N = N(|I|)$ so that $G_{1,N}(I) = \mathbb{T}$.

If $W_1 \cap I \neq \emptyset$, then the intersection with ∂W_1 divide I into at most three components. Let I_1 be the largest component. Then $|I_1| \geq |I|/3$. Consider the image $G_1(I_1)$. Then $|G_1(I_1)| \geq \gamma|I_1|$, since G_1 is continuous both inside and outside of W_1. Next, we choose the largest interval $I_2 \subset G_1(I_1)$, so that either $I_2 \subseteq W_2$ or $I_2 \cap W_2 = \emptyset$. Hence, $|I_2| \geq |G_1(I_1)|/3 > \frac{\gamma}{3}|I_1|$. Repeating this argument, we will obtain a sequence of intervals $(I_n)_{n\geq 1}$, so that

$$|I_{n+1}| > \left(\frac{\gamma}{3}\right)^n |I_n|.$$

Since $\frac{\gamma}{3} > 1$, then the image of I_1 will cover the circle in time $O\left(\ln\left(\frac{1}{|I_1|}\right)\right)$. □

3. Mixing for Model A′.

Let $\bar{P}_n$ and P_n be the transfer operators of T_n and G_n respectively. In the sequel, instead of the notation $W_{n,-1}$ for the backward gate we will use W_n.

By (1.3) we have

$$T_n(W_n) \cap W_{n-1} = \emptyset, \text{ and } T_{n-1}(T_n(W_n)) \cap W_n = \emptyset.$$

This implies that

$$G_n(x) = \begin{cases} T_n(T_{n-1}(T_n(x))) & \text{if} \quad x \in W_n, \\ T_n(x) & \text{if} \quad x \notin \mathbb{T} \setminus W_n. \end{cases} \tag{3.1}$$

Hence the hitting times are

$$\tau_n(x) = \begin{cases} 3 & \text{if} \quad x \in W_n, \\ 1 & \text{if} \quad \text{for } x \in \mathbb{T} \setminus W_n. \end{cases} \tag{3.2}$$

Our main tool for proving the central limit theorem is Theorem 2.1. To apply it we need to verify (**Dec**) and (**Min**) for the operators $\{P_n\}_{n\in\mathbb{Z}}$. We start by verifying (**Dec**) for $\{\bar{P}_n\}$. Once we have established this, we can apply Corollary 2.3 to obtain the same property for $\{P_n\}$ for δ_0 small. The unboundedness of the variance needed in Theorem 2.1 will be established in Section 4.

To show (**Dec**) for arbitrary sequence $T_1, T_2, \dots$ satisfying (1.1), we verify (**C**) for the following class of maps

$$\{T \in C^{1+\mathrm{Lip}}(\mathbb{T}) : T \text{ satisfies (1.1) with constants } \gamma, K, K_1\}. \tag{3.3}$$

To show (**C**) for the class (3.3), we use the Arzela–Ascoli theorem to find a sequence $\{T'_{n_k}\}$ which converges uniformly to a Lipschitz function T' with the same constant. Next, we choose a subsequence of $\{n_k\}$, so that $T_{n_{k_l}}$ converges uniformly to a function T, which will be from class (3.3). It remains to show that any sequence of mappings from (3.3) is exact. We start by obtaining the classical distortion estimate for a composition of expanding maps satisfying the condition (1.1). Note that

$$\left|\frac{T'_n(x)}{T'_n(y)} - 1\right| = \left|\frac{T'_n(x) - T'_n(y)}{T'_n(y)}\right| \le \frac{K_1|x-y|}{\gamma}.$$

We are going to obtain the distortion estimate

$$K_0^{-1} \le \frac{|(T_{1,N})'(x)|}{|(T_{1,N})'(y)|} \le K_0,$$

for all $0 \le n \le N$ and x, y belonging to the same injectivity interval of $T_{1,n}$. We have

$$\frac{|\,(T_{1,N})'(x)|}{\left|(T_{1,N})'\,(y)\right|} = \prod_{n=1}^{N} \frac{|T'_n\,(T_{1,n-1}(x))\,|}{|T'_n\,(T_{1,n-1}(y)||} \le \prod_{n=1}^{N}(1 + C|T_{1,n-1}(x) - T_{1,n-1}(y)|),$$

where $C = K_1/\gamma$. Next,

$$|T_{1,n}(x) - T_{1,n}(y)| \leqslant (1/\gamma)^{N-n}\,|T_{1,N}(x) - T_{1,N}(y)|\,.$$

Hence

$$\frac{|\,(T_{1,N})'(x)|}{\left|(T_{1,N})'\,(y)\right|} \le \prod_{n=0}^{\infty}\left(1 + \frac{C}{\gamma^n}\right) = K_0 < \infty. \tag{3.4}$$

It follows from (3.4) that for any intervals A, B which belong to the same monotonicity interval of $T_{1,N}$, we have

$$K_0^{-1}\frac{|A|}{|B|} \le \frac{|T_{1,N}(A)|}{|T_{1,N}(B)|} \le K_0\frac{|A|}{|B|}. \tag{3.5}$$

We now show the exactness of the sequence $\{T_n\}_{n\ge1}$. Let A be in the asymptotic σ-algebra $\mathcal{A}_\infty$ of $T_{1,n}$, with $|A|>0$. We need to show that $|A|=1$. Since the lengths of injectivity intervals of $T_{1,n}$ go to zero uniformly, as $n\to\infty$, we can find J, so that $T_{1,n}(J)=\mathbb{T}$ and $|J\cap A|>|J|(1-\epsilon)$, where $\epsilon>0$ can be taken arbitrarily small. Next we find an open set $G\subset J$, so that $A^c\cap J\subset G$ and $|G|<2\epsilon|J|$. Using (3.5), we can write

$$|T_{1,N}(A^c\cap J)|\le |T_{1,N}(G)|\le |T_{1,N}(J)|\frac{|G|}{|J|}K_0\le 2\epsilon K_0.$$

Therefore

$$|T_{1,N}(A\cap J)|\ge 1-2\epsilon K_0.$$

Since $A=(T_{1,n})^{-1}(T_{1,n})A$ and $\epsilon>0$ is arbitrary, it follows that $|A|=1$. This finishes the proof of exactness for the sequence $\{T_n\}_{n\ge1}$.

The (**LY**) property for $\{\bar P_n\}$ is shown in Lemma 2.6. It now follows from Proposition 2.2, that any sequence of operators $\{\bar P_n\}_{n\ge1}$ of mappings from (3.3) satisfies (**Dec**).

We next prove, that $\{P_n\}$ also satisfy the property (**Dec**) for δ_0 sufficiently small. Note that (**LY**) for $\{P_n\}$ also follows from Lemma 2.6. In view of Corollary 2.3, it suffices to show that the operators $\bar P_n$ are close to P_n in d_1 metric. To show the closeness we need to check that

$$\sup_{\{f\in\mathcal{V}:|f|_v\le1\}}\left\|P_nf-\bar P_nf\right\|_1\le C\delta_0. \tag{3.6}$$

Indeed, we have that

$$|G_n(W_n)|\le K^3|W_n|\le K^3\delta_0,\quad \forall n\in\mathbb{N}.$$

Hence

$$P_nf(x)=\bar P_nf(x),\quad \forall x\notin G_n(W_n)$$

and so

$$\|P_nf-\bar P_nf\|_1\le\frac{K^3\delta_0}{\gamma^3}\|f\|_\infty\le C'|f|_v\delta_0.$$

Thus, $\{P_n\}_{n\ge0}$ satisfies (**Dec**). The property (**Min**) for $\{P_n\}_{n\ge0}$ follows from Lemma 2.7, as $\gamma>3$ and G_n is continuous everywhere except, possiblly, at the endpoints of W_n.

4. Hitting times for Model A′.

We now prove that the variance in Model A′ grows linearly, if N is large enough and δ_0 is sufficiently small. Recall that

$$\sigma_n^2=\int_{\mathbb{T}}\left(\sum_{i=1}^n\left[\tau_i(G_{1,i}(x))-\int_{\mathbb{T}}\tau_i(G_{1,i}(y))dy\right]^2\right)dx$$

where $\{G_n\}_{n\ge1}$ and $\{\tau_n\}_{n\ge1}$ be given by (3.1) and (3.2).

Lemma 4.1. *There are constants $\bar\delta_0$, N such that if* (1.3_N) *holds and* $\delta_0 \le \bar\delta_0$ *then*

$$\sigma_n^2 \ge Cn$$

for some $C = C(\delta_0, N) > 0$.

The precise restrictions on N and $\bar\delta_0$ will be given later (see (4.5) and (4.6) below).

Proof. Define

$$\tilde\tau_k = \tau_k - \int_{\mathbb{T}} \tau_k(G_{1,k}(x))dx.$$

We have

$$\sigma_n^2 = \sum_{i=1}^n \int_{\mathbb{T}} \tilde\tau_i^2(G_{1,n}(x))dx + 2\sum_{1\le i<j\le n} \int_{\mathbb{T}} \tilde\tau_i(G_{1,i}(x))\tilde\tau_j(G_{1,j}(x))dx. \tag{4.1}$$

By (3.2)

$$\int_{\mathbb{T}} \tau_k(G_{1,k}(x))dx = 1 + 2\int_{\mathbb{T}} 1_{W_k}(G_{1,k}(x))dx.$$

Next,

$$\int_{\mathbb{T}} 1_{W_k}(G_{1,k}(x))dx = \int_{W_k} \mathcal{P}^k \mathbf{1} dx$$

where $\mathcal{P}^k = P_K \dots P_1$ (see (2.1)). Using (2.4), (**Dec**) and (**Min**) we see that there is a constant $\bar C$ such that

$$\frac{\delta_0}{\bar C} \le \int \tau_k(G_{1,k}(x))dx \le \bar C\delta_0.$$

It follows that

$$-\bar C\delta_0 \le \tilde\tau_k \le -\frac{\delta_0}{\bar C} \text{ on } \mathbb{T}\setminus W_k; \quad 1 \le \tilde\tau_k \le 2 \text{ on } W_k. \tag{4.2}$$

Thus

$$\int_{\mathbb{T}} \tilde\tau_i^2(G_{1,i}(x))dx = \int_{\mathbb{T}} \left(\mathcal{P}^i\mathbf{1}\right)\tilde\tau_i^2 dx \ge \sigma\int \tilde\tau_i^2 dx \ge c\delta_0$$

where the first inequality follows from (**Min**) and the second inequality follows from (4.2). Next,

$$\int_{\mathbb{T}} \tilde\tau_i(G_{1,i}(x))\tilde\tau_j(G_{1,j}(x))dx = \int_{\mathbb{T}} \tilde\tau_i\left[P_i\cdots P_{j+1}\left(\tilde\tau_j\mathcal{P}^j\mathbf{1}\right)\right]dx.$$

Note that

$$\int_{\mathbb{T}} [(\mathcal{P}^j\mathbf{1})\tilde\tau_j](y)dy = \int_{\mathbb{T}} \tilde\tau_j(G_{1,j}x)dx = 0$$

by definition of $\tilde\tau$. Hence by (**Dec**)

$$\left|\int \tilde\tau_i P_i\cdots P_{j+1}\left(\tilde\tau_j\mathcal{P}^j\mathbf{1}\right)dx\right| \le D\lambda^{|i-j|}|\tilde\tau_j\mathcal{P}^j\mathbf{1}|_v\|\tilde\tau_i\|_1 \le D'\theta^{|i-j|}\delta_0,$$

for some constants D, D'. Here we used that $\left|\tilde\tau_\ell\mathcal{P}^\ell\mathbf{1}\right|_v \le 3M|\tau|_v$. where M is from (2.4), and $\|\tilde\tau_i\|_1 \le \hat C\delta_0$ where $\hat C = \max(\bar C, 2)$ with $\bar C$ coming from (4.2). Therefore

$$\Big|\sum_{i,j\le n;|i-j|\ge N}\int \tilde\tau_i(G_{1,i}x)\tilde\tau_j(G_{1,j}x)dx\Big| \le \sum_{N\le i\le n}(n-i)D'\theta^i\delta_0 \le D'\delta_0\sum_{N\le i\le n} n\theta^i \le D'n\delta_0\frac{\theta^N}{1-\theta}.$$

We now consider the terms in (4.1) with $|i-j| \le N$. By (2.4)

$$|\{x : G_{1,m}(x) \in W_n\}| = \int_{\mathbb{T}} \mathbb{1}_{W_n}(G_{1,n}(x))dx = \int_{\mathbb{T}} (\mathcal{P}^n \mathbf{1}) \mathbb{1}_{W_n}(x)dx \\ \le |\mathcal{P}^n \mathbf{1}|_v |W_n| \le M\delta_0. \tag{4.3}$$

Recall that by (1.3_N) for all $n \in \mathbb{Z}$ and $0 \le k \le N$

$$G_{n,n+k}(W_n) \cap W_{n+k+1} = \emptyset.$$

Split

$$\int_{\mathbb{T}} \tilde{\tau}_i(G_{1,i}(x))\tilde{\tau}_j(G_{1,j}(x))dx = \left(\int_{I_{ij}} + \int_{\mathbb{T}\setminus I_{ij}} \right) \tilde{\tau}_i(G_{1,i}(x))\tilde{\tau}_j(G_{1,j}(x))dx. \tag{4.4}$$

where

$$I_{ij} = \{x : G_{1,i}(x) \in W_{i+1} \text{ or } G_{1,j}(x) \in W_{j+1}\}.$$

Since $|i-j| \le N$, then $G_{1,j}(x) \in W_{j+1}$ and $G_{1,i}(x) \in W_{i+1}$ can not take place at the same time. Thus, if $G_{1,j}(x) \in W_{j+1}$, then $G_{1,i}(x) \notin W_{i+1}$. Hence by (4.2),

$$|\tilde{\tau}_i(G_{1,j}(x))| \le \bar{C}\delta_0.$$

Also by (4.2) $|\tilde{\tau}_i(G_{1,i}(x))| \le 2$ everywhere. Hence, by (4.3)

$$\int_{I_{ij}} \tilde{\tau}_i(G_{1,i}(x))\tilde{\tau}_j(G_{1,j}(x))dx \le |I_{ij}| \times (\bar{C}\delta_0) \times 2 \le 4\bar{C}M\delta_0^2.$$

For the second term in (4.4), we have by (4.2)

$$\left| \int_{\mathbb{T}\setminus I_{ij}} \tilde{\tau}_i(G_{1,i}(x))\tilde{\tau}_j(G_{1,j}(x))dx \right| \le \bar{C}^2\delta_0^2.$$

Hence

$$\left| \int_{\mathbb{T}} \tilde{\tau}_i(G_{1,i}x)\tilde{\tau}_j(G_{1,j}x)dx \right| \le (4\bar{C}M + \bar{C}^2)\delta_0^2.$$

Since $\mathrm{Card}\{i,j : 1 \le i,j \le n, |i-j| \le N\} \le nN$ we have

$$\sigma_n^2 \ge c\delta_0 n - nN(4\bar{C}M + \bar{C}^2)\delta_0^2 - D'\delta_0 \frac{\theta^N}{1-\theta} n + O(1).$$

Therefore

$$\sigma_n^2 \ge \left(\left(c - D'\frac{\theta^N}{1-\theta} \right) - \left(4\bar{C}M + \bar{C}^2\right)\delta_0 \right) \delta_0 n + O_{N,\delta,M}(1).$$

Take N so large that

$$\theta^N < \frac{c(1-\theta)}{2D'} \tag{4.5}$$

then the first term in parenthesis is at least $c/2$. Hence taking

$$\bar{\delta}_0 = \frac{c}{3(4\bar{C}M + \bar{C}^2)} \tag{4.6}$$

we obtain that for $\delta_0 \le \bar{\delta}_0$

$$\left(c - D'\frac{\theta^N}{1-\theta} \right) > \left(4\bar{C}M + \bar{C}^2\right)\delta_0$$

giving the required estimate $\sigma_n^2 \geq Cn$. □

The estimates established above prove Theorem 1.1(a). We note that the condition (1.3_N) is used only in the proof of Lemma 4.1. Hence part (a') of Theorem 1.1 also holds.

5. Model B. Exponential Mixing.

Let $\bar{P}$ denote the transfer operator of $\bar{T}$. By assumption

$$\bar{T}^p\bar{W} \cap \bar{W} = \emptyset \text{ for } p = 1, 2.$$

We shall use the following notation.

$$\bar{\tau}(x) = \begin{cases} 3 & \text{if} \quad x \in \bar{W}, \\ 1 & \text{if} \quad x \in \mathbb{T} \setminus \bar{W}; \end{cases} \qquad \bar{G}(x) = \begin{cases} \bar{T}(\bar{T}(\bar{T}(x)) & \text{if} \quad x \in \bar{W}, \\ \bar{T}(x) & \text{if} \quad x \notin \mathbb{T} \setminus \bar{W}. \end{cases}$$

Note that if $T_n = \bar{T} + h_n$, where $h_n \in C^{1+\mathrm{Lip}}(\mathbb{T})$, $\|h_n\|_{C^{1+\mathrm{Lip}}} < \delta_0$ and $|\bar{W} \triangle W_n| < \delta_0$, then for δ_0 sufficiently small we will have

$$T_n(W_n) \cap W_{n-1} = \emptyset, \quad T_{n-1}(T_n(W_n)) \cap W_n = \emptyset.$$

Hence, for perturbed maps $T_n = \bar{T} + h_n$, the induced maps G_n in (1.4) and the hitting times functions τ_n are given by the formulas (3.1) and (3.2). As earlier, we denote by P_n the transfer operator of G_n.

We now show that the sequence $\{P_n\}$ satisfies the condition (**Dec**) if δ_0 is sufficiently small. (**Dec**) for $\bar{P}$ follows from Lemma 2.6 and the covering argument in the proof of Lemma 2.7. Applying Corollary 2.3 to the sequence $\bar{P}, \bar{P}, \ldots,$ we can find a neighborhood of $\bar{P}$, where (**Dec**) property is preserved. Hence, to establish (**Dec**) for the sequence $G_1, G_2 \ldots$ for δ_0 small, it suffices to show that the norms $d_1(P_n, \bar{P})$ are small when δ_0 is small.

To this end, we recall the following fact from [3]. Let

$$\widetilde{w}(f, t) = \int_0^1 \sup_{|y-x| \leq t} |f(y) - f(x)| dx. \tag{5.1}$$

Then

$$\widetilde{w}(f, t) \leq 2tV(f). \tag{5.2}$$

As earlier, we need to estimate the norms

$$\|P_n f - \bar{P} f\|_1 = \int_{\mathbb{T}} |P_n f - \bar{P} f| dx.$$

For $x \in W_n \cap \bar{W}$ we have that

$$G_n(x) = T_n(T_{n-1}(T_n(x))) = \bar{T}(\bar{T}(\bar{T}(x)))) + g_n(x),$$

where $\|g_n\|_{C^{1+\mathrm{Lip}}(W_n \cap \bar{W})} < C\delta_0$. Hence, away from a set $A \subset \mathbb{T}$ of measure $O(\delta_0)$ we have $\|\bar{G} - G_n\|_{C^{1+\mathrm{Lip}}(\mathbb{T} \setminus A)} < L\delta_0$, $\forall n \in \mathbb{Z}$. As both $\bar{G}$ and G_n are continuous everywhere away from the endpoints of the intervals $\bar{W}$ and W_n, then for every preimage y_1, with $\bar{G}^{-1}(x) = y_1$, there is a preimage y_2, $G_n^{-1}(x) = y_2$ close to it. The set B of all x for which the pairing described above does not exist between all of its preimages, is of measure $O(\delta_0)$. Since $|G_n'| \le K_1^3$, then for each x there are at most $[K_1^3]+1$ many inverse branches of G_n. One can also see that $|y_1 - y_2| \le L\delta_0/\gamma$. We now write

$$\begin{aligned} P_{G_n} f(x) - P_{\bar{G}} f(x) &= E_0(x) + \sum_{y_1 : G_n(y_1)=x} \frac{f(y_1)}{\bar{G}'(y_1)} - \sum_{y_2 : \bar{G}(y_2)=x} \frac{f(y_2)}{G_n'(y_2)} \\ &= E_0(x) + \sum_{y_1, y_2} \left[\frac{f(y_1)}{\bar{G}'(y_1)} - \frac{f(y_2)}{\bar{G}'(y_1)} \right] + \sum_{y_1, y_2} \left[\frac{f(y_2)}{\bar{G}'(y_1)} - \frac{f(y_2)}{G_n'(y_2)} \right] \end{aligned}$$

where E_0 is supported on B. In particular, $\|E_0\|_1 \le C\delta$. Now note that

$$\left| \frac{1}{\bar{G}'(y_1)} - \frac{1}{G_n'(y_2)} \right| \le \left| \frac{K_1(y_1 - y_2)}{\bar{G}'(y_1)\bar{G}'(y_2)} \right| + \left| \frac{1}{\bar{G}'(y_2)} - \frac{1}{G_n'(y_2)} \right| \le L \left(\frac{K_1 \delta_0}{\gamma^3} + \frac{\delta_0}{\gamma^2} \right).$$

Thus

$$\int_{\mathbb{T}\setminus B} \left| \frac{f(y_1)}{\bar{G}'(y_1)} - \frac{f(y_2)}{G_n'(y_2)} \right| dx \le \int_{\mathbb{T}\setminus B} \left| \frac{f(y_1)}{\bar{G}'(y_1)} - \frac{f(y_2)}{\bar{G}'(y_1)} \right| dx + \int_{\mathbb{T}\setminus B} \left| \frac{f(y_2)}{\bar{G}'(y_1)} - \frac{f(y_2)}{G_n'(y_2)} \right| dx.$$

For the first term on the right, we have by (5.2)

$$\begin{aligned} \int_{\mathbb{T}\setminus B} \left| \frac{f(y_1)}{\bar{G}'(y_1)} - \frac{f(y_2)}{\bar{G}'(y_1)} \right| dx &\le \frac{1}{\gamma} \int_{\mathbb{T}\setminus B} |f(y_1) - f(y_2)| dx \\ &\le \frac{2}{\gamma} \sup |y_1 - y_2| V_{\mathbb{T}}(f) \le \frac{2L\delta_0}{\gamma^2} V_{\mathbb{T}}(f). \end{aligned}$$

Next

$$\int_{\mathbb{T}\setminus B} \left| \frac{f(y_2)}{\bar{G}'(y_1)} - \frac{f(y_2)}{G_n'(y_2)} \right| dx \le \|f\|_\infty \delta_0 L \left(\frac{K_1}{\gamma^3} + \frac{1}{\gamma^2} \right).$$

As the set B is of order $O(\delta_0)$, we also have

$$\int_B |P_n f - \bar{P} f| dx \le L_1 \|f\|_\infty |B| \le L_2 \delta_0 \|f\|_\infty.$$

Recall that $\|f\|_\infty \le |f|_v$. Hence summarizing the estimates above, we finally obtain

$$\|P_n f - \bar{P} f\| \le L' \delta_0 |f|_v.$$

This completes the proof of (**Dec**). The (**Min**) condition for $\mathcal{P}^n \mathbf{1}$ follows from Lemma 2.7. Thus, we have established (**Dec**) and (**Min**) for the sequence $\{P_n\}$.

6. Model B. The growth of variance.

Next we show that the linear growth of variance will persist under small perturbations of $\bar{G}$ and $\bar{\tau}$.

Proposition 6.1. *Assume $\varphi \in \mathcal{V}$ is not cohomologous to a constant under $\bar{G}$. Then there exists $\delta > 0$ and $C_0 > 0$, so that if $P_n \in B_2(\bar{P}, \delta)$, for all $n \ge 1$, then for any sequence of observable $\{f_k\}_{k=1}^\infty$, with $\sup_k |f_k|_v < C_0 < \infty$ and*

$$\sup_{k \ge 1} ||f_k - \varphi||_2 \le \delta, \tag{6.1}$$

we have

$$\operatorname{Var} S_n \ge Cn,$$

where

$$S_n(x) = \sum_{k=0}^{n-1} f_k(G_{1,k}(x)) - \int f_k(G_{1,k}(x))dx.$$

Proof. We will use ideas from [3] and Theorem 4.1 in [11]. Let

$$\tilde{f}_n = f_n - \int f_n(G_{1,n}(x))dx.$$

Note that

$$d_1(P_n, \bar{P}) \leq d_2(P_n, \bar{P}) \leq \delta.$$

Then, in view of Corollary 2.2, we take δ so small that $\{P_n\}_{n\geq 1}$ satisfies (**Dec**).

Consider the following martingale coboundary decomposition. Set

(6.2)
$$\mathbf{H}_n = \frac{1}{\mathcal{P}^n\mathbf{1}} \left[P_n \left(\tilde{f}_{n-1}\mathcal{P}^{n-1}\mathbf{1} \right) + P_n P_{n-1} \left(\tilde{f}_{n-2}\mathcal{P}^{n-2}\mathbf{1} \right) + \cdots + P_n P_{n-1} \ldots P_1 \left(\tilde{f}_0 \mathcal{P}^0 \mathbf{1} \right) \right],$$

$$\psi_n = \tilde{f}_n + \mathbf{H}_n - \mathbf{H}_{n+1} \circ G_{n+1}, \tag{6.3}$$
$$U_n = \psi_n \circ G_{1,n}.$$

Then

$$\sum_{k=0}^{n-1} (\tilde{f}_k \circ G_{1,k}) = \sum_{k=0}^{n-1} U_k + \mathbf{H}_n \circ G_{1,n}.$$

It is shown in [3] that $\{U_k\}$ is a reverse martingale sequence with respect to the filtration $(\mathcal{A}_n)$ defined by (2.2). We thus have

$$\begin{aligned} \left| \|S_n\|_2 - \|\sum_{k=0}^{n-1} U_k\|_2 \right| &= \left| \|S_n\|_2 - \left(\sum_{k=0}^{n-1} \int U_k^2(x)dx \right)^{\frac{1}{2}} \right| \\ &\leq \|S_n - \sum_{k=0}^{n-1} U_k\|_2 \leq \sup_{n\geq 1} |\mathbf{H}_n|_v < \infty. \end{aligned}$$

Therefore it is enough to show linear growth for the sum

$$\sum_{k=0}^{n-1} \int U_k^2(x)dx.$$

Observe that (**Min**) implies that

$$\int U_n^2(x)dx = \int \psi_n^2(G_{1,n}x)dx = \int \psi_n^2(y) \left(\mathcal{P}^n \mathbf{1} \right)(y)dy \geq \sigma \int \psi_n^2(y)dy.$$

We will now show that there exists $\bar{\eta} > 0$ so that for all large $n \in \mathbb{N}$ we have $\|\psi_n\|_2^2 > \bar{\eta}$. This will imply linear growth of the variances $\|S_n\|_2$, i.e.

$$\|S_n\|_2 \geq \sigma\bar{\eta}n + C.$$

We first consider the case where we iterate only one map, i.e. $G_n = \bar{G}$ and $f_n = \varphi$ for all $n \in \mathbb{N}$. Let $\bar{h}$ denote the absolutely continuous invariant density for $\bar{G}$. Let $\varphi_0 = \varphi - \int \varphi(x)\bar{h}(x)dx$. Then

$$\psi_{n,\bar{G}} = \varphi_0 + \mathbf{H}_{n,\bar{G}} - \mathbf{H}_{n+1,\bar{G}} \circ \bar{G}, \tag{6.4}$$

where

$$\mathbf{H}_{n,\bar{G}} = \frac{1}{\bar{P}^n\mathbf{1}}\left[\bar{P}\left(\varphi_0\bar{P}^{n-1}\mathbf{1}\right) + \bar{P}^2\left(\varphi_0\bar{P}^{n-2}\mathbf{1}\right) + \cdots + \bar{P}^n\left(\varphi_0\bar{P}^0\mathbf{1}\right)\right]. \tag{6.5}$$

We now show that there exists $\eta > 0$, so that $||\psi_{n,\bar{G}}||_2 > \eta$ for all large $n \in \mathbb{N}$. Observe that $\bar{P}^n\mathbf{1} \to \bar{h}$ in $|\cdot|_v$ norm and by (**Min**) we also have $\dfrac{1}{\bar{P}^n\mathbf{1}} \to \dfrac{1}{\bar{h}}$ in $|\cdot|_v$. Also note that

$$|P_nP_{n-1}\dots P_{n-k}\left(\tilde{f}_{n-k+1}\mathcal{P}^{n-k}\mathbf{1}\right)|_v \le K\theta^k|\tilde{f}_{n-k+1}\mathcal{P}^{n-k}\mathbf{1}|_v \le K_0\theta^k, \tag{6.6}$$

since $\int \tilde{f}_{n-k+1}\mathcal{P}^{n-k}\mathbf{1}dx = 0$. Hence the $|\cdot|_v$ norm of the terms in (6.2) decays exponentially fast in n. By this and the continuity of $\bar{P}$ in L^2 metric, we have that

$$\mathbf{H}_{n,\bar{G}} \to_{L^2} \tilde{\mathbf{H}} = \frac{1}{\bar{h}}\sum_{n=0}^{\infty}\bar{P}^n\left(\bar{h}\varphi_0\right), \tag{6.7}$$

as $n \to \infty$. Set

$$\psi = \varphi_0 + \tilde{\mathbf{H}} - \tilde{\mathbf{H}} \circ \bar{G}. \tag{6.8}$$

We have $\tilde{\mathbf{H}} \in \mathcal{V}$. Since φ_0 is not cohomologous to 0 under $\bar{G}$, then $\|\psi\|_2 \neq 0$. Hence, in view of (6.7), there exists η such that for all large n

$$||\psi_{n,\bar{G}}||_2 > \eta > 0. \tag{6.9}$$

We now show that φ_0 and $\mathbf{H}_{n,\bar{G}}$ remain uniformly close to $\tilde{f}_n$ and $\mathbf{H}_n$ respectively in L^2 metric as δ goes to zero. This will yield that ψ_n have uniformly large L^2 norms.

Comparing (6.3) and (6.4) we get

$$\psi_n - \psi_{n,\bar{G}} = \left[\tilde{f}_n - \varphi_0\right] + \left[\mathbf{H}_n - \mathbf{H}_{n,\bar{G}}\right] - \left[\mathbf{H}_{n+1}\circ G_{n+1} - \mathbf{H}_{n+1,\bar{G}}\circ\bar{G}\right]. \tag{6.10}$$

By (6.10) and (6.9) it is enough to show that for δ sufficiently small and for all large n, we have

$$\|\mathbf{H}_n - \mathbf{H}_{n,\bar{G}}\|_2 < \frac{\eta}{10}, \quad \|\mathbf{H}_{n+1}\circ G_n - \mathbf{H}_{n+1,\bar{G}}\circ\bar{G}\|_2 < \frac{\eta}{10}, \tag{6.11}$$

and

$$\|\varphi_0 - \tilde{f}_n\|_2 \le \frac{\eta}{10}. \tag{6.12}$$

By Lemma 2.13 of [3], for any $p \le n$ we have the bound

$$\left\|\mathcal{P}^n\mathbf{1} - \bar{P}^n\mathbf{1}\right\|_1 \le C'\left(p\delta + (1-\theta)^{-1}\theta^p\right). \tag{6.13}$$

Taking $p = [\frac{1}{\sqrt{\delta}}] + 1$, we see that for small δ

$$\left\|\mathcal{P}^n\mathbf{1} - \bar{P}^n\mathbf{1}\right\|_1 \le C''\sqrt{\delta}. \tag{6.14}$$

Recalling that $|\mathcal{P}^n\mathbf{1}|_\infty \le M$ and $|\bar{P}^n\mathbf{1}|_\infty \le M$, we obtain

$$\left\|\mathcal{P}^n\mathbf{1} - \bar{P}^n\mathbf{1}\right\|_2 \le \sqrt{2M\left\|\mathcal{P}^n\mathbf{1} - \bar{P}^n\mathbf{1}\right\|_1} \le \sqrt{2MC''\sqrt{\delta}} = C_1\delta^{\frac{1}{4}}. \tag{6.15}$$

Combining (6.14) with (6.1) we get

$$\Big|\int f_n(G_{1,n}(x))dx - \int \varphi(\bar G^n(x))dx\Big| \le C_2\sqrt{\delta}.$$

Using (6.1) once more, we obtain (6.12) if δ is sufficiently small.

Now we estimate the second term in (6.11). We have

$$\begin{aligned}\|\mathbf{H}_{n+1}\circ G_n - \mathbf{H}_{n+1,\bar G}\circ \bar G\|_2 &\le d_2(P_n,\bar P)|\mathbf{H}_{n+1}|_v + \|\mathbf{H}_{n+1}\circ \bar G - \mathbf{H}_{n+1,\bar G}\circ \bar G\|_2\\ &\le \delta\sup_n |\mathbf{H}_n|_v + M\|\mathbf{H}_{n+1} - \mathbf{H}_{n+1,\bar G}\|_2.\end{aligned}$$

Hence, it is enough to bound the norms $\|\mathbf{H}_{n,\bar G} - \mathbf{H}_n\|_2$. To this end note that by (6.6) we can find a number N such that

$$\mathbf{H}_{n,\bar G} = \frac{1}{\bar P^n\mathbf{1}}\left[\bar P\left(\varphi_0 \bar P^{n-1}\mathbf{1}\right) + \bar P^2\left(\varphi_0\bar P^{n-2}\mathbf{1}\right) + \cdots + \bar P^N\left(\varphi_{n-N}\bar P^{n-N}\mathbf{1}\right)\right] + g(N,n),$$

and

$$\mathbf{H}_n = \frac{1}{\mathcal{P}^n\mathbf{1}}\left[P_n\left(\tilde f_{n-1}\mathcal{P}^{n-1}\mathbf{1}\right) + \cdots + P_nP_{n-1}\cdots P_{n-N+1}\left(\tilde f_{n-N}\mathcal{P}^{n-N}\mathbf{1}\right)\right] + h(N,n),$$

satisfy $\|g(N,n)\|_2 < \dfrac{\eta}{40M}$, $\|h(N,n)\|_2 < \dfrac{\eta}{40M}$ for all $n > N$.

We now show that for arbitrary fixed N, there is a constant C, depending only on N and C_0, so that for all $k \le N$ we have

$$\left\|\frac{1}{\mathcal{P}^n\mathbf{1}}P_nP_{n-1}\dots P_{n-k}\left(\tilde f_{n-k+1}\mathcal{P}^{n-k}\mathbf{1}\right) - \frac{1}{\bar P^n\mathbf{1}}\bar P^{k+1}\left(\varphi_0\bar P^{n-k}\mathbf{1}\right)\right\|_2 < C_3\delta^{\frac14}, \tag{6.16}$$

In view of (**Min**) and (6.15) it is enough to show that

$$\left\|P_nP_{n-1}\dots P_{n-k}\left(\tilde f_{n-k+1}\mathcal{P}^{n-k}\mathbf{1}\right) - \bar P^{k+1}\left(\varphi_0\bar P^{n-k}\mathbf{1}\right)\right\|_2 < C_4\delta^{\frac14}.$$

By Lemma 2.4 of [3], we have

$$d_2\left(P_k\cdots P_1,\bar P^k\right) \le M\sum_{j=1}^k d_2\left(P_j,\bar P\right) \le kM\delta, \quad k\ge 1.$$

Therefore, for all $k\le N$ we can write

$$\left\|P_nP_{n-1}\cdots P_{n-k}\left(\tilde f_{n-k-1}\mathcal{P}^{n-k-1}1\right) - \bar P^{k+1}\left(\tilde f_{n-k-1}\mathcal{P}^{n-k-1}1\right)\right\|_2 \le C_5(k+1)\delta.$$

Next, by (6.14) and (6.15) we have

$$\|\tilde f_{n-k-1}\mathcal{P}^{n-k-1}1 - \varphi_0\bar P^{n-k}1\|_2 \le C_6\delta^{\frac14}.$$

Hence, by the continuity of the operator $\bar P^k$ in L^2 norm, it follows that for all $k\le N$ we have

$$\|\bar P^{k+1}\left(\tilde f_{n-k-1}\mathcal{P}^{n-k-1}1\right) - \bar P^{k+1}\left(\varphi_0\bar P^{n-k}1\right)\|_2 \le C_7\delta^{\frac14}. \tag{6.17}$$

Summarizing, we get

$$\|\mathbf{H}_{n,\bar G} - \mathbf{H}_n\|_2 \le C_8\delta^{\frac14} + \frac{\eta}{20M}.$$

Taking δ sufficiently small we will get the estimates (6.11) and (6.12).

Combining the foregoing estimates with (6.10), we see that for δ sufficiently small and n sufficiently large we have

$$\|\psi_n\|_2 > \eta - 3\frac{\eta}{10} = \frac{7\eta}{10},$$

finishing the proof. □

Lemma 6.2. *$\bar{\tau}$ is not cohomologous to a constant under $\bar{G}$.*

Proof. Assume $\bar{\tau}$ is a coboundary for $\bar{G}$. Since $\bar{\tau} \in \mathcal{V}$, then similarly to the proof of Proposition 6.1 (see (6.7) and (6.8)) we can find $g \in \mathcal{V}$, such that the equation

$$\bar{\tau}(x) - \int_{\mathbb{T}} \bar{\tau}(x)h(x)dx = g(x) - g(\bar{G}(x))$$

holds almost surely. Let $A' \subset \mathbb{T}$ be the set of all $x \in A'$ for which the equation above holds for all the forward and backward images of x under $\bar{G}$. Clearly $|A'| = 1$.

Note that there is $x_0 \in \mathbb{T}$, so that $\bar{G}(x_0) = x_0$. Take $x \in A'$. Then

$$\sum_{k=0}^{n} \bar{\tau}(\bar{G}^k(x)) - n\int_{\mathbb{T}} \bar{\tau}(x)h(x)dx = g(x) - g(\bar{G}^n(x)).$$

Observe that $\bar{\tau}(x) = \bar{\tau}(x_0)$ for x sufficiently close to x_0. Hence

$$|n\bar{\tau}(x) - n\int_{\mathbb{T}} \bar{\tau}(x)h(x)dx| = n|\bar{\tau}(x_0) - \int_{\mathbb{T}} \bar{\tau}(x)h(x)dx| \le 2\|g\|_\infty. \tag{6.18}$$

We have $|\bar{\tau}(x_0)| = 1$ or 3. However, $1 < |\int_{\mathbb{T}} \bar{\tau}(x)h(x)dx| < 3$ because h is positive on both the set $\{\bar{\tau} = 1\}$ and the set $\{\bar{\tau} = 3\}$ due to (**Min**). Hence, for large n, (6.18) can not take place. This finishes the proof. □

We have shown that $P_1, P_2, \ldots$ satisfies (**Dec**) and (**Min**) if δ_0 is sufficiently small. Proposition 6.1 gives linear growth of the variance for the sequence

$$\tau_n(x) = \sum_{k=0}^{n-1} r_k(G_{k-1} \circ \cdots \circ G_0 x),$$

for δ_0 sufficiently small. Now Theorem 1.1(c) follows from Theorem 2.1.

7. Walks in random environment. Growth of Variance.

Lemma 7.1. *Let $E = \{(\mathcal{T}_1, \mathcal{W}_1), \ldots, (\mathcal{T}_m, \mathcal{W}_m)\}$ be a collection of maps and gates, so that each sequence $(T_n, W_n)_{n\in\mathbb{Z}}$, with $(T_n, W_n) \in E$, $\forall n \in \mathbb{Z}$ satisfies the conditions of Model A or Model B. Then for all δ_0 sufficiently small, for almost all iid choices of the pairs $(T_n, W_n)_{n\in\mathbb{Z}}$ we have*

$$\sigma_n^2 \ge Cn,$$

where

$$\sigma_n^2 = \int \left[\sum_{i=1}^{n} \left(\tau_i\left(G_{1,i}x\right) - \int_{\mathrm{T}} \tau_i\left(G_{1,i}y\right)dy\right)\right]^2 dx$$

Proof. Since E is finite, then from it follows (3.1) and (3.2) that there are only finitely many induced maps G_n and τ_n appearing in all realizations of environments $(T_n, W_n)_{n\in\mathbb{Z}}$. Let $\{(G_1, \tau_1), \dots, (G_m, \tau_m)\}$ be the set of all possible such pairs. Suppose that (G_i, τ_i) appears with probability q_i. We assume that all q_i are positive (otherwise we drop G_is which have zero probability).

Take δ_0 so small that both (**Dec**) and (**Min**) hold for any realization $(T_n, W_n)_{n\in\mathbb{Z}}$. Set

$$\tilde{f}_n(x) = \tau_n(x) - \int_{\mathbb{T}} \tau_n(G_{1,n}(y))dy = \tau_n(x) - \int_{\mathbb{T}} \tau_n(y)(\mathcal{P}^n \mathbf{1})(y)dy.$$

As before, we consider the martingale coboundary decomposition

$$\mathbf{H}_n = \frac{1}{\mathcal{P}^n\mathbf{1}} \left[P_n\left(\tilde{f}_{n-1}\mathcal{P}^{n-1}\mathbf{1}\right) + P_nP_{n-1}\left(\tilde{f}_{n-2}\mathcal{P}^{n-2}\mathbf{1}\right) + \cdots + P_nP_{n-1}\dots P_1\left(\tilde{f}_0\mathcal{P}^0\mathbf{1}\right)\right],$$

and define the functions

$$\psi_n = \tilde{f}_n + \mathbf{H}_n - \mathbf{H}_{n+1} \circ G_{n+1}. \tag{7.1}$$

As in the proof of Proposition 6.1, it is enough to show that there is a constant C such that for almost all environments if n is large enough then

$$\sum_{k=0}^{n-1} \int \psi_k^2(x)dx \geq Cn.$$

Let $h_{n,k} = P_nP_{n-1}\dots P_{n-k}\mathbf{1}$. We claim that the limit $h_n = \lim_{k\to\infty} h_{n,k}$ exists[3] in BV and moreover there are constants $D > 0, \vartheta < 1$ such that $\|h_n - h_{n,k}\|_v \leq D\vartheta^k$. Indeed

$$h_{n,k} - h_{n,k+1} = P_nP_{n-1}\dots P_{n-k}(\mathbf{1} - P_{n-k-1}\mathbf{1}).$$

Since for each $\int P_m\mathbf{1}dx = \int \mathbf{1}dx = 1$, (**Dec**) implies that $\|h_{n,k} - h_{n,k+1}\|_v$ decays exponentially in k proving both estimates of the claim.

Define

$$\widehat{\mathbf{H}}_n = \frac{1}{h_n}\sum_{k=1}^{\infty} P_nP_{n-1}\dots P_{n-k+1}\left(\hat{f}_{n-k}h_{n-k}\right) \quad \text{where} \quad \hat{f}_m(x) = \tau_m(x) - \int \tau_m(y)h_m(y)dy,$$

$$\hat{\psi}_n = \hat{f}_n + \widehat{\mathbf{H}}_n - \widehat{\mathbf{H}}_{n+1} \circ G_{n+1}$$

Proceeding as in Section 6 we conclude that defini $\|\psi_n - \hat{\psi}_n\|_2$ is exponentially small with respect to n. Accordingly it suffices to show that there is a constant $\hat{C}$ such that for almost all environments if n is large enough then

$$\sum_{k=0}^{n-1} \int \hat{\psi}_k^2(x)dx \geq \hat{C}n. \tag{7.2}$$

[3]We note that changing T_ns for $n < -1$ does not change the distribution of the walk. However, the densities h_n are convenient because their stationarity allows us to use the ergodic theorem. The independence of the walk from the T_n with $n < -1$ will be reflected in the estiamte $|h_n - h_{n,0}|_v \leq D\vartheta^n$ proven below.

Note that the expression in the LHS is a stationary functional of the environment. Therefore applying ergodic theorem we see that (7.2) provided that

$$\mathbf{E}\left(\int \hat{\psi}_k^2(x)dx\right) > 0 \tag{7.3}$$

where $\mathbf{E}$ is the expectation with respect to the environment (with $\hat{C}$ being any number which is smaller that the above integral). It remains to check that the LHS of (7.3). Otherwise we would have that $\psi_k = 0$ almost surely and then the foregoing discussion would imply that there is a constant D such that almost surely $\sigma_n^2 \leq D$ (since $\|\widehat{\mathbf{H}}_n\|_2$ are uniformly bounded). On the other hand if all $(T_k, W_k) = (\mathcal{T}_1, \mathcal{W}_1)$ for all $-1 \leq k \leq n$ then the results of Section 6, in particular, Lemma 6.2 imply that $\sigma_n^2 \geq cn$ for some $c > 0$. If n is so large that $cn \geq D$ we get a contradiction with the assumption that the LHS of (7.3) is zero. This completes the proof. □

8. Natural Scale.

Proof of Theorem 1.3. Observe that

$$\mathbb{P}(\mathcal{S}(z_n) - n > t\hat{\sigma}_n) = \mathbb{P}(\mathcal{S}(z_n) > n + t\hat{\sigma}_n) = \mathbb{P}(z_n > \mathcal{Z}(n + t\hat{\sigma}_n)). \tag{8.1}$$

Let $z_n^* = \max\limits_{k \leq n} z_k$. It is easy to see that

$$z_{n-10}^* \leq z_n \leq z_n^* \tag{8.2}$$

By definition of τ for each p

$$\mathbb{P}(z_{n-p}^* > x) = \mathbb{P}(\tau_{[x]} < n - p). \tag{8.3}$$

Hence

$$\mathbb{P}(z_{n-p}^* > \mathcal{Z}(n + t\hat{\sigma}_n)) = \mathbb{P}(\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)} < n - p)$$

$$= \mathbb{P}\left(\frac{\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)} - n + p + t\sigma_n}{\sqrt{\mathrm{Var}(\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)})}} < -\frac{t\hat{\sigma}_n}{\sqrt{\mathrm{Var}(\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)})}}\right).$$

Recall that

$$\hat{\sigma}_n = \sqrt{\mathrm{Var}(\tau_{\mathcal{Z}(n)})} \leq C\sqrt{\mathcal{Z}(n)} \leq C'\sqrt{n}.$$

It follows that for fixed t we have $|\mathcal{Z}(n + t\hat{\sigma}_n) - \mathcal{Z}(n)| \leq C_1\sqrt{n}$. Since the variance of τ_n grows linearly $\left|\mathrm{Var}(\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)}) - \mathrm{Var}(\tau_{\mathcal{Z}(n)})\right| \leq C_2\sqrt{n}$ and so

$$\frac{t\hat{\sigma}_n}{\sqrt{\mathrm{Var}(\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)})}} = t\left(1 + O\left(n^{-1/2}\right)\right).$$

Therefore, Theorem 1.1 gives

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)} - n + p + t\sigma_n}{\sqrt{\mathrm{Var}(\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)})}} < -\frac{t\hat{\sigma}_n}{\sqrt{\mathrm{Var}(\tau_{\mathcal{Z}(n+t\hat{\sigma}_n)})}}\right) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du. \tag{8.4}$$

Combining (8.2), (8.3) and (8.4) we obtain the result. □

9. Walks in random environment. Control of drift.

Let $\mathcal{F}_{k,n}$ be the σ-algebra generated by $\{T_t, W_t\}_{k\le t\le n}$ and denote $\mathfrak{a}_m = \int_{\mathbb{T}} f_m(G_{1,m-1}x)dx$. The properties of $\mathfrak{a}_m$ are summarized below.

Lemma 9.1. *There are constants $C_1, C_2, C_3 > 0$ $0 < \theta_1, \theta_2, \theta_3 < 1$ such that*

(a) For each k there exist random variables $\mathfrak{a}_{m,k}$ such that $\mathfrak{a}_{m,k}$ is $\mathcal{F}_{m-k,m}$ measurable, for each k the sequence $m \to \mathfrak{a}_{m,k}$ is stationary and $|\mathfrak{a}_{m,k} - \mathfrak{a}_m| < C_1\theta_1^k$.

(b) There exists the limit $\mathbf{a} = \lim_{m\to\infty} \mathbf{E}(\mathfrak{a}_m)$ and moreover $|\mathbf{E}(\mathfrak{a}_m) - \mathbf{a}| \le C_2\theta_2^m$.

(c) $\mathrm{Cov}(\mathfrak{a}_{n_1}, \mathfrak{a}_{n_2}) \le C_3\theta_3^{|n_2-n_1|}$.

(d) There exists $D^2 \ge 0$ such that $\dfrac{\left[\int_{\mathbb{T}} \tau_m(x)dx\right] - m\mathbf{a}}{\sqrt{m}} \Rightarrow \mathcal{N}(0, D^2)$ as $m \to \infty$.

(e) For each $\varepsilon > 0$ there exists $C(\omega)$ such that for each $n_1, n_2 < 10N$ such that $|n_2 - n_1| \le N^{3/4}$ we have

$$|\mathbb{E}(\tau_{n_2}) - \mathbb{E}(\tau_{n_1}) - \mathbf{a}(n_2 - n_1)| \le C(\omega)N^{3/8+\varepsilon}.$$

Remark 9.2. Note that D in part (d) might be equal to zero in some cases.

Proof. Let $\mathfrak{a}_{m,k} = \mathbb{E}[f_m(G_{m-1} \circ \cdots \circ G_{m-k}x)]$. (a) follows by exponential mixing condition (**Dec**). Indeed, note that for $m > k$

$$\Big|\int_{\mathbb{T}} f_m(G_{m-1} \circ \cdots \circ G_{m-k}x)dx - \int_{\mathbb{T}} f_m(G_{m-1} \circ \cdots \circ G_1x)dx\Big|$$

$$\le \int_{\mathbb{T}} [f_m|P_m \dots P_1\mathbf{1} - P_m \dots P_{m-k}\mathbf{1}|]\, dx \le C_1\theta^k.$$

(b) By part (a)

$$\mathbf{E}(\mathfrak{a}_m) = \mathbf{E}(\mathfrak{a}_{m,k}) + O(\theta_1^k) = \mathbf{E}(\mathfrak{a}_{k,0}) + O(\theta_1^k). \tag{9.1}$$

Hence, $|\mathbf{E}(\mathfrak{a}_n) - \mathbf{E}(\mathfrak{a}_m)| < C\theta_1^k$, for $n > m > k$, which shows that the sequence $\{\mathbf{E}(\mathfrak{a}_n)\}_{n\ge1}$ is a Cauchy sequence. Thus, we have the limit $\mathbf{a} = \lim_{m\to\infty} \mathbf{E}(\mathfrak{a}_m)$. Next, by letting $m \to \infty$ in (9.1) we get (b).

(c) using the stationary of $\mathfrak{a}_m$ and (b) we can write

$$\mathbf{E}[(\mathfrak{a}_{n_1} - \mathbf{E}[\mathfrak{a}_{n_1}])(\mathfrak{a}_{n_2} - \mathbf{E}[\mathfrak{a}_{n_2}])] \le C|\mathbf{E}[\mathfrak{a}_{n_1-n_2}] - \mathbf{a}| + C'\theta^{|n_2-n_1|}.$$

$$\le C_3\theta_1^{|n_1-n_2|}.$$

(d) follows from (c), see [8, Chapter XVIII].

(e) also follows from (c) as is shown in [5]. □

We also need

Lemma 9.3 ([9])**.** *There is a constant $\boldsymbol{\sigma}$ such that $\lim_{n\to\infty} \dfrac{\mathrm{Var}(\tau_n)}{n} = \boldsymbol{\sigma}^2$ with probability 1.*

Proof of Theorem 1.4. Let $b_n = \mathcal{Z}(n)$. By Lemma 9.1(e) for $k \le n^{0.6}$

$$\mathcal{S}(b_n + k) = \mathcal{S}(b_n) + k\mathbf{a} + O\left(n^{0.4}\right) = n + k\mathbf{a} + O\left(n^{0.4}\right).$$

Therefore part (a) follows from Theorem 1.3.

To prove part (b) let $v = 1/\mathbf{a}$ and split

$$\frac{z_n - nv}{\sqrt{n}} = \frac{z_n - b_n}{\sqrt{n}} + \frac{b_n - nv}{\sqrt{n}}.$$

By part (a), the first term is asymptotically normal. Also similarly to the proof of Theorem 1.3 we can deduce from Lemma 9.1 that the second term is asymptotically normal. Moreover, those terms are asymptotically independent since the second term depends only on the environment, while the distribution of the first term is asymptotically independent of the environment due to part (a). Since the sum of two independent normal random variables is normal, the sum is asymptotically normal with

$$\sigma^2 = \boldsymbol{\sigma}^2 + D^2$$

where $\boldsymbol{\sigma}$ is from Theorem 1.4(a) and D is from Lemma 9.1(d). □

10. A counter example.

Here we show that the conclusion of Theorem 1.4 does not hold for all environments satisfying Model A and Model B (as opposed to Theorem 1.1). Consider the case where $W_n \in \{W', W''\}$. Suppose that if $W_n \equiv W'$ then $\lim\limits_{m\to\infty} \dfrac{\int_T \tau_m(x)dx}{m} = \mathbf{a}'$ while if $W_n \equiv W''$ then $\lim\limits_{m\to\infty} \dfrac{\int_{\mathbb{T}} \tau_m(x)dx}{m} = \mathbf{a}''$ with $\mathbf{a}' \neq \mathbf{a}''$. Given n let $k(n)$ be such that $k^2 \leq n < (k+1)^2$ and let $W_n = W'$ if $k(n)$ is even, and $W_n = W''$ if $k(n)$ is odd. Using the argument of Lemma 9.1 we see that

$$\int_{\mathbb{T}} \left(\tau_{(k+1)^2}(x) - \tau_{k^2}(x)\right) dx = \begin{cases} 2\mathbf{a}'k + O(1) & k \text{ is even,} \\ 2\mathbf{a}''k + O(1) & k \text{ is odd.} \end{cases}$$

Thus choosing t_k be the largest number such that $\mathbb{E}(\tau_t) \leq k^2$ we get $\mathbb{E}(\tau_t) = k^2 + O(1)$. Since $\mathcal{S}$ has different slopes to the left and to the right of k^2, Theorem 1.3 shows that the distribution of $\frac{z_t - k^2}{k}$ is not close to a normal.

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