SAMPLE PATH PROPERIES OF RANDOM TRANSFORMATIONS.

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1. Models.

Let M be a smooth compact manifold of dimension N and $X_0, X_1 \dots X_d$, $d \geq 2$, be smooth vectorfields on M.

(I) Let $\{w_j\}_{j=-\infty}^{+\infty}$ be a sequence of independent random variables such that w_j is a pair $w_j = (\xi_j, \eta_j)$ uniformly distributed on the set $[-1, 1] \times \{1 \dots d\}$. Let $\phi_k(t)$ denote time t map of the flow generated by X_k and let

$$f_j = \phi_{\eta_j}(\xi_j), \quad F_{m,n} = f_{n-1} \circ \cdots \circ f_{m+1} \circ f_m, \quad F_n = F_{0,n}.$$

(II) Let $w_1(t) \dots w_k(t)$ be independent Brownian motions. Consider Stratanovich differential equation

$$dx_t = X_0(x)dt + \sum_{k=1}^{d} X_k(x) \circ dw_k(t). \tag{1}$$

Let $F_{s,t}$ be the flow of diffeomorphisms generated by (1) and $F_t = F_{0,t}$.

Definition. *We say that either (I) or (II) on satisfy condition (H) on M if the Lie algebra generated by $X_1 \dots X_d$ is TM.*

We assume the following condition

(A) Systems induced by either (I) or (II) on both $M \times M \times M - \text{diag}$ and on Grassmann bundles over M satisfy (H).

Let $\lambda_1 \geq \lambda_2 \geq \lambda_N$ be Lyapunov exponents of our system (given x Lyapunov exponents exist for almost all w and are independent of x see e.g [7]).

Theorem 1. *([11, 4]) (a) Either all exponents coincide or all exponents are different. In the former case F_t preserve a smooth Riemannian metric.*

(b) $\sum_{k=1}^{N} \lambda_k \leq 0$ and if $\sum_{k=1}^{N} \lambda_k = 0$ then F_t preserve a smooth volume form.

We impose the second restriction

(B) $\lambda_1 \neq 0$.

Below we consider only the systems satisfying conditions (A) and (B).

Remark. *If $\{X_j\}$ preserve volume then, by Theorem 1, (B) holds except on a set of infinite codimension. It seems that (B) holds generically also in the dissipative setting but I am not aware of the proof.*

2. SRB measures.

Let m denote Lebesgue measure on M.

Theorem 2. *(a) The following limits exist almost surely*

$$\nu_n = \lim_{k \to -\infty} F_{k,n} m.$$

$\{\nu_n\}$ are invariant in the sense that $F_{k,n}\nu_k = \nu_n$.

(b) ([8]) Given $\alpha > 0$ there exists $\theta < 1, C = C(w)$ such that

$$\forall A \in C^\alpha(M) \quad \left| \int A(F_n(x)) dm(x) - \nu_n(A) \right| \leq C(w) ||A||_{C^\alpha(M)} \theta^n.$$

(c) ([13]) If $\lambda_1 < 0$ then there is a random point $x = x(w)$ such that $\nu_n = \delta_{F_n x(w)}$. Moreover for Lebesgue almost all x $d(F_n x, F_n x(w)) \to 0$ exponentially fast.

(d) ([15]) If $\lambda_1 > 0$ then $\lambda_1 > 0$ then ν_n has positive Hausdorff dimension. Namely let $\Lambda_l = \sum_{k=1}^l \lambda_l$. Let K be the largest number such that $\Lambda_k \geq 0$. Let D be equal to N if $K = N$ and $D = K - (\Lambda_K / \lambda_{k+1})$ otherwise. Then $\mathrm{HD}(\nu_n) = D$.

(e) ([8]) If $\lambda_1 > 0$ then for all $A \in C^3(M)$ there exists $D(A)$ such that for almost all w

$$m \left\{ \frac{\sum_{j=0}^{n-1} [A(F_j x) - \nu_j(A)]}{D\sqrt{n}} < s \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-\xi^2/2} d\xi.$$

Question 1. *What happens if $\lambda_1 = 0$? (see [5] for partial results).*

Question 2. *Can the formula of part (d) be generalized to escape problem from nice domains?*

3. Non-typical points.

Let $\mathcal{E}(A, w) = \{x : \frac{1}{n}[\sum_{j=0}^{n-1} A(F_j x) - \nu_j(A)] \neq 0\}$.

Theorem 3. *([10]) If $A \neq \mathrm{Const}$ then*

(a) if $\lambda_1 > 0$ then $\mathrm{HD}(\mathcal{E}(A)) = N$;

(b) if $\lambda_1 < 0$ then $\mathrm{HD}(\mathcal{E}(A)) < N$.

Question 3. *If $\lambda_1 > 0$ what can be said about*

$$\delta_r = \mathrm{HD}\left\{x : \frac{1}{n}\left[\sum_{j=0}^{n-1} A(F_j x) - \nu_j(A)\right] > r\right\}?$$

Let $\mathcal{E} = \bigcap_{A \in C(M)} \mathcal{E}(A)$.

Question 4. *What can be said about* $\mathrm{HD}(\mathcal{E})$ *in case* $\lambda_1 < 0$?

Question 5. *Let f be a volume preserving diffeomorphism. Suppose that $\lambda_1 > 0$. Is it true that* $\mathrm{HD}(\mathcal{E}) = N$? *More generally, let T be an ergodic automorphism of a probability space (Ω, μ) and*

$$F_n(\omega) = f(T^{n-1}\omega) \circ \cdots \circ f(T\omega) f(\omega)$$

where $f(\omega)$ preserve volume. Suppose that $\lambda_1 > 0$. Is it true that $\mathrm{HD}(\mathcal{E}) = N$?

4. Stable lamination

If ν is a measure on M let

$$I_s(\nu) = \int\int \frac{d\nu(x) d\nu(y)}{d^s(x,y)}.$$

Theorem 4. *(a) ([8]) If ν is a measure on M with $I_s(\nu) < \infty$ for some s then for almost all w $\int A(F_n x) d\nu(x) - \nu_n(A) \to 0$ exponentially fast.*

(b) Stable lamination is transitive on M.

(c) ([9]) If $M = \mathbb{T}^N$ and $\lambda_j \neq 0$ for all j then the lift of stable lamination to $\mathbb{R}^N$ is transitive.

Question 6. *For which over manifolds the lift of the stable lamination to the universal cover must be transitive?*

5. Tools.

The following results play important role in the proofs.

5.1. Two-point motion.

Let $\delta > 0$ be small. Denote

$$\Omega = \{(x,y) \in M \times M : \quad d(x,y) \geq \delta.\}$$

Let $\tau(x,y) = \min\{n \in \mathbb{N} : d(F_n(x), F_n(y) \geq \delta\}$.

Theorem 5. *(a) ([6]) If $\lambda_1 < 0$ then there exists $\theta < 1$ such that for all $(x,y) \in M \times M$*

$$\mathbb{P}\{\tau(x,y) < \infty\} < \theta.$$

(b) ([6]) If $\lambda_1 > 0$ then there exists $r > 1$ such that for all $|\xi| < r$ for all $(x,y) \in \Omega$

$$\mathbb{E}\left(\xi^{\tau(x,y)}\right) \leq \mathrm{Const}$$

(c) ([8]) The the return to Ω process z_{τ_n} is exponentially mixing in the sense that there exists a measure μ on Ω and a number $\theta < 1$ such that for all $A \in C(M)$ for all $(x, y) \in \Omega$

$$\left|\mathbb{E}_{(x,y)}(A(z_{\tau_n}) - \mu(A)\rho^n\right| \leq \text{Const}(\rho\theta)^n ||A||_{C(M)}.$$

Here $\rho = 1$ if $\lambda_1 > 0$ and $\rho < 1$ if $\lambda_1 < 0$.

5.2. **Hyperbolic times.** Given numbers K, α we call a curve γ (K, α)-*smooth* if in the arclength parameterization the following inequality holds

$$\left|\frac{d\gamma}{ds}(s_1) - \frac{d\gamma}{ds}(s_2)\right| \leq K|s_2 - s_1|^\alpha.$$

Theorem 6. *([10]) Fix $\lambda < \lambda_1$ then $\exists r > 0,\ \alpha < 1,\ K > 0$ and $n_0 > 0$ such that for any (K, α)-smooth γ of length between $\frac{r}{100}$ and $100r$ the following holds. $\forall x \in \gamma$ there is a stopping time $\tau(x)$ such that*

(a) $||dF_\tau|T\gamma|| > 100$, $l(F_\tau\gamma) \geq r$;

Let $\bar{\gamma}$ denote a ball of radius r inside $F_\tau\gamma$ centered at $F_\tau(x)$. Then

(b) $\bar{\gamma}$ is (K, α)-smooth;

(c) $\forall k :\ 0 \leq k \leq \left[\frac{\tau}{n_0}\right] \quad \forall y_1, y_2 \in \bar{\gamma}\ d(F_{\tau,\tau-kn_0}y_1, F_{\tau,\tau-kn_0}y_2) \leq d(y_1, y_2)e^{-\lambda k n_0}$;

(d) $\quad \left|\ln ||(dF_\tau^{-1}|T\bar{\gamma})||(y_1) - \ln ||(dF_\tau^{-1}|T\bar{\gamma})||(y_2)\right| \leq \text{Const} d^\alpha(y_1, y_2)$;

(e) $\mathbb{E}(\tau(x)) \leq C_0$; $\mathbb{P}(\tau(x) > N) \leq C_1 e^{-C_2 N}$ where all constants do not depend on γ.

6. Further questions.

In the models described above the distribution of the point $x_n = F_n(x)$ has smooth component. By contrast in the deterministic case (if $F_n = f^n$) then x_n has δ–distribution. The results described above are either unknown or false for the generic deterministic systems.

Question 7. *What can be said in the intermediate cases?*

In other words where is the boundary between truly random and almost deterministic behavior? I believe that very little randomness is needed. For example consider the following model. Let $f_1 \dots f_d$ be smooth diffeomorphisms of M and apply the independently with probabilities $p_1 \dots p_d$.

Conjecture. *The above Markov process is ergodic for generic $f_1 \dots f_d$ in the following cases*

(a) f_j preserve smooth volume;

(b) f_j are close to a given diffeo f.

7. Bibliographical comments.

Properties of SRB measures for random systems are discussed in [13, 14, 15, 18].

Properties of exceptional sets are discussed in [2, 3].

More general classes of random dynamical systems are studied in [1, 11, 16, 17, 18].

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