ON SMALL GAPS IN THE LENGTH SPECTRUM

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Abstract. We discuss upper and lower bounds for the size of gaps in the length spectrum of negatively curved manifolds. For manifolds with algebraic generators for the fundamental gap, we establish the existence of exponential lower bounds for the gaps. On the other hand, we show that the existence of arbitrary small gaps is topologically generic: this is established both for surfaces of constant negative curvature (Theorem 3.1), and for the space of negatively curved metrics (Theorem 4.1). While arbitrary small gaps are topologically generic, it is plausible that the gaps are not too small for almost every metric. One result in this direction is presented in Section 5.

1. Introduction: geodesic length separation in negative curvature

On negatively curved manifolds, the number of closed geodesics of length \( \leq T \) grows exponentially in \( T \) by results of Margulis [Mar04]. (We refer the reader to [Mar04, P-P, P-S] for a comprehensive discussion about the growth and distribution of closed geodesics).

The abundance of closed geodesics leads to the natural question about the sizes of gaps in the length spectrum. In the current note we present a number of results related to this question. In some situations we are able to control the gaps from below, while in other we show that such control is not possible in general.

We note that a presence of exponentially large multiplicities in the length spectrum of a Riemannian manifold (which can be considered as a limiting case of small gaps) changes the level spacings distribution of Laplace eigenvalues on that manifold, see e.g. [LS].

For generic Riemannian metrics, the length spectrum is simple [Abr, A2], so for any geodesic \( \gamma \), only \( \gamma^{-1} \) will have the same length. So, by the Dirichlet box principle, there exist exponentially small gaps between the lengths of different geodesics.

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Accordingly, it seems interesting to investigate manifolds where the gaps between the lengths of different geodesics have exponential lower bound: there exist constants $C, \beta > 0$, such that for any $l_1 \neq l_2 \in \text{Lsp}(M)$ (length spectrum of the hyperbolic surface $M$), we have

$$|l_2 - l_1| > C e^{-\beta \cdot \max(l_1, l_2)}.$$  

This assumption is satisfied for arithmetic hyperbolic groups by the trace separation criterion (cf. [Tak] and [Hej, §18]). In Section 2 we explain (see Theorem 2.1) why the assumption (1.1) holds for hyperbolic surfaces that form a dense set in the corresponding Teichmüller space.

On the other hand the existence of arbitrary small gaps is topologically generic as is shown in Theorem 3.1 for surfaces of constant negative curvature and in Theorem 4.1 for the space of negatively curved metrics endowed with $C^r$-topology, for any $r > 0$.

While arbitrary small gaps are topologically generic, it is plausible that the gaps are not too small for almost every metric. One result in this direction is presented in Section 5 there we obtain an explicit lower bound for the gaps valid for almost every hyperbolic surface.

Length separation between closed geodesics is relevant for the study of wave trace formulas on negatively-curved manifolds: to accurately study contributions from exponentially many closed geodesics to the wave trace formula, it is necessary to separate contributions from geodesics which differ either on the length axis, or in phase space. We remark that a suitable version of (1.1) always holds in phase space: small tubular neighbourhoods of closed geodesics in phase space are disjoint, as shown in [JPT]. Since there exist metrics for which the size of the length gaps cannot be controlled (Theorem 4.1), the authors in [JPT] established microlocal wave trace formula, and used the separation of closed trajectories in phase space in the proof.

2. DIOPHANTINE RESULTS FOR HYPERBOLIC MANIFOLDS WITH ALGEBRAIC GENERATORS OF $\pi_1$

2.1. Length separation for hyperbolic surfaces. We first recall standard results about the fundamental group $\Gamma_g$ of a hyperbolic surface of genus $g \geq 2$: it consists of generators

$$\{A_1, A_2, \ldots, A_{2g} \in PSL_2(\mathbb{R}) : [A_1, A_2] [A_3, A_4] \times \ldots \times [A_{2g-1}, A_{2g}] = 1\}.$$

Here $[A_1, A_2] = A_1 A_2 A_1^{-1} A_2^{-1}$ denotes the commutator.

We formulate the main result of the current section.

**Theorem 2.1.** Assume that $A_j \in PSL_2(\mathbb{Q})$. Then (1.1) holds.
We remark that groups satisfying the assumptions of Theorem 2.1 form a dense set in the corresponding Teichmuller space $T_g$.

The proof of Theorem 2.1 is similar to the proof of Proposition 3 in [GJS], where the authors show that the rotation matrices in $SU(2) \cap M_2(\mathbb{Q})$ satisfy the Diophantine condition defined in [GJS]. Related results for other Lie groups were established in [ABRS, Br11, Var]. Related questions were also discussed in [Glu].

**Proof.** Assume that $A_j \in \text{PSL}_2(\mathbb{Q})$. Then there exists a finite extension $K$ of $\mathbb{Q}$ of degree $d$, and an integer $N > 0$ such that $NA_j \in M_2(\mathcal{O}_K)$, where the $\mathcal{O}_K$ denotes the ring of integers of $K$.

We let $||A_j|| = \text{tr}(A_j A_j^t)$ denote the Hilbert-Schmidt norm of $A_j$. We choose $M > 0$ large enough so that for each of the $d$ embeddings $\sigma_m$ of $K$ into $\mathbb{C}$ we have

$$||\sigma_m(NA_j)|| \leq M, \quad 1 \leq j \leq 2g, \quad 1 \leq m \leq d.$$  

Consider the two closed geodesics $\gamma_k, k = 1, 2$ on $M$, where $l_k = l(\gamma_k), l_1 \neq l_2$. Let $\gamma_k$ be represented by a word $W_k \in \Gamma_g$. We can assume without loss of generality that the word length $|W_k|$ of $W_k$ satisfies $|W_k| \gg 1$. It is known ([Miln, Lemma 2]) that there exists $C_1 > 1$ such that

$$\frac{1}{C_1} \leq \frac{l_k}{|W_k|} \leq C_1. \quad (2.2)$$

We let $b_k = |W_k|$, and $\alpha_k = \text{tr} W_k(A_1, \ldots, A_{2g})$. Since $\alpha_k = 2 \cosh(l_k/2)$, it suffices to show that there exists $C, D > 0$ such that

$$|\alpha_1 - \alpha_2| \geq \frac{1}{(D \cdot \max(|\alpha_1|, |\alpha_2|))^{C'}}. \quad (2.3)$$

We know that $N^{b_k} \alpha_k \in \mathcal{O}_K$. We consider two cases:

1. $N^{b_1} \alpha_1 = N^{b_2} \alpha_2$;
2. $N^{b_1} \alpha_1 \neq N^{b_2} \alpha_2$.

**Case (1):** We assume without loss of generality that $|\alpha_1| > |\alpha_2|$. We have $\alpha_1 \neq \alpha_2, \alpha_2 = N^{b_1-b_2} \alpha_1$; by assumption, $b_1 - b_2 < 0$. It follows that

$$|\alpha_1 - \alpha_2| = |\alpha_1|(1 - N^{b_1-b_2}) \geq \frac{|\alpha_1|}{2},$$

implying (2.3).

**Case (2):** We may assume without loss of generality that $0 < \alpha_1 - \alpha_2 < 1/2$, otherwise (2.3) holds trivially. We may also assume that $\alpha_1 \gg 1$. Let $b = \max(b_1, b_2)$. 

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implying (2.3).

**Case (2):** We may assume without loss of generality that $0 < \alpha_1 - \alpha_2 < 1/2$, otherwise (2.3) holds trivially. We may also assume that $\alpha_1 \gg 1$. Let $b = \max(b_1, b_2)$.
We know, by the property of the norm of an algebraic integer, that
\[ \prod_{m=1}^{d} |\sigma_m(N^{b_1}\alpha_1 - N^{b_2}\alpha_2)| \geq 1. \]

We also know that for every embedding \( \sigma_m : K \rightarrow \mathbb{C} \), we have \( |\sigma_m(N^{b_1}\alpha_1 - N^{b_2}\alpha_2)| \leq 2M^b \). It follows that
\[ |N^{b_1}\alpha_1 - N^{b_2}\alpha_2| \geq \frac{1}{2^{d-1}M^{b(d-1)}}. \]

We consider two further cases:
(2a) \( b_1 = b_2 \);
(2b) \( b_1 \neq b_2 \).

**Case 2a:** It follows from (2.4) that
\[ |\alpha_1 - \alpha_2| \geq \frac{1}{2^{d-1}M^{b_1}M^{b_2(d-1)}}, \]
and it follows from (2.2) that
\[ \frac{1}{(M^{d-1}N)^{b_1}} \geq \frac{1}{\alpha_1^C} \]
for some \( C > 0 \), implying (2.3).

**Case 2b:**
We assume without loss of generality that \( b_1 > b_2 \), hence \( b = b_1 \).
Since \( N^{b_2}\alpha_2 \in \mathcal{O}_K \), it follows that \( N^{b_1}\alpha_2 = N^{b_1-b_2}N^{b_2}\alpha_2 \in \mathcal{O}_K \) as well. Now, since \( \alpha_1 \neq \alpha_2 \), we have \( N^{b_1}\alpha_1 \neq N^{b_1}\alpha_2 \). Accordingly, we can proceed as in Case 2a: one can show similarly to (2.4) that
\[ |N^{b_1}\alpha_1 - N^{b_1}\alpha_2| \geq \frac{1}{2^{d-1}M^{b_1(b-1)}}, \]
hence
\[ |\alpha_1 - \alpha_2| \geq \frac{1}{N^{b_2d-1}M^{b(d-1)}} \]
and (2.3) follows as in Case 2a.

**Remark 2.5.** A suitable version of Theorem 2.1 should hold for many higher-dimensional manifolds of non-positive curvature. Indeed, it is known ([MR, Chapter 3], [Thur]) that generators of a fundamental group of a finite volume hyperbolic 3-manifold can be chosen to be algebraic. Hence we have

**Corollary 2.6.** (1.1) holds for finite volume hyperbolic 3-manifolds.

Also, many hyperbolic manifolds of dimension \( n \geq 4 \) are arithmetic.
We finally note that arithmetic manifolds appear as fundamental domains \( G/\Gamma \) where \( G \) is a connected semi-simple algebraic \( \mathbb{R} \)-group without compact factors of \( \mathbb{R} \)-rank \( \geq 2 \), and \( \Gamma \) is a lattice in \( G \) (cf. [Mar74, Mar75, Mar77]).
3. **Small gaps for surface of constant negative curvature.**

**Theorem 3.1.** The set of tuples \( \{A_1, A_2 \ldots A_{2g}\} \) where (1.1) fails is topologically generic.

**Proof.** Let \( \gamma_A \) denote the closed geodesic whose lift to the fundamental cover joins \( q \) and \( Aq \). Let \( \mathbb{L} \) denote the length spectrum of the geodesics \( \gamma_A \) where \( A \) belongs to a subgroup generated by \( A_1 \) and \( A_2 \). Note that for a dense set of tuples it holds that for each \( \delta \) there exists \( L \) such that for \( l > L \) the set \([l, l + \delta]\) intersects \( \mathbb{L} \) (to see that it suffice to consider the geodesics \( g_{A_1^kA_2^m} \) under the assumption that the eigenvalues of \( A_1 \) and \( A_2 \) are non commensurable. Consider a geodesic \( \tilde{\gamma} = \gamma_{A_3A_1^p} \) where \( n \) is very large. By the foregoing discussion there exists \( l \in \mathbb{L} \) such that \(|l - L_3| < \delta\). Now consider the perturbations of \( A_3 \) of the form \( A_3(\eta) = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} A_3 \). Assume that \( A_3A_1^p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) After applying a small perturbation if necessary we can assume that all entries of this matrix have the order as its trace. Then

\[
\text{tr}(A_3(\eta)A_1^p) = \text{tr}(A_3A_1^p) + \eta c,
\]

so by a small perturbation we can make \( L_{\gamma_{A_3(\eta)A_1^p}} \) as close to \( l \) as we wish. Now the result follows by a standard Baire category argument (cf section 4).

4. **Constructing metrics with small gaps in the length spectrum**

We prove the following proposition:

**Theorem 4.1.** For any \( r > 0 \) for any negatively-curved \( C^r \) metric \( g \), there exists \( C > 0 \) satisfying the following property: for any function \( F(t) \) (which we assume is monotone and fast decreasing), and a number \( \delta > 0 \), there exists a metric \( \bar{g} \), such that \(||\bar{g} - g||_{C^r} < \delta\) and such that \( \bar{g} \) satisfies the following property. There exists an infinite sequence of pairs of closed geodesics \( \gamma_{1,j}, \gamma_{2,j} \) with \( L_{\bar{g}}(\gamma_{i,j}) \to \infty \) as \( j \to \infty \), and

\[
(4.2) \quad |L_{\bar{g}}(\gamma_{1,j}) - L_{\bar{g}}(\gamma_{2,j})| < \min\{F(L_{\bar{g}}(\gamma_{1,j})), F(L_{\bar{g}}(\gamma_{2,j}))\}.
\]

This shows that, in general, one cannot obtain good lower bounds for gaps in the length spectrum for an everywhere \( C^r \)-dense set of negatively curved metrics.

Theorem 4.1 follows from the lemma below by a standard Baire category argument.
Lemma 4.3. Given a metric $g$ and number $L$ and $\delta$ there is a metric $\tilde{g}$ such that $\|g - \tilde{g}\|_{C^r} \leq \delta$ and there are two $\tilde{g}$-geodesics $\gamma_1$ and $\gamma_2$ such that

$$L_{\tilde{g}}(\gamma_1) = L_{\tilde{g}}(\gamma_1) > L.$$ 

Proof of Theorem 4.1. We claim that given metric $g$ and numbers $k \in \mathbb{N}$ and $\delta > 0$ there exists a metric $\bar{g}$ such that $\|\bar{g} - g\|_{C^r} < \delta$ and for each $j = 1\ldots k$ there are geodesics $\gamma_{1,j}, \gamma_{2,j}$ such that

$$(4.4) \quad L_{\bar{g}}(\gamma_{1,j}) > j, \quad |L_{\bar{g}}(\gamma_{1,j}) - L_{\bar{g}}(\gamma_{2,j})| \leq F(\max(L_{\bar{g}}(\gamma_{1,j}), L_{\bar{g}}(\gamma_{2,j}))).$$

It follows that the space of metrics satisfying $(4.2)$ is topologically generic and hence dense.

It remains to construct $\bar{g}$ satisfying $(4.4)$. By Lemma 4.3 we can find $g_1$ such that $\|g - g_1\|_{C^r} < \frac{\delta}{2}$ and there are two geodesics $\gamma_{1,1}, \gamma_{2,1}$ such that

$$L_{g_1}(\gamma_{1,1}) > 1 \quad \text{and} \quad L_{g_1}(\gamma_{1,1}) = L_{g_1}(\gamma_{2,1}).$$

For $j \geq 1$ we apply Lemma 4.3 to find $g_j$ such that

$$\|g_j - g_{j-1}\|_{C^r} \leq \min\left(\frac{\delta}{2^j}, \min_{l=1}^{j} F(L_{g_l}(\gamma_{1,l})) + 1\right)$$

and there are two geodesics $\gamma_{1,j}, \gamma_{2,j}$ such that

$$L_{g_j}(\gamma_{1,j}) = L_{g_j}(\gamma_{2,j}) > j.$$ 

Then $g_k$ satisfies the required properties since the lengths of $g_{1,l}$ have changed by less than $F(L_{g_1}(\gamma_{1,l}) + 1)/2$ in the process of making consecutive inductive steps.

Remark 4.5. In particular if we continue the above procedure for the infinite number of steps then the limiting metric will satisfy the conditions of theorem 4.1.

The proof of Lemma 4.3 relies on two facts. If $\gamma$ is a closed geodesic let $\nu_\gamma$ denote the invariant measure for the geodesic flow supported on $\gamma$. Let $h$ denote the topological entropy of the geodesic flow.

Lemma 4.6. [P-P] $L^h e^{-Lh} \sum_{L(\gamma) \leq L} \nu_\gamma$ converges as $L \to \infty$ to the Bowen-Margulis measure $\mu$.

Lemma 4.7. For each $q_0 \in M$ there exists $\varepsilon$ such that for each $L$ there is periodic geodesic $\gamma$ such that $L(\gamma) > L$ and $\gamma$ does not visit an $\varepsilon$ neighborhood of $q_0$.

Proof of Lemma 4.3. Pick a small $\bar{\delta}$ and large $L$. Let $\gamma_1$ be a closed geodesic such that $L_g(\gamma_1) > L$ and $d(q(\gamma_2(t)), q_0) > \varepsilon$. Let $\gamma_2$ be a closed geodesic such that $L_g(\gamma_1) < L_g(\gamma_2) < L_g(\gamma_1) + \bar{\delta}$ and $\gamma$ spends
at least time \( \mu(B(q_0, \varepsilon/2)/2L_g(\gamma_1) \) inside \( B(q_0, \varepsilon/2) \) (the existence of such a geodesic follows from Lemma 4.6). Take \( \tilde{g}^n = (1 - \eta z(q))g \) where \( z(q) = 1 \) on \( B(q_0, \varepsilon/2) \) and \( z(q) = 0 \) outside \( B(q_0, \varepsilon) \). We can choose \( z \) so that \( ||z||_{C^r} = O(\varepsilon^\gamma) \). Then \( ||g - \tilde{g}^n|| = O(\eta/\varepsilon^\gamma) \) and

\[
L_{\tilde{g}^n}(\gamma_1) \equiv L_g(\gamma_1) \text{ and } L_{\tilde{g}^n}(\gamma_2) \leq L_g(\gamma_1) + \delta - \frac{\mu(B(q_0, \varepsilon/2)L_g(\gamma_1)\eta)}{2}.
\]

Accordingly there exists \( \eta < \frac{2\delta}{L\mu(B(q_0, \varepsilon/2))} \) such that \( L_{\tilde{g}^n}(\gamma_1) = L_{\tilde{g}^n}(\gamma_2) \) as claimed.

**Proof of Lemma 4.7.** We first show how to construct a not necessary closed geodesic avoiding \( B(q_0, \varepsilon) \) and then upgrade the result to get the existence of a closed geodesics.

The first part of the argument is similar to [BS, D]. Pick a small \( \kappa > 0 \). Take an unstable curve \( \sigma \) of small length \( \kappa \). We show that if \( \kappa \) and \( \varepsilon \) are sufficiently small then \( \sigma \) contains a point such that the corresponding geodesic avoids \( B(q_0, \varepsilon) \). Let \( T_1 \) be a number such that \( |\phi_{T_1}(\sigma)| = 1 \) where \( \phi \) denotes the geodesic flow. Note that \( T_1 = O(|\ln \kappa|) \). Also observe that there exists a number \( r_0 \) such that if \( \tilde{\sigma} \) is an unstable curve and \( x \in \tilde{\sigma} \) is such that \( d(q(x), q_0) < \varepsilon \) then \( d(q(\phi_t y), q_0) > \varepsilon \) if \( |t| < r_0 \) and \( C\varepsilon \leq d(y, x) \leq r_0 \) where \( d \) denotes the distance in the phase space (just take \( r_0 \) much smaller than the injectivity radius of \( q_0 \)).

Thus the set

\[
\{ y \in \phi_{T_1}(\sigma) : d(q(\phi_{-t} y), q_0) \leq \varepsilon \text{ for some } 0 \leq t \leq T_1 \}
\]

is a union of \( O(|\ln \kappa|/r_0) \) components of length \( O(\varepsilon/\kappa^a) \) for some \( a > 0 \). Therefore if \( \kappa \ll 1 \) and \( \varepsilon \ll \kappa \) then the average distance between the components is much larger than \( \kappa \). So we can find \( \sigma_1 \subset \phi_{T_1} \sigma \) such that \( |\sigma_1| = \kappa \), and if \( y \in \sigma_1 \) then \( d(q(\phi_{-t} y), q_0) > \varepsilon \) for each \( 0 \leq t \leq T_1 \). Take \( T_2 \) such that \( |\phi_{T_2} \sigma_1| = 1 \). Then we can find \( \sigma_2 \subset \phi_{T_2} \sigma_1 \) such that \( |\sigma_2| = \kappa \), and if \( y \in \sigma_2 \) then \( d(q(\phi_{-t} y), q_0) > \varepsilon \) for each \( 0 \leq t \leq T_2 \). We continue this procedure inductively to construct arcs \( \sigma_j \) for all \( j \in \mathbb{N} \). Taking

\[
x = \bigcap_{j} \phi_{-(T_1 + T_2 + \ldots + T_j)} \sigma_j
\]

we obtain a geodesic avoiding \( B(q_0, \varepsilon) \). To complete the proof we need

**Lemma 4.8.** [Anosov Closing Lemma][H-K] Given \( \eta > 0 \) there exists \( \delta > 0 \) such that if for some \( t_1, t_2 \) such that \( |t_2 - t_1| \) is sufficiently large we have \( d(\gamma(t_1), \gamma(t_2)) < \delta \) then there exists a closed geodesic \( \tilde{\gamma} \) such that \( |L(\tilde{\gamma}) - |t_2 - t_1| | < \eta \) and for each \( t \in [t_1, t_2] \) there exists \( s \) such that \( d(\gamma(t), \tilde{\gamma}(s)) < \eta \).
Take $\delta$ corresponding to $\eta = \varepsilon/2$. Consider points $\gamma(jL)$ where $j = 1\ldots K$. By pigeonhole principle if $K$ is sufficiently large we can find $j_1, j_2$ such that $d(\gamma(j_1 L), \gamma(j_2 L)) < \delta$ and so there exists a closed geodesic $\tilde{\gamma}$ avoiding $B(q_0, \varepsilon/2)$. Since $\varepsilon$ is arbitrary, Lemma 4.7 follows.

Suppose now that dim$(M) = 2$. Let $\mathcal{H}(M)$ denote the space of metrics with positive topological entropy. This set is $C^r$ open ([K]) and dense. (If genus$(M) \geq 2$ then every metric has positive topological entropy [K], while for torus the density of $\mathcal{H}(M)$ follows from [Ban] and for sphere it follows from [KW]).

**Theorem 4.9.** The set of metrics satisfying (4.2) is topologically generic in $\mathcal{H}(M)$.

**Corollary 4.10.** The set of metrics satisfying (4.2) is topologically generic in the space of all metrics on $M$.

**Proof of Theorem 4.9.** By [K] if $g \in \mathcal{H}(M)$ then there is a hyperbolic basic set $\Lambda$ for the geodesic flow. Since Lemmas 4.3, 4.6, 4.7 and 4.8 remain valid in the setting of hyperbolic sets the proof proceeds similarly to the proof of Theorem 4.2. (In the proof of Lemma 4.3 we need to take $\sigma_1$ so that it crosses completely an element of some Markov partition $\Pi$ such that all elements of $\Pi$ have unstable length between $\kappa$ and $C\kappa$. The number of eligible segments now is not $O(1/\kappa)$ but $O(1/\kappa^a)$ for some $a > 0$ but this is still much larger than $|\ln \kappa|$.)

5. Small gaps for hyperbolic surfaces, continued

Here we show that for Lebesgue-typical hyperbolic surface the gaps in the length spectrum cannot be too small. Our argument in similar to [KR]. Related results are obtained in [Var].

**Proposition 5.1.** (see e.g [M-H, Section 3.2]) Consider a degree $D$ polynomial $P(x) = a_D x^D + a_{D-1} x^{D-1} + \ldots a_0$. Then

$$\sup_{[-1,1]} |P(x)| \geq \frac{|a_D|}{2^{D-1}}.$$

**Corollary 5.2.** Let $0 \neq P \in \mathbb{Z}[x_1, x_2 \ldots x_n]$, deg$(P) = D$ then

$$\sup_{[-1,1]^n} |P(x)| \geq \frac{1}{2^{D-1}}.$$

**Proof.** By induction. For $n = 0$ or $1$ the result follows from Proposition 5.1.
Next, suppose the statement is proven for polynomials of \( n - 1 \) variables. If \( P \) does not depend on \( x_n \) then we are done. Otherwise let \( k > 0 \) be the degree of \( P \) with respect to \( x_n \). Then
\[
P(x) = a_k(x_1, \ldots, x_{n-1})x_n^k + a_{k-1}(x_1, \ldots, x_{n-1})x_n^{k-1} + \cdots + a_0(x_1, \ldots, x_{n-1})
\]
where \( a_k \) is the polynomial with integer coefficients of degree \( D - k \).
Let
\[
(\bar{x}_1, \ldots, \bar{x}_{n-1}) = \arg \max_{[-1,1]^{n-1}} |a_k(x_1, \ldots, x_{n-1})|.
\]
Then
\[
\sup_{[-1,1]^n} |P(x_1, \ldots, x_{n-1}, x_n)| \geq \max_{x_n \in [-1,1]} |P(\bar{x}_1, \ldots, \bar{x}_{n-1}, x_n)| \geq |a(\bar{x}_1, \ldots, \bar{x}_{n-1})| 2^{1-k}
\]
completing the proof.

**Proposition 5.3.** (Remez Inequality) (see [BG] or [Yom, Theorem 1.1]) Let \( B \) be a convex set in \( \mathbb{R}^n \), \( \Omega \subset B \), and \( P \) be a polynomial of degree \( D \). Then
\[
\sup_B |P| \leq C_B \text{mes}^{-D}(\Omega) \sup_{\Omega} |P|.
\]

**Corollary 5.4.** Under the conditions of Proposition 5.3
\[
\text{mes}(x \in B : |P(x)| \leq \varepsilon) \leq \left( \frac{C_B \varepsilon}{\sup_B |P|} \right)^{1/D}.
\]

**Proof.** Apply Proposition 5.3 with \( \Omega = \{ x \in B : |P(x)| \leq \varepsilon \} \).

**Corollary 5.5.** If \( P_N \in \mathbb{Z}[x_1, x_2, \ldots, x_n] \) are polynomials of degree \( D_N \) and \( \varepsilon_N \) is a sequence such that \( \sum_N \varepsilon_N^{1/D_N} < \infty \) then \( |P(x_1, \ldots, x_n)| < \varepsilon_N \) has only finitely many solutions for almost every \( (x_1 \ldots x_n) \in \mathbb{R}^n \).

**Proof.** It suffices to show this for a fixed cube \( B \) with side 2. Then
\[
\text{mes}(x \in B : |P_N(x)| \leq \varepsilon_N) \leq \left( \frac{C \varepsilon_N}{2^{D_N}} \right)^{1/D_N} = C \varepsilon_N^{1/D_N}
\]
so the statement follows from Borel-Cantelli Lemma.

**Corollary 5.6.** Let \( m \) be a fixed number.
(a) Let \( P_N \in \mathbb{Z}((a_1, b_1, c_1, d_1), \ldots, (a_m, b_m, c_m, d_m)) \) be polynomials of degree \( D_N \). For \( A_1, \ldots A_m \in SL_2(\mathbb{R}) \) with \( A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \) let
\[
H_N(A_1, \ldots, A_m) = P_N((a_1, b_1, c_1, d_1), \ldots, (a_m, b_m, c_m, d_m))
\]
If $\varepsilon_N^{1/(m+2)D_N} < \infty$ then

$$|H_N(A_1 \ldots A_m)| < \varepsilon_N$$

for only finitely many $N$ for almost every $(A_1, \ldots A_m) \in (SL_2(\mathbb{R}))^m$.

(b) Given $g \in \mathbb{N}$ let

$$G_g = \{(A_1, \ldots A_{2g}) \in (SL_2(\mathbb{R}))^{2m} : [A_1, A_2][A_3, A_4] \ldots [A_{2g-1}, A_{2g}] = I\}.$$ Let $P_N \in \mathbb{Z}((a_1, b_1, c_1, d_1), \ldots, (a_{2m}, b_{2m}, c_{2m}, d_{2m}))$ be polynomials of degree $D_N$. Let

$$H_N(A_1, \ldots A_{2g}) = P_N((a_1, b_1, c_1, d_1), \ldots, (a_{2g}, b_{2g}, c_{2g}, d_{2g})).$$

If $\varepsilon_N^{1/(4g-2)(g+2)D_N} < \infty$ then

$$|H_N(A_1 \ldots A_{2g})| < \varepsilon_N$$

for only finitely many $N$ for almost every $(A_1, \ldots A_{2g}) \in G_g$.

**Proof.** (a) It suffices to prove the statement under the assumption that $|a_j| > \delta$ for some fixed $\delta > 0$. Then $d_j = \frac{1+b_jc_j}{a_j}$ and so

$$H(A_1, \ldots A_m) = \frac{\tilde{P}_N((a_1, b_1, c_1), \ldots, (a_m, b_m, c_m))}{\prod_{j=1}^{m} a_j^{d_j}}$$

where $\tilde{P}_N$ is a polynomial of degree $d_n \leq (m+2)d_N$. Thus if $|P_N| \leq \varepsilon_N$ then $|\tilde{P}_N| \leq \tilde{\varepsilon}_N := \frac{\varepsilon_N}{\delta^{(m+2)D_N}}$. Since

$$\sum_N \varepsilon_N^{1/d_N} \leq \frac{1}{\delta} \sum_N \varepsilon_N^{1/(m+2)D_N} < \infty$$

the result follows from Corollary 5.5.

(b) Rewriting the equations defining $G_m$ in the form

$$[A_1, A_2] \ldots [A_{2g-3}, A_{2g-2}]A_{2g-1}A_{2g}A_{2g-1}^{-1} = A_{2g}$$

we can express the entries of $A_{2g}$ as rational functions of the entries of the other matrices. Arguing as in part (a) we can reduce the inequality $|P_N(A_1, \ldots A_{2g-1}, A_{2g})| < \varepsilon$ to $|\tilde{P}_N(A_1, \ldots A_{2g-1})| < \tilde{\varepsilon}_N$ where $\tilde{P}_N$ is the polynomial of degree $(4g-2)D_N$. Now the result follows from part (a). \qed

**Corollary 5.7.** For each $\eta > 0$ for almost every $A_1, \ldots A_m \in SL_2(\mathbb{R})$ the inequality

$$||W(A_1, \ldots A_m) - I|| > (2m - 1)^{-|W|^2(m+2+\eta)}$$

holds for all except for finitely many words $W$. 
Proof. If $||W(A_1, \ldots, A_m) - I|| \leq \varepsilon$ then all entries of $W - I$ are $\varepsilon$ close to $I$. Considering for example, the condition $W_{11}(A_1, \ldots, A_m) - 1$ we get a polynomial of degree $|W|$. Therefore, by Corollary 5.6 it suffices to check that

$$\sum_w (2m - 1)^{-\frac{|W|^2(m+2)}{|W|(m+2)}} < \infty$$

but the above sum equals to

$$\sum_{D}(2m - 1)^D (2m - 1)^{-D - D(\eta/(m+2)} = \sum_{D}(2m - 1)^{-\eta D/(m+2)} < \infty \quad \Box$$

**Corollary 5.8.** For $A = (A_1, \ldots, A_{2g}) \in G_g$ let $S_A$ be the surface defined by $A$. Given a word $W$ let $l(W, A)$ be the length of the closed geodesic in the homotopy class defined by $W$. Then for each $\eta > 0$ the following holds for almost all $A \in G_g$

There exists a constant $K = K(A)$ such that for each pair $W_1, W_2$

either

$$l(W_1, A) = l(W_2, A)$$

or

$$|l(W_1, A) - l(W_2, A)| \geq K(4g - 1)^{-\frac{1}{2(2g+4)(4g-2)+\eta}} \max^2(|W_1|, |W_2|).$$

**Remark 5.10.** Recall that [Ran] shows that for any hyperbolic surface the length spectrum has unbounded multiplicity so there are many pairs of non conjugated words there the first alternative of the corollary holds.

**Remark 5.11.** Note that (2.2) shows that $l(W_1, A)$ can be close to $l(W_2, A)$ only if the lengths of $W_1$ and $W_2$ are of the same order. Thus (5.9) implies that

$$|l(W_1, A) - l(W_2, A)| \geq Ke^{-R(A)l(W_1, A)}.$$

**Proof.** Let $P_W(A) = \text{tr}(W(A))$. Since $P_W(A) = 2\cosh(l(W, A)/2)$, it follows that if $l(W_1(A))$ is close to $l(W_2(A))$ then

$$l(W_1, A) - l(W_2, A) \geq C|P_{W_1}(A) - P_{W_2}(A)|e^{-|W_1|^D}.$$

Therefore it suffices to show that if $l(W_1, A) \neq l(W_2, A)$ then

$$|P_{W_1}(A) - P_{W_2}(A)| \geq Ke^{-|W_1|^D (4g - 1)^{-\frac{1}{2(2g+4)(4g-2)+\eta}} \max^2(|W_1|, |W_2|)}.$$

Since $\eta$ is arbitrary, we can actually check that

$$|P_{W_1}(A) - P_{W_2}(A)| \geq K(4g - 1)^{-\frac{1}{2(2g+4)(4g-2)+\eta}} \max^2(|W_1|, |W_2|).$$

To verify this we will show that for almost all $A \in G_m$ the inequality

$$|P_{W_1}(A) - P_{W_2}(A)| < (4g - 1)^{-\frac{1}{2(2g+4)(4g-2)+\eta}} \max^2(|W_1|, |W_2|).$$
has only finitely many solutions. Let $P_{W_1,W_2}(A) = P_{W_1}(A) - P_{W_2}(A)$. It is a polynomial of degree $\max(|W_1|, |W_2|)$. So by Corollary 5.6(b) it suffices to check that
\[ \sum_{W_1,W_2} (4g-1) - \frac{[(2g+4)(4g-2)+\eta] \max(|W_1|, |W_2|)}{(4g-2)(g+2)} < \infty \]
There are at most $(4g-1)^{2k}$ pairs $(W_1, W_2)$ with $k = \max(W_1, W_2)$ so the last sum is estimated by
\[ \sum_k (4g-1)^{2k} (4g-1)^{-\frac{[(2g+4)(4g-2)+\eta]k}{(4g-2)(g+2)}} = \sum_k (4g-1)^{-\frac{\eta k}{(4g-2)(g+2)}} < \infty \]
proving the result.

Remark 5.12. It is likely that an explicit lower bound for the gaps in the length spectrum could also be obtained for prevalent set of negatively curved metrics (see [Kal] for related results) but we do not pursue this question here.

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