Math 404 – Spring 2025 – Harry Tamvakis
PROBLEM SET 1 – Due February 6, 2025

Reading for this week: Review of the definitions of rings, fields, vector spaces, and basic linear algebra.

Problems

1) Let V be a vector space and let $H_1, \ldots, H_n$ be subspaces of V. Prove that the intersection $H_1 \cap \cdots \cap H_n$ is a subspace of V.

2) (a) Let V be a vector space over the field $\mathbb{R}$ of real numbers. Suppose that u, v and w are linearly independent vectors in V. Show that the vectors $u+v$, $v+w$ and $w+u$ are also linearly independent. What about the vectors $u-v$, $v-w$ and $w-u$? Justify your answer.

(b) Now suppose that V is a vector space over a field F other than $\mathbb{R}$ and u, v and w are linearly independent vectors in V. Then it can happen that the vectors $u+v$, $v+w$, and $w+u$ are linearly dependent. Give an example where this actually occurs.

3) Find a basis for the (real) subspace of $\mathbb{R}^4$ spanned by the vectors $(1,2,4,0)$, $(3,1,7,1)$, $(7,-6,8,4)$, and $(-1,8,6,-2)$.

4) Let $H \subset \mathbb{R}^5$ be the space of solutions of the system of linear equations $Ax = 0$, where $A = \begin{pmatrix} 3 & 1 & 2 & 5 & -1 \\ 1 & 4 & 3 & 0 & 7 \end{pmatrix}$. Find a basis for H.

5) Let F be any field and V be the F-vector space of all 3×3 symmetric matrices A with entries in F (recall that a matrix A is *symmetric* if $A = A^t$, i.e., A is equal to its own transpose). Determine a basis for V.

6) Solve completely the systems of linear equations $Ax = b$, where

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad b := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad b := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

and we are looking for solutions:

(a) in $\mathbb{Q}$; (b) in $\mathbb{Z}/2\mathbb{Z}$; (c) in $\mathbb{Z}/3\mathbb{Z}$; (d) in $\mathbb{Z}/7\mathbb{Z}$.

7) Let A be an $n \times n$ matrix with entries in a field F. Show that the matrices $I, A, A^2, \ldots$ are linearly dependent. Deduce that there is a non-zero polynomial $f(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$ in $F[x]$ which has A as a root, that is, such that

$$a_r A^r + a_{r-1}A^{r-1} + \cdots + a_1 A + a_0 I = 0.$$

8) Let A be an $n \times n$ matrix with entries in a field F. Referring to the previous problem, suppose that $f(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$ is a non-zero polynomial in $F[x]$ which has A as a root.

(a) If $a_0 \neq 0$, prove that A is an invertible matrix.

(b) Show that the converse of part (a) is false. Specialize the definition of the polynomial $f(x)$ so that it is uniquely determined by the matrix A and such that the converse of part (a) is true.

Extra Credit Problems.

EC1) Let V be a vector space over a field F.

(a) Suppose H_1 and H_2 are subspaces of V. Prove that their union $H_1 \cup H_2$ is a subspace of V if and only if $H_1 \subset H_2$ or $H_2 \subset H_1$.

(b) Suppose H_1, H_2, and H_3 are subspaces of V such that $H_i \neq V$ for each i. Can we have $H_1 \cup H_2 \cup H_3 = V$? Justify your answer.

EC2) Let V be a vector space over the field $\mathbb{R}$ of real numbers.

(a) Let $H_1, \ldots, H_n$ be a finite collection of subspaces of V so that $H_i \neq V$ for each $i \geq 1$. Prove that $H_1 \cup \cdots \cup H_n \neq V$.

(b) Suppose $H_1, H_2, \ldots, H_n, \ldots$ is an infinite sequence of proper subspaces of V. Is it true that the union $\bigcup_i H_i = H_1 \cup \cdots \cup H_n \cup \cdots$ can never equal V? Give a proof of this statement or find a counterexample.

EC3) Let p be a prime number and F be a finite field with p elements. Let k and n be integers with $1 \leq k \leq n$ and consider the set $V := F^n$ as a vector space over F. Let $S(k, n)$ be the set of all k-dimensional linear subspaces of F^n. Compute (with proof) the number of elements of $S(k, n)$. [Hint: Do the case $k = 1$ first, then generalize.]