Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 1 – Due September 5, 2024

Reading for this week: Sections 1.1–1.40, 2.1–2.18.

Problems

From the textbook: Section 1.B, Exercise 6, Section 1.C, Exercises 7, 8, 19. In addition, do the following problems:

A1) Let V be the set of all pairs (x, y) of real numbers and let $F = \mathbb{R}$. Define the operations $(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, 0)$ and $\lambda \cdot (x, y) := (\lambda x, 0)$. Is V, with these operations, a vector space over F? Justify your answer.

A2) Which of the following sets of vectors $v = (x, y, z)$ are subspaces of the real vector space $\mathbb{R}^3$? Justify your answer.

(a) All v such that $x \geq 0$; (b) All v such that $x + 2y = 3z$;

(c) All v such that $xz = 0$; (d) All v such that $y \in \mathbb{Q}$.

A3) Let V be the real vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following sets of functions are subspaces of V?

(a) All f such that $f(x^2) = f(x)^2$; (b) All f such that $f(3) = 0$;

(c) All f such that $f(2) = 1 + f(5)$; (d) All f which are continuous.

A4) Determine whether the vector $(3, -1, 0, -1)$ lies in the subspace of $\mathbb{R}^4$ which is spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, -5)$.

A5) Let H be the subspace of all vectors (x, y, z, w) in $\mathbb{R}^4$ such that $x + y - z - w = 0$. Find a finite set of vectors whose span is H.

A6) Find three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.

Extra Credit Problems.

EC1) Let V be a vector space over the field $\mathbb{R}$ of real numbers. We proved in class that if H_1 and H_2 are subspaces of V such that $H_1 \neq V$ and $H_2 \neq V$ (that is, the H_i are *proper* subspaces), then $H_1 \cup H_2 \neq V$.

(a) Suppose that H_1, H_2, and H_3 are three proper subspaces of V. Prove that

$$H_1 \cup H_2 \cup H_3 \neq V.$$

(b) Suppose that $H_1, \ldots, H_n$ are proper subspaces of V, for some $n \geq 1$. Prove that $H_1 \cup \cdots \cup H_n \neq V$.

(c) Suppose that $H_1, H_2, \ldots, H_n, \ldots$ is a sequence of proper subspaces of V. Is it true that the union $\bigcup_{n=1}^{\infty} H_n$ can never equal V? Give a proof or find a counterexample.

EC2) Prove that part (a) of the previous problem is false for general vector spaces. That is, give an example of a field F, a vector space V over F, and three proper subspaces H_1, H_2, H_3 of V such that

$$H_1 \cup H_2 \cup H_3 = V.$$