Math 620 – Fall 2024 – Harry Tamvakis

PROBLEM SET 1 – Due September 12, 2024

1) (a) Show that, in the ring $\mathbb{Z}[i]$, the relation $xy = \epsilon z^n$, for x, y relatively prime numbers and ϵ a unit, implies $x = \epsilon' u^n$ and $y = \epsilon'' v^n$, with ϵ', ϵ'' both units.

(b) Use part (a) to prove that the integer solutions of the equation

$$x^2 + y^2 = z^2$$

such that $x, y, z \geq 1$ and $(x, y, z) = 1$ (the so-called 'Pythagorean triples') are all given, up permutation of x and y, by the formulas

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2,$$

where $m, n \in \mathbb{Z}$, $m > n \geq 1$, $(m, n) = 1$, m and n not both odd.

2) Recall that two algebraic numbers are *conjugate* if they have the same minimal polynomial over $\mathbb{Q}$. Suppose that α is a non-zero algebraic integer, all of whose conjugates (including α) have absolute value ≤ 1. Prove that α must be a root of unity. [Hint: Show that for each fixed n, there are only finitely many α of degree n over $\mathbb{Q}$ with the required properties. Deduce that the powers of α are restricted to a finite set].

3) Let K be a number field and $\mathcal{O}_K$ be the ring of algebraic integers in K. This problem gives a direct proof that the group $(\mathcal{O}_K, +)$ is finitely generated.

(a) Suppose that $(A, +)$ is an additive subgroup of $(\mathbb{R}^m, +)$, and assume that A intersects any compact subset of $\mathbb{R}^m$ in a finite set of points. Prove that A is a free abelian group with at most m generators.

(b) Let $\sigma_1, \ldots, \sigma_n$ be the distinct embeddings of K into $\mathbb{C}$. Embed $\mathcal{O}_K$ into a Euclidean space E by the map $\alpha \mapsto (\sigma_1\alpha, \ldots, \sigma_n\alpha)$. Show that $\mathcal{O}_K$ intersects any compact subset of E in a finite set.

(c) Deduce that $(\mathcal{O}_K, +)$ is a free abelian group of rank n, and show that a basis of $\mathcal{O}_K$ over $\mathbb{Z}$ is also a basis of K over $\mathbb{Q}$.

4) (a) Let $K := \mathbb{Q}(\alpha)$, where α is a root of the irreducible cubic polynomial $x^3 - 3x + 1$. Compute the trace and the norm of α^2.

(b) Compute the trace and norm of the n-th root of unity $\zeta_n := e^{2\pi i/n}$ in the cyclotomic number field $\mathbb{Q}(\zeta_n)$ when (i) $n = 6$ and (ii) $n = 12$.