Math 404 – Spring 2025 – Harry Tamvakis
PROBLEM SET 10 – Due April 24, 2025

Reading for this week: Sections 8.1, 9.1, 9.3, 9.4 and Theorems 8.18 – 8.20 and 10.1–10.3 (as covered in class, with different proofs).

Problems

1) (a) Let $K = \mathbb{Q}(\sqrt{2+\sqrt{2}})$. Show that $K \supset \mathbb{Q}$ is a Galois extension and that the Galois group $G(K/\mathbb{Q})$ is cyclic. List all the intermediate fields between K and $\mathbb{Q}$.

(b) If $L = \mathbb{Q}(\sqrt{3+\sqrt{3}})$, is $L \supset \mathbb{Q}$ a Galois extension? Justify your answer.

2) Let F be a field of characteristic $p > 0$ and let $f(x) = x^p - x + 1$ be in $F[x]$.

(a) Prove that if α is a root of $f(x)$ in some field extension K of F, then so are $\alpha+1, \alpha+2, \ldots, \alpha+p-1$.

(b) Assume that $f(x)$ is irreducible in $F[x]$, and let G be the Galois group of $f(x)$ over F. Prove that G is a cyclic group.

3) Let G be a finite group and N a normal subgroup of G.

(a) Prove that if N and G/N are both solvable, then G is solvable.

(b) If H is any subgroup of G and we know that H and N are both solvable, prove that HN is solvable.

4) Let G be a group. In the lecture, we defined the decreasing sequence of subgroups $G^{(k)}$ of G, where $G^{(0)} = G$ and for each $k \geq 1$, $G^{(k)}$ is the subgroup of G generated by all commutators $[a,b] := aba^{-1}b^{-1}$ where $a, b \in G^{(k-1)}$. We showed that G is solvable if and only if $G^{(k)} = \{e\}$ for some k.

(a) Prove that each $G^{(k)}$ is a normal subgroup of G.

(b) Prove that a non-trivial solvable group G always has an abelian normal subgroup $N \neq \{e\}$.

5) For $n \geq 2$ the *dihedral group* D_{2n} of order $2n$ is the group

$$D_{2n} = \{1, x, x^2, \ldots, x^{n-1}, y, xy, x^2y, \ldots, x^{n-1}y\}$$

generated by elements x, y with $x^n = 1$, $y^2 = 1$, and $yx = x^{-1}y$.

(a) Determine the commutator subgroup G' when (i) $G = D_4$; (ii) $G = D_6$; (iii) $G = D_8$; (iv) $G = D_{2n}$ for $n \geq 5$.

(b) Prove that D_{2n} is a solvable group.

6) For any two subgroups H, K of a group G, their *commutator* $[H, K]$ is the subgroup generated by all $[h, k]$ with $h \in H$ and $k \in K$. Define subgroups G^i of G recursively by setting $G^0 := G$ and $G^i := [G, G^{i-1}]$ for $i \geq 1$. A group G is called *nilpotent* if $G^n = \{e\}$ for some $n \geq 0$. Any nilpotent group is solvable, but the converse is not true. Determine all integers $n \geq 2$ such that the dihedral group D_{2n} is nilpotent.

7) (a) Let G be a group and H be a subgroup of index n. Prove that there is a group homomorphism $\phi : G \to S_n$ such that $\text{Ker}(\phi) \subset H$. [Hint: If X is the family of all left H-cosets in G, consider the function $\phi : G \to S(X)$ with $\phi(g)(aH) := gaH$.]

(b) Prove that the alternating group A_6 has no subgroups of prime index. [Hint: If H is such a subgroup, show that $|A_6 : H|$ equals 2, 3, or 5, and use part (a).]

8) (a) Let K/F be a finite Galois extension of fields. Must there be an intermediate field E such that $[E : F]$ is a prime number?

(b) The same question as in (a) with the added hypothesis that $G(K/F)$ is a solvable group.

<u>Extra Credit Problem.</u>

EC) (a) Suppose that G is a group of order p^n (where p is a prime). Prove that G is nilpotent [Hint: First prove that the center

$$Z(G) := \{x \in G \mid gx = xg \text{ for all } g \in G\}$$

has at least p elements.]

A subgroup P of a finite group G is called a *Sylow subgroup* if $|P| = p^k$ for some prime p and p^k is the largest power of p that divides $|G|$. A theorem of Sylow states that G has a Sylow subgroup for every prime p dividing $|G|$.

(b) Suppose that $G = P_1 \times \cdots \times P_k$ is the direct product of its Sylow subgroups $P_1, \ldots, P_k$. Prove that G is nilpotent. (With more work, one can show that the converse is also true).