Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 10 – Due November 21, 2024

Problems

From the textbook: Section 7A, Exercise 23, Section 7B, Exercises 10, 21, 24, Section 9A, Exercise 4. In addition, do the following problems:

A1) Let $T : V \to V$ be a normal linear operator on a finite dimensional complex inner product space V. Prove that T is self-adjoint (respectively, unitary) if and only if every eigenvalue of T is real (respectively, of absolute value 1). Give examples to show that this is false if T is not normal.

A2) Let $T : V \to V$ be a linear operator on a finite dimensional complex inner product space V, and let λ and μ be complex numbers such that $|\lambda| = |\mu| = 1$. Prove that $\lambda T + \mu T^*$ is normal.

A3) For which values of $a \in \mathbb{R}$ is the matrix $\begin{pmatrix} a & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ positive definite? Positive semi-definite?

A4) Find necessary and sufficient conditions on the real numbers a and b such that the matrix $\begin{pmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{pmatrix}$ is (i) positive definite; (ii) positive semi-definite.

A5) A matrix $A \in M_n(\mathbb{C})$ is called *nilpotent* if there exists an integer $k \geq 1$ such that $A^k = 0$.

(a) Suppose that $A, B \in M_n(\mathbb{C})$ are nilpotent matrices such that $AB = BA$. Prove that AB and $A + B$ are nilpotent matrices.

(b) Give example(s) to show that if A and B are nilpotent matrices, then AB and $A + B$ need not be nilpotent.

Extra Credit Problem

EC) (a) Let $A \in M_n(\mathbb{R})$ be a positive definite symmetric matrix, and consider the ellipsoid in $\mathbb{R}^n$ defined by the equation $\langle Ax, x\rangle = 1$. Let

$u \in \mathbb{R}^n$ be any vector of length 1 and c be a real number. Prove that the $(n-1)$-dimensional hyperplane with equation

$$\langle u, x \rangle = c$$

is tangent to the ellipsoid if and only if $|c| = \sqrt{\langle A^{-1}u, u \rangle}$.

[Hint: Write $A = P^2$ for P symmetric and positive, and change coordinates to $y = Px$. In the new coordinates, the ellipsoid becomes the sphere $\langle y, y \rangle = 1$, and the hyperplane is given by the equation $\langle P^{-1}u, y \rangle = c$.]

(b) Use part (a) to find the equations of the two tangent lines to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which are perpendicular to the unit vector $u = (\cos\theta, \sin\theta)$.