Math 404 – Spring 2025 – Harry Tamvakis
PROBLEM SET 11 – Due May 1, 2025

Reading for this week: Section 8.1 and 8.5-8.12 from Section 8.2.

Problems

1) Let $f(x) := x^3 + ax + b \in \mathbb{Q}[x]$ be an irreducible cubic polynomial with Galois group G and discriminant $d = -4a^3 - 27b^2$. Prove that

(a) $f(x)$ has exactly one real root if and only if $d < 0$, in which case $G \cong S_3$. [Hint: Use the definition of d and the fact that the roots of f sum to zero.]

(b) $f(x)$ has three real roots if and only if $d > 0$. In this case, either $\sqrt{d} \in \mathbb{Q}$ and $G \cong C_3$ is a cyclic group or $\sqrt{d} \notin \mathbb{Q}$ and $G \cong S_3$.

2) Let $f \in \mathbb{Q}[x]$ be a polynomial of degree 4, and let K be the splitting field of f over $\mathbb{Q}$. Assume that $[K : \mathbb{Q}] = 24$. Prove that for any positive divisor d of 24, there is an element $\alpha \in K$ such that α has degree d over $\mathbb{Q}$.

3) Prove that the transitive subgroups of S_4 are (i) S_4, (ii) A_4,

(iii) $V := \{1, (12)(34), (13)(24), (14)(23)\}$,

(iv) $C := \{1, (1234), (13)(24), (1432)\}$ and its conjugates, and

(v) $D := V \cup \{(12), (34), (1423), (1324)\}$ and its conjugates.

4) Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 4 with Galois group G, and let α be a root of f. Prove that if there is no field properly between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$ then $G = A_4$ or $G = S_4$.

5) Let F be a subfield of the real numbers and $f \in F[x]$ be an irreducible polynomial of degree 4. Let G be the Galois group of f over F. Prove that if f has exactly two real roots, then G is the symmetric group S_4 or G has order 8.

6) Compute the cyclotomic polynomials $\Phi_9(x)$, $\Phi_{10}(x)$, and $\Phi_{2^n}(x)$ for each $n \geq 1$. Show all of your work.

7) Let $n \geq 3$ be an odd natural number.

(a) Prove that if a field F contains the n-th roots of unity, then F also contains the $2n$-th roots of unity.

(b) Let $\Phi_n(x)$ be the n-th cyclotomic polynomial. Prove that

$$\Phi_{2n}(x) = \Phi_n(-x).$$

8) Let $L \supset F$ be a field extension and suppose that K_1 and K_2 are two subfields of L containing F. Assume that K_1/F and K_2/F are both Galois extensions.

(a) Prove that K_1K_2 and $K_1 \cap K_2$ are Galois extensions.

(b) Prove that the map $\sigma \mapsto (\sigma|_{K_1}, \sigma|_{K_2})$ induces group homomorphism

$$\phi : G(K_1K_2/F) \to G(K_1/F) \times G(K_2/F)$$

which maps $G(K_1K_2/F)$ isomorphically onto the subgroup

$$H := \{(\sigma_1, \sigma_2) \mid \sigma_1|_{K_1\cap K_2} = \sigma_2|_{K_1\cap K_2}\}$$

of $G(K_1/F) \times G(K_2/F)$.

[Hint: For part (b), show that (i) ϕ is 1-1 and (ii) $|G(K_1K_2/F)| = |H|$. To compute $|H|$, prove that each $\sigma_1 \in G(K_1/F)$, $\sigma_1|_{K_1\cap K_2}$ has exactly $[K_2 : K_1 \cap K_2]$ extensions to an element of $G(K_2/F)$. Deduce that $|H| = [K_1 : F][K_2 : K_1 \cap K_2]$, and use a Corollary from class.]

Extra Credit Problems.

EC1) Prove the converse of Problem 4 above: If $f \in \mathbb{Q}[x]$ is an irreducible quartic polynomial with Galois group A_4 or S_4, and α is a root of f, then there is no field properly between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$.

EC2) Show that the discriminant of the quartic polynomial x^4+ax+b is equal to $-27a^4 + 256b^3$.