Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 11 – Due December 5, 2024

Reading for this week: Sections 5.19-5.30, 8.1-8.25, 8.35-8.38, 8.42-8.46.

Problems

From the textbook: Section 5B, Exercises 6, 11, 14, Section 8A, Exercises 2, 20, Section 8B, Exercise 17.

A1) (a) Determine the minimal polynomial of an $n \times n$ diagonal matrix whose diagonal entries are the complex numbers $\lambda_1, \ldots, \lambda_n$.

(b) If a matrix $A \in M_n(\mathbb{C})$ has minimal polynomial in the form you found in part (a), can you conclude that A is a diagonal matrix? Justify your answer.

A2) Suppose that V is a vector space over F, $T : V \to V$ is a linear map, and $v \in V$.

(a) Prove that there exists a unique monic irreducible polynomial p of smallest degree such that $p(T)v = 0$.

(b) Prove that p divides the minimal polynomial of T.

A3) (a) Let α be a complex number. Prove that the matrices

$$\begin{pmatrix} 0 & 1 & \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

are similar.

(b) Discover and prove a generalization of part (a) to $n \times n$ matrices, for $n \geq 4$.

A4) Let N_1 and N_2 be 6×6 nilpotent matrices with entries in a field F. Suppose that N_1 and N_2 have the same minimal polynomial and the same rank. Prove that N_1 and N_2 are similar. Show that this is not true for 7×7 nilpotent matrices.

Extra Credit Problem

EC) (a) Suppose that V is a finite dimensional vector space over a field F, and $T : V \to V$ is a linear transformation. Prove that we have a decomposition $V = U \oplus U'$ of V as a direct sum of two T-invariant subspaces U and U', such that $T|_U$ is an *isomorphism*, while $T|_{U'}$ is *nilpotent*.

(b) Suppose that $F = \mathbb{C}$, so that the vector space V in part (a) decomposes as a direct sum of generalized eigenspaces for T. Describe the subspaces U and U' in terms of these eigenspaces.