Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 2 – Due September 12, 2024

Reading for this week: Sections 2.19–2.32, 2.34–2.42.

Problems

From the textbook: Section 2.A, Exercises 5, 9, 13, Section 2.B, Exercises 6, 7, 8, Section 2.C, Exercises 5(a)(b), 8, 9. In addition, do the following problems:

A1) Find a basis for the real subspace of $\mathbb{R}^4$ spanned by the vectors $(1,2,-1,0)$, $(4,8,-4,-3)$, $(0,1,3,4)$, and $(2,5,1,4)$.

A2) Let $V \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $Ax = 0$, where $A = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$. Find a basis for V.

A3) Let V be the real vector space of all functions $f : [0,1] \to \mathbb{R}$. Prove that the functions x^2, $\sin x$, and $\cos x$ are linearly independent in V.

Extra Credit Problems.

EC1) Let V be a finite dimensional vector space over a field F. Suppose we are given a rule which associates to each *subset*, S, of V a *subspace*, $[S]$ of V, and we are told that this rule obeys the laws:

(1) For each subset S, S is contained in $[S]$;

(2) If $S \subset T$, then $[S] \subset [T]$;

(3) If W is any subspace of V and $W \neq V$, then $[W] \neq V$.

Prove that for every subset S of V, we have $[S] = \mathrm{Span}(S)$.

[Hint: First try to prove this when $\dim V = 1$ and $\dim V = 2$ to get some intuition about what is going on; then do the general case.]

EC2) Let $V = \mathbb{R}$ be the set of real numbers. Consider V as a vector space over the field $\mathbb{Q}$ of rational numbers, with the usual operations. Prove that V is *not* finite dimensional. [Hint: You will need to know about countable and uncountable sets to do this problem.]