Math 620 – Fall 2024 – Harry Tamvakis

PROBLEM SET 2 – Due September 26, 2024

1) (Problem 2 in Section 2.15) Let K be a number field with $2r_2$ complex embeddings into $\mathbb{C}$ and let $\mathcal{O}$ be an order of K. Prove that the sign of $\Delta(\mathcal{O})$ is equal to $(-1)^{r_2}$.

2) Let $x_1, \ldots, x_n$ be arbitrary integers in a number field K, and consider the determinant of the matrix $\{\sigma_i(x_j)\}_{1\leq i,j\leq n}$ considered in class. Let P denote the sum of the $n!/2$ terms in the expansion of the determinant which are prefixed by plus signs, and N denote the sum of the $n!/2$ terms prefixed by minus signs, so that

$$\Delta(x_1, \ldots, x_n) = (P - N)^2.$$

(a) Prove that $P + N$ and PN are fixed by each embedding $\sigma_i : K \to \mathbb{C}$, and deduce that they are both rational numbers.

(b) Show that $P + N$ and PN are rational integers.

(c) Prove that $\Delta(x_1, \ldots, x_n)$ is congruent to 0 or 1 mod 4.

3) Let a and b be positive integers which are not squares. Show that every unit of the ring $\mathbb{Z}[\sqrt{a},\sqrt{-b}\,]$ is a unit of $\mathbb{Z}[\sqrt{a}\,]$.

4) Let $\{a_{ij}\}$ be a matrix of real numbers such that $\det(a_{ij}) \neq 0$. Consider the linear forms

$$L_i(x_1, \ldots, x_n) := \sum_{j=1}^{n} a_{ij} x_j$$

for $i = 1, \ldots, n$ and let $c_1, \ldots, c_n$ be positive real numbers such that $c_1 \cdots c_n > |\det(a_{ij})|$. Show that there exist integers $k_1, \ldots, k_n$ not all zero such that

$$|L_i(k_1, \ldots, k_n)| < c_i \quad \text{for} \quad i = 1, \ldots, n.$$

5) Let $\alpha = \sqrt[3]{2}$ and $K = \mathbb{Q}(\alpha)$.

(a) Let $x = a + b\alpha + c\alpha^2$ be an element of K. Calculate the trace, norm, and characteristic polynomial of x over $\mathbb{Q}$.

(b) Consider the order $\mathcal{O} = \mathbb{Z}[\alpha]$ of K. One can show that $\mathcal{O} = \mathcal{O}_K$ is the ring of integers in K. Calculate the discriminant $d_K = \Delta(\mathcal{O})$.

(c) Show that every positive (respectively, negative) unit is of norm 1 (respectively, -1). Show that $u = \alpha^2 + \alpha + 1$ is a unit of $\mathcal{O}$. It is a fact that u is a generator of the group of positive units of $\mathcal{O}$.

(d) Calculate four solutions in integers a, b, c of the equation

$$a^3 + 2b^3 + 4c^3 - 6abc = 1.$$

Explain how one can find all its rational integer solutions, and also all its positive integer solutions.