Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 3 – Due September 19, 2024

Reading for this week: Sections 3.1–3.20, 3.29–3.51, 3.59–3.62, 3.69–3.72, 3.87–3.94, 1.41–1.46.

Problems

From the textbook: Section 1.C, Exercises 21, 24. Section 3.B, Exercises 3, 5, Section 3.E, Exercise 1. In addition, do the following problems:

A1) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$T(x, y, z) = (x - y + 2z,\, 2x + y,\, -x - 2y + 2z).$$

(a) Verify that T is a linear transformation.

(b) What are the conditions on the real numbers a, b, and c so that the vector (a, b, c) lies in the image of T? What is the rank of T?

(c) What are the conditions on the real numbers a, b, and c so that the vector (a, b, c) lies in the kernel of T? What is the nullity of T?

A2) Find an explicit linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that the image of T is spanned by the vectors $(1, 2, 4)$ and $(3, 6, -1)$.

A3) Let $V = \mathbb{C}$ be the field of complex numbers regarded as a vector space over the field of *real* numbers (with the usual operations). Find a function $T : V \to V$ such that T is a linear transformation on the real vector space V, but such that T is not a linear transformation when V is regarded as a vector space over the field of *complex* numbers. Make sure to justify your answer.

A4) Let T_A be the linear operator on $\mathbb{R}^3$ which is represented by the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}.$$

Find a basis for the image of T_A and a basis for the kernel of T_A.

A5) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (2x,\, x - y,\, 3x + y + z). \tag{1}$$

If T invertible? If so, find a formula for the inverse transformation T^{-1} similar to the equation (1) which defines T.

A6) Let V and W be finite dimensional vector spaces over the same field F. Suppose that $T : V \to W$ is a linear transformation and let $v_1, \ldots, v_n$ be a basis of V.

(a) If T is an isomorphism, prove that $T(v_1), \ldots, T(v_n)$ is a basis of W.

(b) Conversely, if $T(v_1), \ldots, T(v_n)$ is a basis of W, prove that T is an isomorphism.

(c) Does the conclusion of part (a) remain true if T is only 1-1 on V? What if T is only onto W? In each case, give a proof or provide a counterexample.

A7) Suppose that V is a vector space (possibly infinite dimensional) and S and T are two linear transformations $V \to V$. We denote the composite map $S \circ T$ by ST, and the identity map on V by I. If $ST = I$, then S is called a *left inverse* of T and T is called a *right inverse* of S. Prove that if S has exactly one right inverse, then S is an isomorphism. [Hint: If T is the unique right inverse of S, then consider the map $TS + T - I$.]

Extra Credit Problems.

EC1) Consider the set of all those vectors in $\mathbb{R}^3$ all of whose coordinates are either 0 or 1. How many different bases of $\mathbb{R}^3$ does this set contain?

EC2) Let F denote the field $\{0, 1\}$ with two elements, and fix $n \geq 1$.

(a) How many matrices are there in $M_n(F)$?

(b) How many matrices in $M_n(F)$ are *invertible*? For example, there are 6 invertible matrices in $M_2(F)$, namely, the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

[Hint: For part (b), a matrix A is invertible if and only if its columns form a basis of F^n. Choose the columns one by one and record how many choices you have at each step, then multiply the resulting numbers together.]