Math 620 – Fall 2024 – Harry Tamvakis

PROBLEM SET 3 – Due October 10, 2024

1) Suppose the M is a compact convex body in $\mathbb{R}^n$ which is symmetric about the origin and has volume $\mathrm{vol}(M) \geq m2^n$ for some natural number m. Prove that besides the origin there are at least m other lattice points which lie in the set M (here a lattice point means an element of $\mathbb{Z}^n$).

2) [Koch, Exercise 3.13.2] Determine a value of $\alpha \in \mathbb{Z}[i]$ such that

$$\alpha \equiv 1 \pmod{4+i} \quad \text{and} \quad \alpha \equiv i \pmod{3+2i}.$$

3) Let d be a negative, square-free integer which has at least two prime factors. Show that $\mathbb{Z}[\sqrt{d}\,]$ is not a principal ideal domain.

4) Suppose that $\mathcal{O}$ is a Dedekind domain.

(a) If I, J are non-zero ideals of $\mathcal{O}$, prove that there exists an $\beta \in I$ such that $\beta I^{-1} + J = \mathcal{O}$.

(b) Let I be a non-zero ideal of $\mathcal{O}$, and suppose $\alpha \in I \smallsetminus \{0\}$. Show that there exists an $\beta \in I$ such that $I = \langle \alpha, \beta \rangle \mathcal{O}$.

5) Consider the factorization of 18 into elements of the ring $\mathcal{O} = \mathbb{Z}[\sqrt{-17}]$:

$$18 = 2 \cdot 3 \cdot 3 = (1 + \sqrt{-17})(1 - \sqrt{-17}). \tag{1}$$

Determine the unique factorization of the ideals $2\mathcal{O}$, $3\mathcal{O}$, $(1+\sqrt{-17})\mathcal{O}$, and $(1-\sqrt{-17})\mathcal{O}$ as a product of prime ideals of $\mathcal{O}$, and combine these to refine (1) into the (unique) factorization of $18\mathcal{O}$ into prime ideals.