Math 404 – Spring 2025 – Harry Tamvakis
PROBLEM SET 4 – Due February 27, 2025

Reading for this week: Sections 3.2, 3.3, and 4.1.

Problems

1) Let K and L be subfields of a field M and suppose that K and L both contain the field F. Let KL denote the subfield of M generated by $K \cup L$. Write $[K : F] = m$, $[L : F] = n$ and $[KL : F] = t$ (these cardinalities might be infinite).

(a) Prove that t is finite if and only if both m and n are finite.

(b) In this case show that both m and n divide t, and $t \leq mn$.

(c) If m and n are relatively prime, show that $t = mn$.

2) Let α be a complex number which is a root of $x^3 - 3x + 4$, an irreducible polynomial over $\mathbb{Q}$. Express the inverse of $1+\alpha+\alpha^2$ in the form $x + y\alpha + z\alpha^2$ for some rational numbers x, y, and z.

3) Determine the minimal polynomial of $\alpha = \sqrt{3}+\sqrt{5}$ over each of the following fields: (a) $\mathbb{Q}$ (b) $\mathbb{Q}(\sqrt{5})$ (c) $\mathbb{Q}(\sqrt{10})$ (d) $\mathbb{Q}(\sqrt{15})$.

4) Let $n \geq 1$ be positive integer and

$$\zeta_n := e^{2\pi i/n} = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n}).$$

The complex number ζ_n is an n-th root of unity which generates the group of all n-th roots of unity in $\mathbb{C}$. Find the minimal polynomial over $\mathbb{Q}$ of (a) ζ_6 (b) ζ_9 (c) ζ_{11} (d) ζ_{12}.

5) Decide whether or not i is in the field (a) $\mathbb{Q}(\sqrt{-2})$, (b) $\mathbb{Q}(\sqrt[4]{-2})$, (c) $\mathbb{Q}(\alpha)$, where $\alpha \in \mathbb{C}$ satisfies $\alpha^3 + \alpha + 1 = 0$.

6) Suppose that α and β have minimal polynomials $x^2 + a_1x + a_2$ and $x^2 + b_1x + b_2$ over a field F, respectively.

(a) Use the method explained in class (or your own approach) to construct a polynomial in $F[x]$ which has $\alpha\beta$ as a root.

(b) Which polynomial does part (a) produce when $F = \mathbb{Q}$, $\alpha = \sqrt{m}$, and $\beta = \sqrt{n}$, where m and n are positive integers which are not perfect squares?

7) Consider the polynomial

$$f(x) = x^5 - \sqrt{3}\,x^4 + \sqrt{5}\,x^2 + \sqrt{15}\,x - 4$$

and let α be complex number with $f(\alpha) = 0$. Prove that α is algebraic over $\mathbb{Q}$ and that the degree of α over $\mathbb{Q}$ is less than or equal to 20.

8) (a) Prove that if K is a field containing $\mathbb{C}$ and $[K : \mathbb{C}]$ is finite, then $K = \mathbb{C}$.

(b) Prove that if K is a field containing $\mathbb{R}$ and $[K : \mathbb{R}]$ is finite, then $K = \mathbb{R}$ or K is isomorphic to $\mathbb{C}$.

Extra Credit Problems.

EC1) Determine all natural numbers n such that the polynomial

$$x^n + 1$$

is irreducible over $\mathbb{Q}$. You must *prove* that your answer is correct.

EC2) Show that there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ of degree > 1 that is irreducible modulo p for all prime numbers p.