Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 4 – Due September 26, 2024

Reading for this week: Sections 2.33, 2.43, 3.21–3.28, 3.63–3.68, 3.93–3.116.

Problems

From the textbook: Section 3.B, Exercises 17, 25. Section 3.D, Exercises 9, 10. Section 3.E, Exercises 6, 14, Section 3.F, Exercises 4, 9. In addition, do the following problems:

A1) Let P denote the real vector space of all polynomials with real coefficients. For each of the subspaces U of P which follow, decide (with proof) whether or not the quotient space P/U is finite dimensional. (a) $U := P_n$ is the space of polynomials of degree at most n; (b) U is the space of even polynomials, that is, $p \in P$ such that $p(-x) = p(x)$ for all x; (c) U is the set of all polynomials which are divisible by x^2, that is, $p \in P$ such that $p(x) = x^2 q(x)$ for some polynomial $q \in P$.

A2) Let U and V be subspaces of a vector space W and suppose that $U \subset V$. The quotient space V/U is a then a subspace of the quotient space W/U in a natural way. Prove that we have an isomorphism

$$(W/U)/(V/U) \cong W/V.$$

A3) The vectors $v_1 = (1, 1, 1)$, $v_2 = (1, 1, -1)$, and $v_3 = (1, -1, -1)$ form a basis of $\mathbb{R}^3$. If v_1^*, v_2^*, v_3^* denotes the dual basis, and if $x = (0, 1, 0)$, compute the values $v_1^*(x)$, $v_2^*(x)$, and $v_3^*(x)$.

A4) Let P_2 be the real vector space of all polynomials p with real coefficients of degree at most 2. Define three linear functionals on P_2 by

$$f_1(p) = \int_0^1 p(x)\,dx, \quad f_2(p) = \int_0^2 p(x)\,dx, \quad f_3(p) = \int_0^{-1} p(x)\,dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for the dual space P_2^* by exhibiting the basis for P_2 of which it is the dual.

Extra Credit Problems.

EC1) From Section 2.C, Exercise 19.

EC2) Let F be a field and V be a vector space over F.

(a) Suppose that f is a linear functional on V and v is a non-zero vector in V which is not in $\mathrm{Ker}(f)$. Prove that

$$V = \mathrm{Ker}(f) \oplus \{\lambda v \mid \lambda \in F\}.$$

(b) Suppose that f and g are linear functionals on V such that $\mathrm{Ker}(f) = \mathrm{Ker}(g)$. Prove that there exists a constant $\lambda \in F$ such that $f = \lambda g$.

(c) Give an example of two linear maps $S, T : F^5 \to F^2$ such that $\mathrm{Ker}(S) = \mathrm{Ker}(T)$ but S is not a scalar multiple of T.