Math 404 – Spring 2025 – Harry Tamvakis
PROBLEM SET 5 – Due March 6, 2025

Reading for this week: Sections 4.1, 4.2, and Chapter 5.

Problems

1) Decide whether the following pairs of fields are isomorphic:

(a) $\mathbb{Q}(\sqrt[4]{2})$ and $\mathbb{Q}(i\sqrt[4]{2})$;

(b) $\mathbb{Q}(\sqrt[3]{1+\sqrt{3}})$ and $\mathbb{Q}(\sqrt[3]{1-\sqrt{3}})$;

(c) $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$.

2) Let a and b be real numbers. Show that the complex number $z = a + bi$ is algebraic (over $\mathbb{Q}$) if and only if a and b are algebraic.

3) Let r be a rational number. Show that the numbers $\sin(r\pi)$ and $\cos(r\pi)$ are algebraic (over $\mathbb{Q}$).

4) In class we explained that a real number a is called *constructible* if, given the origin $(0, 0)$ and the point $(1, 0)$, we can use a straight edge and compass to construct the point $(a, 0)$. Prove that the set of constructible real numbers is a subfield of $\mathbb{R}$.

5) Follow up on the previous problem and prove that the field K of constructible numbers is the smallest subfield of $\mathbb{R}$ with the property that if $a > 0$ and $a \in K$, then $\sqrt{a} \in K$.

6) Determine (with proof) whether or not a regular 9-gon can be constructed with straight edge and compass.

7) Prove that it is possible to construct a regular pentagon with straight edge and compass (this result is found in Euclid's *Elements*). [Hint: If $\zeta := e^{2\pi i/5}$, then $\zeta + \zeta^{-1} = 2\cos(2\pi/5)$. Use this to show that $\cos(2\pi/5)$ has degree 2 over $\mathbb{Q}$.]

8) Determine the degree and a basis of the splitting field over $\mathbb{Q}$ of the following polynomials.

(a) $x^4 - 2$ (b) $x^3 + 27$ (c) $x^5 - 1$ (d) $x^4 + x^2 - 1$.

Extra Credit Problems.

E1) A complex number z is called an *algebraic integer* if z is a root of a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients a_i, for some $n \geq 1$. Prove that if a and b are algebraic integers, then $a + b$ and ab are algebraic integers. It follows that the set $\mathcal{O}$ of all algebraic integers is a *subring* of $\mathbb{C}$.

EC2) (a) Prove that it is possible to divide a 19° angle into 19 equal parts with straight edge and compass.

(b) In part (a) you constructed a 1° angle, which can then be used to construct a 20° angle. Why doesn't this contradict the discussion in class on trisecting an angle of 60° with straight edge and compass?