Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 5 – Due October 3, 2024

Reading for this week: Sections 3.29 – 3.33 (the matrix of a linear transformation with respect to different bases), 3.108 – 3.117 (dual vector space and dual basis), and Chapter 4 (polynomials and complex numbers).

Problems

From the textbook: Section 3.C, Exercise 4, Section 3.F, Exercise 30, Section 4, Exercises 8, 11, 13, 14. In addition, do the following problems:

A1) (a) From Section 3.B, Exercise 23.

(b) Let $m < n$ and suppose that A is an $n \times m$ matrix and B is an $m \times n$ matrix. Prove that the $n \times n$ matrix AB is not invertible. [Hint: Try to use part (a).]

A2) Let V be a vector space over a field F. The *double dual space* V^{**} is defined to be the dual space of V^*; that is, $V^{**} := (V^*)^*$. Define a map $T : V \to V^{**}$ by $T(v)(f) := f(v)$ for any $v \in V$ and $f \in V^*$.

(a) Prove that T is a linear map $V \to V^{**}$.

(b) Assume that V is finite dimensional. Prove that T is an isomorphism from V to V^{**}. [Hint: Show that $\dim(V^{**}) = \dim(V)$ and prove that T is 1-1 on V.]

A3) Consider the vector space $V = M_2(\mathbb{R})$ of real 2×2 matrices.

(a) Write down an ordered basis $\mathcal{B}$ for V.

(b) Suppose $P = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ is given and define a linear transformation $T : V \to V$ by $T(X) = PXP^{-1}$ for each $X \in V$. Find the matrix $(T)_{\mathcal{B}}$ of T relative to the basis $\mathcal{B}$.

(c) Is the matrix $(T)_{\mathcal{B}}$ invertible? Why?

A4) The linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, 0)$ is represented in the standard ordered basis by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This operator satisfies $T^2 = T$. Prove that if S is a linear transformation on $\mathbb{R}^2$ such that $S^2 = S$, then $S = 0$, or $S = I$, or there is an ordered basis $\mathcal{B}$ for $\mathbb{R}^2$ such that $(S)_{\mathcal{B}} = A$ (above).

Extra Credit Problems.

EC1) Let P_n be the vector space of all polynomials of degree at most n with real coefficients. Let $x_1, \ldots, x_t$ be distinct real numbers, and write $P_n(x_1, \ldots, x_t)$ for the set of polynomials f in P_n for which we have

$$f(x_1) = \cdots = f(x_t) = 0.$$

This is a subspace of P_n (as you can easily check); compute its dimension as a function of n and t. Exhibit an explicit basis for the subspace $P_n(x_1, \ldots, x_t)$ - prove that it is a basis.

EC2) (a) In the notation of the previous problem, for which n and t can we find polynomials $g_k \in P_n$, for $1 \leq k \leq t$, such that

$$g_k(x_1) = \cdots = g_k(x_{k-1}) = 0,\ g_k(x_k) = 1,\ g_k(x_{k+1}) = \cdots = g_k(x_t) = 0\,?$$

Given numbers $\lambda_1, \ldots, \lambda_t$, say that it is required to find a polynomial $F \in P_n$ with $F(x_j) = \lambda_j$ for all $j \in [1, t]$. Suppose that you know the g_k. Write a simple formula in terms of the λ_j's and g_j's for F.

(b) Fix t and let n be the smallest integer for which the g's of part (a) exist. Are they then unique? Write down the g's explicitly - prove all your assertions. Based on this, can one find a unique F (as in part (a)) for this case?