Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 6 – Due October 17, 2024

Reading for this week: Sections 9.24-9.53, except 9.48 and 9.51.

Problems

From the textbook: Section 9.C, Exercises 5, 12.

A1) Let σ denote the 6-cycle (123456). Compute the powers

$$\sigma^2, \sigma^3, \sigma^4, \sigma^5, \sigma^6$$

and write each of them as a product of disjoint cycles in S_6.

A2) Prove that every permutation in S_n is the product of transpositions of the form $(i, i+1)$, where $1 \leq i \leq n-1$. Is this factorization unique?

A3) Suppose that V is a finite dimensional vector space and that S, T are two linear operators in $\mathrm{Hom}(V, V)$. Suppose that $ST = 0$ but $S \neq 0$ and $T \neq 0$. Prove that $\det S = \det T = 0$.

A4) The *transpose* of an $n \times n$ matrix $A = \{a_{ij}\}$ is the matrix $A' = \{a'_{ij}\}$ with $a'_{ij} := a_{ji}$ for each i, j with $1 \leq i, j \leq n$. Using the definition of the determinant given in class, prove that $\det(A) = \det(A')$.

A5) (a) Let a, b, c be real numbers. Compute the 3×3 determinant

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

and factor the resulting polynomial in a, b, c.

(b) Use induction on n to generalize the result of part (a) to $n \times n$ matrices, where n is any fixed positive integer.

A6) Let F be any field, A be a 2×3 matrix with entries in F, and (D_1, D_2, D_3) be the vector in F^3 defined by

$$D_1 = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Prove that

(a) The rank of A is two if and only if $(D_1, D_2, D_3) \neq (0, 0, 0)$;

(b) If A has rank two, then (D_1, D_2, D_3) is a basis for the solution space of the system of equations $AX = 0$.

Extra Credit Problems.

EC1) Let V be an n-dimensional real vector space. An alternating multilinear map $V^p \to \mathbb{R}$ is called an *alternating p-linear form* on V. If f is an alternating p-linear form on V and g an alternating q-linear form on V, define a map $\phi : V^{p+q} \to \mathbb{R}$ as follows:

$$\phi(x_1, \ldots, x_{p+q}) :=$$
$$\sum_{\substack{\sigma(1)<\cdots<\sigma(p)\\ \sigma(p+1)<\cdots<\sigma(p+q)}} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) g(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)})$$

where the sum is over all permutations $\sigma \in S_{p+q}$ of the numbers $1, \ldots, p+q$ which preserve the order of the first p and the last q numbers.

(a) Show that ϕ is an alternating $(p+q)$-linear form on V. It is called the *exterior product* of the forms f and g, and is denoted by $f \wedge g$.

(b) (Associativity of the exterior product) Show that if f, g and h are three alternating multilinear forms on V, then

$$f \wedge (g \wedge h) = (f \wedge g) \wedge h.$$

(c) If $u_1, \ldots, u_p$ are linear functionals in V^* then define a p-linear form (also called the exterior product) $u_1 \wedge \cdots \wedge u_p$ by

$$(u_1 \wedge \cdots \wedge u_p)(x_1, \ldots, x_p) = \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) u_1(x_{\sigma(1)}) \cdots u_p(x_{\sigma(p)}).$$

Suppose that $v_1, \ldots, v_q$ are also linear functionals on V. In the formula defining $\phi = f \wedge g$, take

$$f = u_1 \wedge \cdots \wedge u_p, \qquad g = v_1 \wedge \cdots \wedge v_q.$$

Show that

$$\phi = u_1 \wedge \cdots \wedge u_p \wedge v_1 \wedge \cdots \wedge v_q.$$

EC2) This problem gives Laplace's general rule for expanding a determinant. Let $A = \{a_{ij}\}_{1 \le i,j \le n}$ be an $n \times n$ real matrix. For each subset I of the set $\{1, \ldots, n\}$ with cardinality $|I| = p$, let A_I denote the matrix formed by the a_{ij} for which $i \in I$ and $1 \le j \le p$, and let A_I^c denote the 'complementary' matrix, formed by the a_{ij} for which $i \notin I$

and $p+1 \leq j \leq n$. Finally, let $n(I)$ denote the number of ordered pairs (i, j) such that $i \in I$, $j \notin I$ and $i > j$. Prove *Laplace's formula*

$$\det(A) = \sum_{|I|=p} (-1)^{n(I)} \det(A_I) \det(A_I^c)$$

the sum being over all *subsets* $I \subset \{1, \ldots, n\}$ such that $|I| = p$. [Hint: Use the results of problem (EC1)]

EC3) Let V be an n-dimensional real vector space. Denote by $L^p(V)$ the vector space of all alternating p-linear forms on V. In class we proved that $\dim L^n(V) = 1$. Prove that

$$\dim L^p(V) = \binom{n}{p}$$

for all $p \geq 0$, where $\binom{n}{p} = \dfrac{n!}{p!(n-p)!}$ is a binomial coefficient.