Math 620 – Fall 2024 – Harry Tamvakis

PROBLEM SET 6 – Due December 3, 2024

1) Let K be a complete valuated field, and L be an algebraic and separable extension of K. This problem shows that if L is not a finite extension of K, then L, with the (unique) extension of the valuation on K, is not *complete*.

(a) Let $\overline{K}$ be an algebraic closure of K with $L \subset \overline{K}$. Let $\alpha \in \overline{K}$ be separable over K and r be the minimal distance from α to its conjugates over K. Show that if the distance from α to the subspace L is strictly less than r, then $\alpha \in L$.

(b) Suppose L is an infinite separable extension of K. Choose a sequence of elements $\{\alpha_n\}$ of $L \smallsetminus K$ which are not all contained in some finite extension of K. Let r_n be the minimum distance from α_n to its conjugates over K. Choose $b \in K$ with $0 < |b| < 1$ and sequences of positive integers $\{p_n\}$ and $\{q_n\}$ such that

$$|b|^{p_n} < \frac{\min\{r_n, 1\}}{2} \qquad \text{and} \qquad |\alpha_{n+1}| \cdot |b|^{q_n} < 1.$$

Set $\lambda_n := b^{p_1+\cdots+p_n+q_1+\cdots+q_n}$. Prove that the infinite series

$$\alpha_1 + \lambda_1\alpha_2 + \cdots + \lambda_n\alpha_{n+1} + \cdots \tag{1}$$

converges to a limit s in the completion of L.

(c) If s_n is the sum of the first n terms of the series (1), show that

$$|s - s_n| < |\lambda_{n-1}|\, r_n.$$

(d) Assume s is algebraic over K. Use parts (a), (c) and induction to prove that $\alpha_n \in K(s)$, for all n, and thus arrive at a contradiction. [Hint: Consider the distance from α_n to $(s - s_{n-1})/\lambda_{n-1}$].

2) Consider the Möbius function $\mu : \mathbb{N} \to \mathbb{Z}$ defined by $\mu(1) = 1$, $\mu(n) = 0$ if n is not square free, and $\mu(p_1 \cdots p_r) = (-1)^r$ if $p_1, \ldots, p_r$ are distinct primes. Let $a : \mathbb{N} \to \mathbb{C}$ be a *completely multiplicative* function, which means that $a(1) = 1$ and $a(mn) = a(m)a(n)$ for any positive integers m and n.

Assume that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ converges absolutely for $\mathrm{Re}(s) > \sigma_0 > 0$. Prove that $g(s) = \sum_{n=1}^{\infty} \frac{\mu(n)a(n)}{n^s}$ also converges absolutely for $\mathrm{Re}(s) > \sigma_0$ and that $f(s)g(s) = 1$.

3) Compute the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ when K is (i) $\mathbb{Q}(\sqrt{-1})$, (ii) $\mathbb{Q}(\sqrt{-2})$, (iii) $\mathbb{Q}(\sqrt{-3})$, (iv) $\mathbb{Q}(\sqrt{2})$, (v) $\mathbb{Q}(\sqrt{5})$.

4) Show that the Dedekind zeta function of $\mathbb{Q}(i)$ satisfies

$$\zeta_{\mathbb{Q}(i)}(s)/\zeta_{\mathbb{Q}}(s) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{(-1)^{(n-1)/2}}{n^s}.$$

5) Let $\mathbb{F}_q$ be a finite field with q elements and consider the zeta function of the ring $\mathbb{F}_q[t]$:

$$\zeta_{\mathbb{F}_q[t]}(s) := \sum_{I \neq 0} \frac{1}{N(I)^s}$$

where the sum runs over all non-zero ideals I and $N(I) = [\mathbb{F}_q[t] : I]$. Prove that

$$\zeta_{\mathbb{F}_q[t]}(s) = \frac{1}{1 - q^{1-s}}.$$