Math 404 – Spring 2025 – Harry Tamvakis
PROBLEM SET 7 – Due April 3, 2025

Reading for this week: Section 7.2, except Theorem 7.12. The statement of the Fundamental Theorem 7.34. We gave some first examples in the class lectures; read Section 7.7 for a more involved one.

Problems

1) Let $q = p^n$ be a prime power and $\mathbb{F}_q$ denote the field with q elements. Prove that the algebraic closure of $\mathbb{F}_q$ is the union of the finite fields $\mathbb{F}_{q^k}$ over all $k \geq 1$.

2) Let $K \supset F$ be a finite field extension. Prove that the Galois group $G(K/F)$ is a finite group (for this problem, you may only use the results that we have proved thus far in class/lecture).

3) Let K be the splitting field of the polynomial (a) $x^6 - 27$; (b) $x^6 + 27$ over $\mathbb{Q}$. In both cases, compute $[K : \mathbb{Q}]$ and determine the Galois group of K over $\mathbb{Q}$ up to isomorphism.

4) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible cubic polynomial with exactly one real root, and let K be the splitting field of f over $\mathbb{Q}$. Prove that $[K : \mathbb{Q}] = 6$.

5) Let $f(x)$ be an irreducible cubic polynomial in $\mathbb{Q}[x]$ with exactly one real root α. Prove or disprove: the other two roots of f form a complex conjugate pair β, $\overline{\beta}$, and the field $\mathbb{Q}(\beta)$ has an automorphism σ which interchanges β with $\overline{\beta}$.

6) Find all subgroups of the Galois group $G(K/\mathbb{Q})$ of the splitting field K of the polynomial $f(x)$ over $\mathbb{Q}$ as well as all corresponding subfields E between $\mathbb{Q}$ and K when

(a) $f(x) = (x^2 - 5)(x^2 - 7)$; (b) $f(x) = x^4 + 1$.

7) Let K be the splitting field of the polynomial $x^5 - 1$ over $\mathbb{Q}$.

(a) Show that the Galois group $G(K/\mathbb{Q})$ is a cyclic group of order 4.

(b) Determine all fields E which are contained in K.

8) Let $\mathbb{C}(x)$ be the field of rational functions with complex coefficients (the fraction field of $\mathbb{C}[x]$, where x is an indeterminate). For each of the following four sets S of automorphisms of the field $\mathbb{C}(x)$, determine the group $H = \langle S \rangle$ of automorphisms which is generated by S, and compute the fixed field $\mathbb{C}(x)^H$ explicitly.

(a) $\sigma(x) = x^{-1}$ (b) $\sigma(x) = ix$ (c) $\sigma(x) = -x,\ \tau(x) = x^{-1}$

(d) $\sigma(x) = ix,\ \tau(x) = x^{-1}$.

Extra Credit Problems.

EC1) Let F be a finite field, and let $f(x)$ be a non-constant polynomial whose derivative is the zero polynomial. Prove that f is not irreducible over F. [Hint: First show that for any $a \in F$, the equation $x^p = a$ has a solution in F, where p denotes the characteristic of F.]

EC2) (a) If F is a field of characteristic $p > 0$ and if L is an extension of F, let

$$K = \{a \in L \mid a^{p^n} \in F \text{ for some } n\}.$$

Prove that K is a subfield of L.

(b) With F, K, and L as above, prove that any automorphism of L which leaves every element of F fixed also leaves every element of K fixed.