Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 7 – Due October 24, 2024

Reading for this week: Sections 5.1-5.12, 5.19-5.20, 5.35-5.42, 5.47-5.59, 8.47-8.54, 9.48, 9.51, 9.63, 9.65.

Problems

From the textbook: Section 5.A, Exercises 4, 22, 26, 27, Section 5.C, Exercise 6, Section 5.D, Exercise 21.

A1) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}.$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$ consisting of eigenvectors for T.

A2) Determine the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}$$

and conclude that every monic polynomial of positive degree is the characteristic polynomial of some linear transformation.

A3) True or false? If a triangular matrix A is similar to a diagonal matrix, then A is already diagonal. If the statement is true, prove it. If it is false, then provide an example and justify your answer.

A4) We are given two jars, the first containing 1 liter of liquid A and the second 1 liter of liquid B. Also provided is a cup which has a capacity of k liters, where $0 < k < 1$. We first fill the cup from the first jar and transfer the content to the second jar, stirring thoroughly afterwards. Next we dip the cup in the second jar and transfer k liters of liquid back to the first jar. This operation is repeated again and again. Prove that as the number of iterations n of the operation tends to infinity, the concentrations of A and B in both jars tend to equal each other. [I do not expect any physical or chemical arguments in this problem; rather, rephrase it in mathematical terms and proceed from there].

Extra Credit Problem.

EC1) Determine all matrices $A \in M_n(\mathbb{R})$ such that $AB = BA$ for any matrix B in $M_n(\mathbb{R})$. Justify your answer.

EC2) Suppose that $f : M_n(\mathbb{R}) \to \mathbb{R}$ is a linear functional on $M_n(\mathbb{R})$ such that $f(AB) = f(BA)$ for any two matrices A and B in $M_n(\mathbb{R})$. Prove that there exists a $c \in \mathbb{R}$ such that $f = c\,\mathrm{Tr}$, that is, f is a scalar multiple of the trace function. If, in addition, we have $f(I_n) = n$, then $f = \mathrm{Tr}$.