Math 404 – Spring 2025 – Harry Tamvakis

PROBLEM SET 8 – Due April 10, 2025

Reading for this week: Sections 7.1, 7.2, 7.4, 7.7, Theorem 10.4, Example 10.5, and Theorems 10.8 and 10.10 (as discussed in class).

Problems

1) (a) Let p be a prime number and let S_p be the symmetric group of $p!$ permutations of $\{1, 2, \ldots, p\}$. Prove that any transposition (ij) and p-cycle $(a_1 a_2 \ldots a_p)$ together generate S_p.

(b) Find an integer $n \geq 2$ and a transposition and an n-cycle in S_n that together do not generate S_n. This shows that the assumption that p is prime in part (a) is necessary.

2) Construct a polynomial of degree 7 in $\mathbb{Q}[x]$ whose Galois group over $\mathbb{Q}$ is the symmetric group S_7.

3) Let σ be an automorphism of a field K. Suppose that $\sigma^4 = 1$ and

$$\sigma(\alpha) + \sigma^3(\alpha) = \alpha + \sigma^2(\alpha)$$

for all $\alpha \in K$. Prove that $\sigma^2 = 1$.

4) Let $K \supset F$ be a finite extension two fields. We defined $G(K/F)$ to be the group of all F-automorphisms of the field K. Prove that the order of $G(K/F)$ divides the degree $[K : F]$.

5) Let p be a prime number and $\zeta := e^{2\pi i/p}$ be a primitive p-th root of unity in $\mathbb{C}$. Prove that the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ is a cyclic group of order $p - 1$.

6) Let F be any field, let $x_1, \ldots, x_n$ be independent variables, and set $K = F(x_1, \ldots, x_n)$. Let S be the subfield of K consisting of symmetric rational functions in $x_1, \ldots, x_n$, that is, the fixed field for the natural action of the symmetric group S_n on K. Suppose that E is a field with $S \subset E \subset K$ and $[E : S] = 2$. Prove that $E = S(\alpha)$ where

$$\alpha(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

7) Suppose $x_1, \ldots, x_n$ are n independent variables. For each k with $1 \le k \le n$ let

$$e_k = e_k(x_1, \ldots, x_n) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

be the k-th elementary symmetric polynomial, and define the k-th power sum polynomial by

$$p_k = p_k(x_1, \ldots, x_n) := x_1^k + x_2^k + \cdots + x_n^k.$$

Set $e_0 := 1$, let t be a formal variable, and consider the *generating function* $E(t)$ for the elementary symmetric polynomials, defined by

$$E(t) := \sum_{k=0}^{n} e_k t^k = 1 + e_1 t + e_2 t^2 + \cdots + e_n t^n.$$

(a) Show that $E(t) = \prod_{i=1}^{n} (1 + x_i t)$.

(b) Prove that we have

$$\text{(1)} \qquad \frac{E'(t)}{E(t)} = \sum_{i=1}^{n} \frac{x_i}{1 + x_i t} = \sum_{i=1}^{n} \sum_{k=0}^{\infty} (-1)^k x_i^{k+1} t^k = \sum_{k=0}^{\infty} (-1)^k p_{k+1} t^k$$

in the ring of formal power series $\mathbb{Q}(x_1, \ldots, x_n)[[t]]$.

(c) Prove *Newton's identities:*

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^{k-1} p_1 e_{k-1} + (-1)^k k e_k = 0,$$

for $1 \le k \le n$. [Hint: Multiply the identity (1) by $E(t)$ and equate the like powers of t on both sides.]

8) In the situation of Problem 7, show that p_k is equal to the determinant of the $k \times k$ matrix

$$\begin{pmatrix} e_1 & 2e_2 & 3e_3 & \cdots & & ke_k \\ 1 & e_1 & e_2 & \cdots & & e_{k-1} \\ 0 & 1 & e_1 & \cdots & & e_{k-2} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & e_1 \end{pmatrix}.$$