Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 8 – Due October 31, 2024

Reading for this week: Sections 6.1-6.36, 6.39-6.45, 6.46-6.56, 7.44-7.53.

Problems

From the textbook: Section 6.A, Exercises 25, 26, 27.

A1) Let P_3 be the vector space of polynomials with real coefficients of degree at most 3, equipped with the inner product

$$\langle f, g\rangle := \int_0^1 f(x)g(x)\,dx.$$

(a) Find the orthogonal complement of the subspace of scalar polynomials.

(b) Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$.

A2) Let V be a finite dimensional real inner product space, W a subspace of V, and v an arbitrary vector in V. A *best approximation* to v by vectors in W is a vector $v' \in W$ such that $\|v - v'\| \leq \|v - w\|$ for every $w \in W$. Prove that there is a *unique* best approximation v' to v by vectors in W and in fact $v' = \mathrm{proj}_W v$ is the orthogonal projection of v onto W.

A3) The distance $\|v - v'\|$ between v and v' in Problem (A2) is the distance from v to the subspace W. Note that $v - v'$ is the component of v in $W^{\perp}$. Use this fact to prove that the distance d from a point $(x_0, y_0, z_0) \in \mathbb{R}^3$ to the plane with equation $ax + by + cz = 0$ is given by

$$d = \frac{|ax_0 + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}}.$$

A4) Let $V = M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices over $\mathbb{R}$, with the inner product $\langle A, B\rangle := \mathrm{Tr}(AB^t)$. Find the orthogonal complement in V of the subspace of all diagonal matrices.

A5) Find a 3×3 orthogonal matrix whose first row is a multiple of the vector $(1, 1, 1)$.

A6) Prove that if an upper triangular matrix A is orthogonal, then A is a diagonal matrix.

A7) Equip $M_n(\mathbb{C})$ with the inner product $\langle A, B\rangle := \mathrm{Tr}(AB^*)$. For any matrix M in $M_n(\mathbb{C})$, let $T_M : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the linear operator on $M_n(\mathbb{C})$ defined by $T_M(A) = MA$. Prove that T_M is unitary if and only if M is a unitary matrix.

Extra Credit Problems.

EC1) Suppose A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. In case the equation $Ax = b$ has no solution, it is often desirable to find the next best thing, an $\widehat{x}$ that makes $A\widehat{x}$ as close as possible to b. Call $\widehat{x}$ a *nearest solution* if

$$\|A\widehat{x} - b\| \leq \|Ax - b\|$$

for all x in $\mathbb{R}^n$ (where the norm is the usual one coming from the dot product). Prove that $\widehat{x}$ is a nearest solution of $Ax = b$ if and only if $A^t A\widehat{x} = A^t b$.

EC2) Suppose that A and b are as in Problem (EC1). Show that $A^t A$ is invertible if and only if A has linearly independent columns. In this case there is a unique nearest solution of $Ax = b$, namely $\widehat{x} = (A^t A)^{-1} A^t b$.

EC3) Suppose that we are given points $(x_1, y_1), \ldots, (x_n, y_n)$ in the Euclidean plane $\mathbb{R}^2$. A line $y = \alpha x + \beta$ passes through each point exactly when the system

$$\alpha x_i + \beta = y_i, \qquad 1 \leq i \leq n$$

has a solution in α and β. If there are more than two points then this system may not have a solution. What condition on the points guarantees that there is a unique nearest solution $(\widehat{\alpha}, \widehat{\beta})$ for the system? In the latter case derive formulas for $\widehat{\alpha}$ and $\widehat{\beta}$ in terms of the x_i's and y_i's.