Math 405 – Fall 2024 – Harry Tamvakis
PROBLEM SET 9 – Due November 7, 2024

Reading for this week: Sections 7.1-7.5, 7.7-7.12, 7.18-7.20, 7.22-7.25, 7.29-7.33.

Problems

From the textbook: Section 7.A, Exercises 1, 6, Section 7.B, 6, 11, 14.

A1) Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map defined by $T(e_1) = (1 + i, 2)$ and $T(e_2) = (i, i)$, where $E = (e_1, e_2)$ is the standard ordered basis of $\mathbb{C}^2$. Determine the matrix of T^* in the standard basis E. Is T a normal operator?

A2) Let V be a finite-dimensional complex inner product space and $T : V \to V$ a linear operator on V.

(a) Prove that the image of T^* is the orthogonal complement of the kernel of T.

(b) If T is invertible, prove that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

A3) Let $A \in M_n(\mathbb{C})$ be a skew-hermitian matrix, that is, a matrix such that $A^* = -A$.

(a) If n is even, prove that $\det(A) \in \mathbb{R}$, while if n is odd, prove that $\det(A) \in i\mathbb{R}$.

(b) Prove that A is diagonalizable.

(c) Prove that any matrix in $M_n(\mathbb{C})$ is the sum of two diagonalizable matrices.

A4) Let V be a finite dimensional inner product space and let W be a subspace of V. Then $V = W \oplus W^\perp$, that is, every vector v in V can be written uniquely as $v = v_1 + v_2$, with $v_1 \in W$ and $v_2 \in W^\perp$. Define a linear operator $U : V \to V$ by $U(v) = v_1 - v_2$.

(a) Prove that U is both self-adjoint and unitary.

(b) If $V = \mathbb{R}^3$ with the standard inner product and W is the subspace spanned by the vector $(1, 0, 1)$, find the matrix of U in the standard ordered basis.

A5) The function $z \mapsto \dfrac{z+1}{z-1}$ maps the imaginary axis $i\mathbb{R}$ in the complex plane once around the unit circle, missing the point 1. The inverse function is given by the same formula. Prove each of the following statements, which are the analogues of these facts for matrices $A \in M_n(\mathbb{C})$.

(a) If A is skew-hermitian, then $A - I$ is invertible.

(b) If $U := (A+I)(A-I)^{-1}$, then U is unitary. [Hint: Show that $\|(A+I)x\|^2 = \|(A-I)x\|^2$ for every x.]

(c) The matrix $U - I$ is invertible.

(d) If U is any unitary matrix such that $U - I$ is invertible, and if $A := (U+I)(U-I)^{-1}$, then A is skew-hermitian.

<u>**Extra Credit Problem**</u>

EC) Let V be a finite dimensional complex inner product space. In Problem (A4), we constructed some linear operators on V which are both self-adjoint and unitary. Prove that there are no others, i.e., that every self-adjoint and unitary operator on V arises from some subspace W as described in Problem (A4).