Countable Markov partitions suitable for thermodynamic formalism

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To the memory of Roy Adler

Abstract

We study hyperbolic attractors of some dynamical systems with apriori given countable Markov partitions. Assuming that contraction is stronger than expansion we construct new Markov rectangles such that their crossections by unstable manifolds are Cantor sets of positive Lebesgue measure. Using new Markov partitions we develop thermodynamical formalism and prove exponential decay of correlations and related properties for certain Hölder functions. The results are based on the methods developed by Sarig [26] - [28].

1 Introduction

Examples of one-dimensional maps with countable Markov partitions go back to the Gauss transformation, and further developments appeared in particular in [24], [3], [31]. Beginning in [4], theorems about ergodic properties of such maps are often referred to as *Folklore*.

More recently an interest in such maps was motivated by works on ergodic and statistical properties of quadratic-like and Hénon-like maps. The study of such maps

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is typically based on various tower constructions, see in particular [16], [17], [7], [8], [32]. Power maps defined on the tower satisfy hyperbolicity and distortion estimates.

Although the following remark is not directly related to the results of our paper, it might be useful for the further work in the area under discussion.

Remark 1.1 *Since numerical evidence for the existence of a strange attractor was presented in the original Hénon paper [13], rigorous results were only obtained in unspecified small neighborhoods of one-dimensional maps.*
One possible approach to that problem is to prove that in a sufficiently small neighborhood of the classical Hénon values, $a_H = 1.4, b_H = .3$*, there is a positive Lebesgue measure set M, such that for* $(a,b) \in M$*, Hénon maps* $f_{a,b}$ *have SRB measures with strong mixing properties.*
The main difficulty in that direction is to design a set of checkable numeric estimates which can be maintained through the induction. In the one-dimensional case such estimates were used in [22], [15], [11].

We study apriori given two-dimensional systems with countable Markov partitions satisfying hyperbolicity and distortion conditions. In [18] we proved strong mixing properties of such systems assuming distortion condition D2, requiring boundedness of the quotient of the second derivatives over the first derivative. This condition is too strong for our purposes since it does not hold for power maps induced by quadratic and Hénon maps.

Here we return to the more general setting of [20], [21] where only boundedness of the quotient of the second derivatives over the square of first derivatives is assumed. In order to study the decay of correlations we require additionally that contraction in our models grows faster than expansion, see condition H5 below. Condition H5 naturally holds for power maps generated by Hénon-like maps with Jacobian less than 1. We also require distortions of our initial maps to be uniformly bounded, see condition BIV below. That is a standard requirement for maps defined on a tower.
The main new idea of the paper is to develop thermodynamic formalism by using special Markov rectangles such that their intersections with unstable manifolds are Cantor sets of positive Lebesgue measure.

2 Description of models and statement of results.

1. **Description of the model.**

   The setting for our model is the same as in [18] - [21]. To summarize, let Q be the unit square. Let $\xi = \{E_1, E_2, \ldots\}$ be a countable collection of full-height, closed, curvilinear rectangles in Q. Hyperbolicity conditions that we will recall below imply that the left and right boundaries of E_i are graphs of smooth functions $x^{(i)}(y)$ with $\left|\frac{dx^{(i)}}{dy}\right| \leq \alpha$ for $0 < \alpha < 1$.

   Assume that each E_i lies inside a domain of definition of a C^2 diffeomorphism F which maps E_i onto its image $S_i \subset Q$. The images $F|_{E_i}(E_i) = f_i(E_i) = S_i$ are disjoint, full-width strips of Q which are bounded from above and below by the graphs of smooth functions $y^i(x), |\frac{dy^{(i)}}{dx}| \leq \alpha$.

   We recall geometric and hyperbolicity conditions from [21].

2. **Geometric conditions.**

   For $z \in Q$, let ℓ_z be the horizontal line through z. We define $\delta_z(E_i) = diam(\ell_z \bigcap E_i)$, $\delta_{i,max} = \max_{z \in Q} \delta_z(E_i)$, $\delta_{i,min} = \min_{z \in Q} \delta_z(E_i)$.

   G1. For $i \neq j$ holds $int\ E_i \cap int\ E_j = \emptyset$, $int\ S_i \cap int\ S_j = \emptyset$.

   G2. $mes(Q \setminus \cup_i\ int\, E_i) = 0$ where mes stands for Lebesgue measure.

   G3. for some $0 < a \leq b < 1$ and some $\tilde{C} \geq 1$ it holds that

   $$\tilde{C}^{-1} a^i \leq \delta_{i,min} \leq \delta_{i,max} \leq \tilde{C} b^i.$$

3. **Hyperbolicity conditions.**

   Let $J_F(z)$ be the absolute value of the Jacobian determinant of F at z.

   There exist constants $0 < \alpha < 1$ and $K_0 > 1$ such that for each i the map

   $$F(z) = f_i(z) \text{ for } z \in E_i$$

   satisfies

H1. $| F_{2x}(z) | + \alpha | F_{2y}(z) | + \alpha^2 | F_{1y}(z) | \leq \alpha | F_{1x}(z) |$

H2. $| F_{1x}(z) | - \alpha | F_{1y}(z) | \geq K_0$.

H3. $| F_{1y}(z) | + \alpha | F_{2y}(z) | + \alpha^2 | F_{2x}(z) | \leq \alpha | F_{1x}(z) |$

H4. $| F_{1x}(z) | - \alpha | F_{2x}(z) | \geq J_F(z) K_0$.

We recall some notation.

Given a finite string $i_0 \ldots i_{n-1}$, $n \geq 1$, we define inductively

$$E_{i_0 \ldots i_{n-1}} = E_{i_0} \bigcap f_{i_0}^{-1} E_{i_1 i_2 \ldots i_{n-1}} \tag{1}$$

Then, each set $E_{i_0 \ldots i_{n-1}}$ is a full height subrectangle of E_{i_0}.

Analogously, for a string $i_{-m} \ldots i_{-1}$ we define

$$S_{i_{-m} \ldots i_{-1}} = f_{i_{-1}} (S_{i_{-m} \ldots i_{-2}} \bigcap E_{i_{-1}})$$

and get that $S_{i_{-m} \ldots i_{-1}}$ is a full width strip in Q. It is easy to see that $S_{i_{-m} \ldots i_{-1}} = f_{i_{-1}} \circ f_{i_{-2}} \circ \ldots \circ f_{i_{-m}} (E_{i_{-m} \ldots i_{-1}})$ and that $f_{i_0}^{-1} (S_{i_{-m} \ldots i_{-1} i_0})$ is a full-width substrip of E_{i_0}.

We also define curvilinear rectangles $R_{i_{-m} \ldots i_{-1}, i_0 \ldots i_{n-1}}$ by

$$R_{i_{-m} \ldots i_{-1}, i_0 \ldots i_{n-1}} = S_{i_{-m} \ldots i_{-1}} \bigcap E_{i_0 \ldots i_{n-1}} \tag{2}$$

If there are no negative indices then respective rectangle is full height in Q.

The following is a well known fact in hyperbolic theory, see [21].

Proposition 2.1 *Any C^1 map F satisfying the above geometric conditions G1–G3 and hyperbolicity conditions H1–H4 has a "topological attractor"*

$$\Lambda = \bigcup_{\ldots i_{-n} \ldots i_{-1}} \bigcap_{k \geq 1} S_{i_{-k} \ldots i_{-1}}.$$

The infinite intersections

$$\bigcap_{k=1}^{\infty} S_{i_{-k} \ldots i_{-1}}$$

define C^1 curves $y(x)$, $|dy/dx| \leq \alpha$, which are the unstable manifolds for the points of the attractor.

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$$\bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} R_{i_{-m} \ldots i_{-1}, i_0 \ldots i_{n-1}}$$

define points of the attractor.

4. **Distortion condition.**

We formulate certain assumptions on the second derivatives. We use the distance function $d((x,y),(x_1,y_1)) = \max(| x - x_1 |, | y - y_1 |)$ associated with the norm $| v | = \max(| v_1 |, | v_2 |)$ on vectors $v = (v_1, v_2)$.

Our first condition is the same as in [21]; we recall it below.

Let $f_i(x,y) = (f_{i1}(x,y), f_{i2}(x,y))$. We use $f_{ijx}, f_{ijy}, f_{ijxx}, f_{ijxy}$, etc. to denote partial derivatives of f_{ij}, $j = 1,2$.

We define

$$| D^2 f_i(z) | = \max_{j=1,2,(k,l)=(x,x),(x,y),(y,y)} | f_{ijkl}(z) |.$$

We assume that there exists a constant $C_0 > 0$ such that the following *Distortion Condition* holds:

D1. $\sup_{z \in E_i, i \geq 1} \dfrac{| D^2 f_i(z) |}{| f_{i1x}(z) |} \delta_z(E_i) < C_0.$

5. **A result about systems satisfying geometric, hyperbolicity and distortion conditions:**

Our conditions imply the following Theorem proved in [20] and [21].

Theorem 2.2 *Let F be a piecewise smooth mapping as above satisfying the geometric conditions G1–G3, the hyperbolicity conditions H1–H4, and the distortion condition D1.*

Then, F has an SRB measure μ supported on Λ whose basin has full Lebesgue measure in Q. The dynamical system (F,μ) satisfies the following properties.

(a) *(F,μ) is measure-theoretically isomorphic to a Bernoulli shift.*

(b) *F has finite entropy with respect to the measure μ, and the entropy formula holds*

$$h_\mu(F) = \int log|D^u F|d\mu \tag{3}$$

where $D^u F(z)$ is the norm of the derivative of F in the unstable direction at z.

(c)

$$h_\mu(F) = \lim_{n\to\infty} \frac{1}{n} \log|\, DF^n(z)\,| \tag{4}$$

where the latter limit exists for Lebesgue almost all z and is independent of such z.

6. **Additional distortion and hyperbolicity conditions and statement of the main theorems.**

(a) Properties of the function $\phi(z) = -\log(D^u F(z))$ are important when applying thermodynamic formalism to hyperbolic attractors. We consider systems satisfying conditions of Theorem 2.2 and some extra hyperbolicity conditions, which can be used for power maps arising from Henon-type diffeomorphisms.

We explore a general principle that can be stated as : *Contraction increases faster than expansion* - see hyperbolicity condition H5 below. For such systems we construct new Markov partitions such that the pullback of $\phi(z)$ into a respective symbolic space is a locally Hölder function.

New Markov rectangles are Cantor sets, such that their one-dimensional crossections by $W^u(z)$ have positive Lebesgue measure.

(b) We consider a class of system which satisfy conditions of the Theorem 2.2 as well as the following additional assumptions.

   i. **Bounded Initial Variation**.

   BIV. There exists $B_0 > 0$ such that for all i and all

$$\{z_1 = (x_1, y_1), z_2 = (x_2, y_2)\} \in E_i$$

   holds

$$|\log f_{i1x}(z_1) - \log f_{i1x}(z_2)| < B_0. \tag{5}$$

   BIV does not allow unbounded oscillations of widths for initial rectangles.

   ii. **Contraction grows faster than expansion**.
   We assume that there is a constant a_1 satisfying

$$0 < a_1 < a \tag{6}$$

   where a is from G3, such that for each j, for each $z \in E_j$, and for any vector v in the stable cone $K^s_\alpha(z)$ holds

   H5. $|Df_j^{-1} v| \geq a_1^{-j} |v|$.

   Condition H5 means that up to a uniform factor, contractions of f_j grow faster than expansions. In particular it implies that up to a uniform factor, heights of $S_i \cap E_k$ are smaller than widths of E_k for all $k \leq i$.

Remark 2.3 *Uniform hyperbolicity and distortion conditions were used in [20], [21] to extend the classical approach of [6] and to study ergodic properties of systems with countable Markov partitions. By combining D1 and H5 we can add methods of thermodynamic formalism.*

(c) Let $\mathscr{H}_\gamma$ be the space of functions on Q satisfying Hölder property with exponent γ

$$| \phi(x) - \phi(y) | \leq c| x - y |^\gamma.$$

We state our main Theorems.

Theorem 2.4 *(***Exponential Decay of Correlations***)*
Let F be a piecewise smooth mapping as above, satisfying geometric conditions G1– G3, hyperbolicity conditions H1–H5, distortion condition D1, and the BIV condition. Then the system (F, μ) has exponential decay of correlations for $f \in \mathscr{H}_\gamma$ and $g \in L^\infty(\mu)$. Namely there exists $0 < \eta < 1$, $\eta = \eta(\gamma)$, such that

$$\left| \int f\,(g \circ F^n)\, d\mu - \int f\, d\mu \int g\, d\mu \right| < C(f, g)\eta^n. \tag{7}$$

Theorem 2.5 *(***Central Limit Theorem***)*
Let (F, μ) satisfy the assumptions of Theorem 2.4 and suppose that $f \in \mathscr{H}_\gamma$. If $\int f d\mu = 0$ and f cannot be expressed as $g - g \circ F$ for g continuous, then there is a positive constant $\sigma = \sigma(f)$, such that for every $t \in \mathbf{R}$,

$$\lim_{n\to\infty} \mu \left\{ x : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ F^k(x) < t \right\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-t}^{\infty} e^{\frac{-s^2}{2\sigma^2}}\, ds.$$

3 Hölder properties of $\log(D^u F(z))$ and Markov partitions in the phase space.

1. The key step toward the proof of Theorem 2.4 is to establish that the pullback of the function $\log D^u F$ into the respective symbolic space is Hölder continuous. Then one can follow the Ruelle-Bowen approach ([25], [9]), in particular using results of Sarig [26] - [28] to develop thermodynamic formalism for the systems under consideration. Hölder properties of the

pullback of $\log D^u F$ into symbolic space follow from Hölder properties of $\log D^u F$ in the phase space.
In order to get an appropriate symbolic space, we construct a partition of a subset of positive measure $\mathscr{C} \subset \Lambda$, such that the first return map to $\mathscr{C}$ is Markov. Elements of the Markov partition of Λ are elements of the Markov partition of $\mathscr{C}$ and their orbits before the first return.
Elements C_i of the Markov partition $\mathscr{C}$ have the following property. For $z \in C_i$ the crossection $C_i^u(z)$ of C_i by $W^u(z)$ is a Cantor set of positive linear Lebesgue measure.

2. Cantor sets that we construct are inscribed in curvilinear rectangles $R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}$. Recall that $R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}$ are bounded from above and below by arcs of unstable curves Γ^u, which are images of some pieces of the top and bottom of Q, and from the left and right by arcs of stable curves Γ^s which are preimages of some pieces of the left and right boundaries of Q .

   For $x \in R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}} \cap \Lambda$ let

$$\Gamma^s(x, R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}) = W^s(x) \cap R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}$$

$$\Gamma^u(x, R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}) = W^u(x) \cap R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}.$$

   We define the height of $R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}$ as

$$H(R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}) = sup_x \left|\Gamma^s(x, R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}})\right|$$

   The width $W(R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}})$ is defined similarly.

   Hyperbolicity conditions imply that stable boundaries of rectangles belong to stable cones. Since standard horizontal lines belong to unstable cones, and stable and unstable cones are separated, we get the following.

   For every ε_1 there is an ε_0 such that if

$$H(R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}) < \varepsilon_0 w_{min}(R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}}) \tag{8}$$

   then for all $m \geq 1$, $n \geq 1$, the ratio of lengths of any two unstable curves $\Gamma^u(x, R_{i_{-m}\ldots i_{-1},i_0\ldots i_{n-1}})$ is bounded by $1 \pm \varepsilon_1$. Similarly it follows from hyperbolicity conditions, that if l is the length of a standard horizontal crossection of $E_{i_0\dots i_{n-1}}$ through a point $x \in R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$, then for some c_0,

$$\mid \Gamma^u(x, R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}) \mid < c_0 l. \tag{9}$$

3. **Admissible objects**.

We define the following strings of indices as *admissible*:

A1. A string $\bar{\alpha} = [i_1 \dots i_k]$ is admissible if for each $l = 1, 2, \dots k-1$ it holds that

$$\sum_{m=1}^{l} i_m \geq i_{l+1} \tag{10}$$

A rectangle $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}} =$ is admissible if the string $[i_{-m} \dots i_{-1} i_0 \dots i_{n-1}]$ is admissible. It follows from definition that if $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$, $m \geq 0, n \geq 1$ is admissible, then all rectangles obtained by moving the comma in the index to the left or to the right are admissible. A one-sided sequence $i_1 i_2 \dots i_n \dots$ is admissible if all strings $[i_1 \dots i_n]$ are admissible.

In particular all strings $[ij], i \geq j$ are admissible, and thus, the respective rectangles are admissible.

Note that distortion estimates may be satisfied on non-admissible rectangles if their heights are small enough, but we ignore that possibility, and organize our construction based on condition A1.

4. We estimate the variation of $\log D^u F$ on admissible two-dimensional curvilinear rectangles $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$. For any function $a(x,y)$ the variation of $a(x,y)$ over a rectangle R is defined as

$$var(a(x,y))|R = \sup_{(x_1,y_1)\in R,(x_2,y_2)\in R} \mid a(x_1,y_1) - a(x_2,y_2) \mid \tag{11}$$

The function $\log D^u F$ is **locally Hölder** on admissible rectangles $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$ if for $m \geq 0$, $n \geq 1$ the variation of $\log D^u F$ on $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$ satisfies

$$var(\log D^u F)|R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}} < C\theta_0^{min(m,n)} \tag{12}$$

for some $C > 0$, $\theta_0 < 1$.

Note that on initial rectangles E_i estimate (12) is satisfied because of BIV.

5. The proof of the following Proposition is similar to the proof of Proposition 5.1 in [18].

Proposition 3.1 *For any admissible string* $[i_{-m} \dots i_{-1} i_0 \dots i_{n-1}]$ *the variation of* $\log D^u F$ *on* $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$ *satisfies (12) with some C and* θ_0 *independent of* m, n *and determined by hyperbolicity and distortion conditions.*

Note that in [18] and [19] we proved Hölder property of $\log D^u F$ on arbitrary rectangles $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$. Here we prove it for admissible rectangles.

Proof.

(a) Admissible rectangles $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$ are bounded from above and below by some arcs of two unstable curves $\Gamma^u_{i_{-m} \dots i_{-1}}$ which are images of some pieces of the top and bottom of Q and from left and right by some arcs of two stable curves $\Gamma^s_{i_0 \dots i_{n-1}}$ which are preimages of some pieces of the left and right boundaries of Q.

Let $z_1, z_2 \in R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}} \cap \Lambda$. We consider two points z_3, z_4 such that $W^s(z_3) = W^s(z_4)$ and for which we can connect z_1 to z_3 and z_2 to z_4 along their respective unstable manifolds. We define the following curves inside $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$,

$$\gamma_1 = \gamma(z_1, z_3) \subset W^u(z_1)$$

$$\gamma_2 = \gamma(z_2, z_4) \subset W^u(z_2)$$

$$\gamma_3 = \gamma(z_3, z_4) \subset W^s(z_3).$$

Now we bound each term on the right hand side of the inequality

$$\begin{aligned} |logD^uF(z_1) - logD^uF(z_2)| &\leq \\ |logD^uF(z_1) - logD^uF(z_3)| &+ \\ |logD^uF(z_3) - logD^uF(z_4)| &+ \\ |logD^uF(z_4) - logD^uF(z_2)| & \end{aligned} \tag{13}$$

Estimates of $|logD^uF(z_1) - logD^uF(z_3)|$ and $|logD^uF(z_4) - logD^uF(z_2)|$ are the same as estimates (15) – (28) in the proof of Proposition 5.1 in [18]. Then we get

$$|logD^uF(z_1) - logD^uF(z_3)| < C_2\frac{1}{K_0^n}. \tag{14}$$

Similarly,

$$|logD^uF(z_2) - logD^uF(z_4)| < C_2\frac{1}{K_0^n}. \tag{15}$$

(b) The second part of the proof, depending on m, follows again the ideas in [18] and [19] but also utilizes condition H5. We are left with estimating the difference

$$|logD^uF(z_3) - logD^uF(z_4)|. \tag{16}$$

Note that the BIV condition implies that the above difference is uniformly bounded on full-height rectangles. From [21] we get that the hyperbolicity conditions imply that any unit vector in K_α^u at a point $z \in E_i$, in particular a tangent vector to $W^u(z)$, has coordinates $(1, a_z)$ with $|a_z| < \alpha$. Thus we need to estimate

$$log|F_{1x}(z_3) + a_{z_3}F_{1y}(z_3)| - log|F_{1x}(z_4) - a_{z_4}F_{1y}(z_4)|. \tag{17}$$

Now we are moving along $\gamma_3 \subset W^s(z_3)$ connecting z_3 and z_4.
We cover γ_3 by rectangles $\tilde{R}_k$ for which the widths Δx and lengths Δy satisfy $|\Delta x| < \alpha|\Delta y|$. As in [18], we get (17) by estimating differences

$$|logF_{1x}(\tilde{z}) - logF_{1x}(\tilde{z}')| \tag{18}$$

for $\tilde{z}, \tilde{z}' \in \tilde{R}_k \cap W^s(z_3)$, and

$$|a_{z_3} - a_{z_4}|. \tag{19}$$

To estimate (18) we use the mean value theorem and get on each rectangle $\tilde{R}_k$ an estimate not exceeding

$$Const \cdot \sup_{z \in \tilde{R}_k} \frac{|f_{1ij}(z)|}{|f_{1x}(z)|}\Delta y. \tag{20}$$

Let $\Gamma_3 = W^s(z_3) \cap R_{i_{-1},i_0}$. The sum of the contributions from (20) is bounded by

$$Const \sup_{z \in R_{i_{-1},i_0}} \frac{|f_{1ij}(z)|}{|f_{1x}(z)|} |\gamma_3|. \tag{21}$$

Since the distortion condition D1 is expressed using the width of E_{i_0}, we use condition H5.

On an admissible rectangle $R_{i_{-m}\dots i_{-1},i_0\dots i_{n-1}}$, the string $[i_{-m}\dots i_{-1}i_0]$ is admissible. Let h_{max} be the maximal height of $S_{i_{-m}\dots i_{-1}}$ and let w_{min} be the minimal width of E_{i_0}. From condition H5 we know that contraction of f_i is stronger than a_1^i. Since the rectangle is admissible, the sum of the indices satisfies $i_{-m} + \dots i_{-1} \geq i_0$. Since contraction of the composition is stronger than $a_1^{i_0}$, it follows that

$$\frac{h_{max}}{w_{min}} < C\left(\frac{a_1}{a}\right)^{i_0}. \tag{22}$$

As each index is at least 1 we get that $i_0 \geq m$ and thus

$$h_{max} < \left(\frac{a_1}{a}\right)^m \tilde{C}^{-1} w_{min}. \tag{23}$$

Therefore the heights of rectangles $R_{i_{-m}\dots i_{-1},i_0}$ decay exponentially comparatively to the width of E_{i_0}. Since $|\gamma_3| < h_{max}$ and $w_{min} < \delta_z(E_{i_0})$ for any $z \in E_{i_0}$, we can apply D1 and obtain the following bound for the sum of the contributions from (18),

$$C_3\left(\frac{a_1}{a}\right)^m \tag{24}$$

We estimate (19) as in [18], [19]. We assume by induction

$$|a_{z_3} - a_{z_4}| < c_1\theta_1^m \tag{25}$$

As in [19] one can assume by taking if needed instead of F some power of F

$$\frac{1}{K_0^2} + \alpha^2 < 1 \tag{26}$$

Note that differently from [19] the variation of $\log D^u F$ along stable manifolds inside admissible rectangles is controlled not by using bounded distortions of inverse maps, but from (23) and (26).

With that modification we prove like in Lemma 5.2 from [18]

$$|a_{F(z_3)} - a_{F(z_4)}| < c_1 \theta_1^{m+1}. \tag{27}$$

where

$$max\{\frac{1}{K_0^2} + \alpha^2, \frac{a_1}{a}\} < \theta_1 < 1 \tag{28}$$

Combining (24) and (27) gives us

$$|logD^u F(z_3) - logD^u F(z_4)| < C_4 \theta_1^m. \tag{29}$$

Finally combining (14), (15), and (29) concludes the proof of Proposition 3.1, if we take $\theta_0 < 1$ satisfying

$$\theta_0 > max\{\frac{1}{K_0}, \theta_1\}. \tag{30}$$

6. **Construction of full height Cantor sets**.

We define full height Cantor sets C_n inside full height rectangles E_n, $n \geq 1$ by

$$C_n = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{m} E_{i_0 \ldots i_{k-1}} \tag{31}$$

where $i_0 = n$, $k \geq 1$ and all $[i_0 \ldots i_{k-1}]$ are admissible strings.

As an example let us consider several rectangles with indices starting from 1. It follows from the definition that $[11]$ is admissible and $[1i], i > 1$ are not. Then $R_{[11]}$ is the only defining rectangle of order two for the Cantor set C_1 and that the other $R_{[1i]}$ are gaps. The next defining rectangles of order three for C_1 are $R_{[111]}$ and $R_{[112]}$, of order four $R_{[1111]}, R_{[1112]}, R_{[1113]}, R_{[1121]}, R_{[1122]}, R_{[1123]}, R_{[1124]}$ and so on.

As each index is at least 1 we get from the definition of admissible rectangles that inadmissible indices satisfy $i_N > N$. Geometric condition G3

implies that on any unstable manifold, the relative measure of the union of rectangles with indices greater than N decays exponentially.
Then uniformly bounded distortion implies that the total relative linear measure of gaps in an unstable manifold of any defining rectangle of order N has uniform exponential decay.

This implies the following Corollary.

Corollary 3.2 *There is a $c_0 > 0$ such that for any initial full height rectangle E_n and respective full height Cantor set C_n constructed inside E_n and for any $z \in C_n \cap \Lambda$ the relative linear measure of C_n in $W^u(z, E_n)$ is greater than c_0. Moreover that relative measure tends to one when $n \to \infty$.*

Also as in [18], Remark 5.10, we get the following Corollary from distortion estimates.

Corollary 3.3 *Let $E_{i_0 \dots i_{n-1}}$ be a full height admissible rectangle of order n. Then for any two points z_1, $z_2 \in E_{i_0 \dots i_{n-1}} \cap \Lambda$, it holds that*

$$\frac{\mid W^u(z_1, E_{i_0 \dots i_{n-1}}) \mid}{\mid W^u(z_2, E_{i_0 \dots i_{n-1}}) \mid} < c \tag{32}$$

As Corollary 3.3 is valid for all defining rectangles we get

Corollary 3.4 *Let C_n be the full height Cantor set constructed inside E_n, let $z \in C_n \cap \Lambda$, and let $\mid W^u(z, C_n) \mid$ be the linear Lebesgue measure of C_n in $W^u(z, E_n)$. Then for any two points z_1, $z_2 \in C_n \cap \Lambda$, it holds that*

$$\frac{\mid W^u(z_1, C_n) \mid}{\mid W^u(z_2, C_n) \mid} < c \tag{33}$$

7. **Markov properties of C_n.**

Every Cantor set C_i is determined by its defining rectangles and equivalently by its gaps. Defining rectangles are labeled by admissible strings $[i_1 i_2 \dots i_n]$ satisfying A1. Gaps are labeled by nonadmissible strings $[i_1 i_2 \dots i_n j]$, where

$[i_1 i_2 \dots i_n]$ is admissible and $j > i_1 + i_2 + \dots + i_n$.

For example gaps of C_3 are:

E_{3k}, $k > 3$ - gaps of order 1,

E_{31k}, $k > 4$, E_{32k}, $k > 5$, E_{33k}, $k > 6$ - gaps of order 2,

E_{311k}, $k > 5$ - gaps of order 3, and so on.

The following example illustrates Markov relations between Cantor sets C_i.

Consider an admissible rectangle E_{112}. Then $F(E_{112}) = R_{1,12}$ is a sub-rectangle of a non-admissible rectangle E_{12}. However, because of H5, $F^2(E_{112}) = R_{11,2}$ has height less than the width of E_2.

We will use the notation

$$C_{112} = C_1 \cap E_{112}$$

and in general,

$$C_{i_1 \dots i_s} = C_{i_1} \cap E_{i_1 \dots i_s},$$

$$C_{i_1 \dots i_k, i_{k+1} \dots i_l} = C_{i_{k+1} \dots i_l} \cap R_{i_1 \dots i_k, i_{k+1} \dots i_l}.$$

We have that

$$F^2(C_{112}) \supset C_{11,2}. \tag{34}$$

As the sum of indices in [112] is greater than 2, the inclusion in (34) is not an equality. Namely strings [23] and [24] are not admissible, but [1123] and [1124] are admissible. The image $F^2(C_{112})$ covers respective slice of C_2 and also covers some parts of gaps C_{23} and C_{24} used in the construction of the Cantor set C_2.

Consider the union of all full height Cantor sets C_n :

$$\mathscr{C} = \bigcup_{n=1}^{\infty} C_n. \tag{35}$$

For $z \in C_{i_1 \ldots i_n}$ let us denote $W^u(z, C_{i_1 \ldots i_n}) = C_{i_1 \ldots i_n} \cap W^u(z, E_{i_1 \ldots i_n})$.

Let T be the first return map to $\mathscr{C}$ generated by F . Similarly to (34) above we get the following Markov properties.

Proposition 3.5 *If $[i_1 \ldots i_n]$ is admissible, $z \in C_{i_1 \ldots i_n}$, $T(z) \in C_{i_n}$, then*

$$T(W^u(z, C_{i_1 \ldots i_n})) \supset W^u(T(z), C_{i_n}). \tag{36}$$

$$T(W^s(z, C_{i_1 \ldots i_n})) \subset W^s(T(z), C_{i_1 \ldots i_{n-1}, i_n}) \tag{37}$$

8. **Estimates of measure**.

Let T_1 be the first return map to $C_1 \subset \mathscr{C}$. Next we estimate the measure of points in C_1 which return to C_1 after at least n iterates of F.

Proposition 3.6 *Let B_n be the set of points in the domain of T_1 such that the return time for $z \in B_n$ is greater than n. For some $C > 0$ and $0 < \beta < 1$,*

$$\mu(B_n) < C\beta^n. \tag{38}$$

We begin by proving (38) for the first return map T onto $\mathscr{C}$.

Suppose $x = (x_0 x_1 \ldots x_n \ldots) \in \mathscr{C}$, $y = Fx = (x_1 x_2 \ldots x_n x_{n+1} \ldots) \notin \mathscr{C}$, and $F^n x \in \mathscr{C}$ is the first return.

As $y \notin \mathscr{C}$, y_k such that $y_k = x_{k+1} > y_0 + \ldots + y_{k-1} = x_1 + \ldots + x_k$. If $k \geq n$, then

$$x_{k+1} > x_1 + \ldots x_k \geq x_n + \ldots + x_k$$

which contradicts that $F^n(x) \in \mathscr{C}$.

Suppose $k < n$ satisfies

$$x_1 + \ldots + x_k < x_{k+1}$$

Then $x_{k+1} \geq k+1$ because each coordinates is at least 1.

As all images $F^k x,\ 1 \leq k < n$ do not belong to $\mathscr{C}$, there is a coordinate $x_N,\ N \geq k$ such that

$$x_k + \ldots + x_N < x_{N+1}.$$

If $N \geq n$, then

$$x_n + \ldots + x_N < x_{N+1}$$

and we get a contradiction as above to $F^n x \in \mathscr{C}$. So $N < n$ and we get

$$N+1 = 1 + (k+1) + (N-k-1) \leq x_k + x_{k+1} + \ldots + x_N < x_{N+1} \tag{39}$$

Proceeding as above we get

$$x_n \geq n. \tag{40}$$

As widths of E_i decay exponentially, the measure of the collection of x satisfying (40) decays exponentially. Since the measure of $\mathscr{C}$ is positive, we get that the measure of points which do not return to $\mathscr{C}$ after n iterates is less than

$$C_1 \beta_1^n. \tag{41}$$

for some $C_1 > 0$ and $0 < \beta_1 < 1$.

Next we note that if $z \in C_1$ is mapped by T into C_i, then because all transitions from C_i to C_1 are admissible, z will be mapped into C_1 by the next iterate of F. Therefore points which do not return into C_1 after n iterates are subdivided into two subsets: points which did not return into $\mathscr{C}$ after n iterates and points which returned into $C_i,\ i > 1$, after $n-1$ iterates and at the next iterate were not mapped into C_1. Because of uniformly bounded distortion the measure of the second set is less than

$$C_1 \beta_1^{n-1}(1-\gamma_0) \tag{42}$$

where $0 < \gamma_0 < 1$. Then (38) follows from (41) and (42) if we take $C = 2C_1$ and $\beta = max\{\beta_1, 1-\gamma_0\}$.

This concludes the proof of Proposition 3.6.

9. **First return maps and respective transition matrices.**

Note that on every unstable leaf, relative measures of C_i inside E_i are uniformly bounded away from 0. Together with uniformly bounded distortion it implies that in the orbits of the first return map each gap is substituted by a union of new gaps and a Cantor set of relative measure greater than some uniform $c_0 > 0$. Then at the end of that construction we get, up to a set of measure zero,

$$C_i = \bigcup_k \{T^{-1}(TC_i \cap C_k) = C_{ik}\} \tag{43}$$

Note that C_{ik} can be unions of several Cantor sets which belong to disjoint full height rectangles. For example consider $C_{1113} \subset E_{1113}$. Admissible rectangle E_{1113} is mapped as follows,

$$E_{1113} \to E_{113} \to E_{13} \to E_3$$

As E_{113}, E_{13} are inadmissible, we get that $T = F^3$ maps C_{1113} onto C_3 in a Markov way. Similarly $T = F^3$ maps C_{1123} onto C_3 in a Markov way.
The correct labeling is provided by respective strings of the original alphabet starting width i and ending with k.

To get an authentic Markov partition which generates a transition matrix of 0's and 1's we partition each $C_{j_0 j_1}$ into subsets

$$C_{i_0 i_1 \ldots i_{n-1}} \tag{44}$$

where

$$i_0 = j_0, i_1, \ldots, i_{n-1} = j_1 \tag{45}$$

is admissible, and for all $k > 0$

$$i_k, \ldots, i_{n-1} \tag{46}$$

are not admissible.
In other words the first return map T maps $C_{i_0 i_1 \ldots i_{n-1}}$ onto a full width substrip of $C_{i_{n-1}}$ and all intermediate images of $C_{i_0 i_1 \ldots i_{n-1}}$ belong to various gaps.

As in the proof of (38) we get that the length of the above strings starting from j_0 and ending with j_1 is at most $j_1 - j_0 + 2$.
The union of $C_{j_0 i_1 \ldots i_{n-2} j_1}$ forms a Markov partition

$$\mathscr{M}\mathscr{P} = \{C_{j_0 i_1 \dots i_{n-2} j_1}\} \tag{47}$$

of $\mathscr{C}$. Using Proposition 3.5 we get

Lemma 3.7 *To any one-sided T-admissible sequence of transitions*

$$C_{j_0 i^1_1 \dots i^1_{n_1-2} j_1} \to C_{j_1 i^2_1 \dots i^2_{n_2-1} j_2} \to \dots$$

corresponds a unique one-sided sequence of the original alphabet

$$j_0,\, i^1_1, \dots, i^1_{n_1-1} = j_1,\, i^2_1, \dots, i^2_{n_2-1} = \dots \tag{48}$$

such that

$$C_{j_0 i^1_1 \dots i^1_{n_1-2} j_1} \cap T^{-1}(C_{j_1 i^2_1 \dots i^2_{n_2-1} j_2}) \cap \dots$$

coincides with the stable manifold labeled by (48).

The union of elements of $\mathscr{M}\mathscr{P}$ and all intermediate iterates of $C_{j_0 i_1 \dots i_{n-2} j_1}$ form a tower over $\mathscr{C}$. Elements of this tower form a Markov partition of the attractor Λ.

Up to a set of μ measure zero, any point of the attractor is uniquely labeled by a two-sided sequence of admissible transitions

$$\dots \to C_{j_{-1} i^{-1}_1 \dots i^{-1}_{n_{-1}-2} j_0} \to C_{j_0 i^1_1 \dots i^1_{n_1-2} j_1} \to C_{j_1 i^2_1, \dots i^2_{n_2-2} j_2} \to \dots \tag{49}$$

We consider a new alphabet Ω corresponding to the elements of the tower from Lemma 3.7 and get a subshift

$$(\Omega, X, \sigma) \tag{50}$$

Recall that a subshift is topologically mixing if for any states a and b there is $n(a,b)$ such that for $n \geq n(a,b)$ there is an admissible word of length n starting from a ending with b.

We will need the following Proposition.

10. **Proposition 3.8** *Subshift* (Ω, X, σ) *is topologically mixing.*

Note that although our original map is clearly topologically mixing elements of the Markov partition are Cantor sets, so the statement is not obvious.

Proof of Proposition 3.8. By construction any Cantor set C which coincides with some element of the tower is mapped by some iterate of F onto a full width substrip of some Cantor set C_i. As the image of any C_i (including C_1) contains a full with substrip of C_1, we get that all consecutive images of C_i have Markov intersections with C_1.

It remains to prove that for any element Δ of the tower there is an $n(\Delta)$ such that $F^n C_1$ has Markov intersection with Δ for $n > n(\Delta)$. By construction any $\Delta = F^{k(\Delta)}P$ where P is a full height Cantor subrectangle of some C_i. So it is enough to prove that $F^n C_1$ intersects C_i for $n > n(i)$. But C_1 contains a full height Cantor subset $C_{11\dots1i}$ and all images of C_1 have Markov intersections with $C_{11\dots i}$. That proves Proposition 3.8.

Next we consider the first return map T_1 induced by T on C_1. Consider the Markov partition $\mathcal{M}\mathcal{P}_1$ of C_1

$$\mathcal{M}\mathcal{P}_1 = \{C_{1i_1\dots i_m 1}\} \tag{51}$$

generated on C_1 by T_1. By construction T_1 maps its domains (which are full height Cantor sets) onto full width substrips of C_1. Therefore the transition matrix corresponding to the map T_1 on $\mathcal{M}\mathcal{P}_1$ consists of all 1's.

Remark: $[1]$ *will correspond to our choice of state* $[a]$ *in the symbolic dynamics in later sections.*

4 Thermodynamic formalism, Reduction of Theorems 2.4 and 2.5 to results for functions defined on one-sided sequences.

1. **Reduction Arguments.**

By first reducing to functions defined on one-sided sequences, we will show that our transfer operator (53) has the spectral gap property (56) on a particular Banach Space. This property implies exponential decay of correlations and Central Limit Theorem for functions defined on one-sided sequences. Then we can extend these results to certain functions defined on two-sided sequences, Theorems 2.4 and 2.5, respectively. This exchange between the two settings is a consequence of the reduction from two-sided shifts to one-sided shifts following from the classical arguments of Ruelle and Bowen, see [25], [9]. In the case of an infinite alphabet, detailed reduction arguments can be found in sections 4 and 5 of [33].

If we restrict our consideration to Hölder functions on Q, then we are left with the proof of the spectral gap property (SGP) of the transfer operator acting on a suitable space $\mathscr{L}$ of functions defined on one-sided sequences of the alphabet Ω. We do this in the next sections, following [10] and [28].

2. **Thermodynamic Formalism.**

Now by following [25], [9] we develop thermodynamic formalism on the space of one-sided sequences for the function $\Phi(x,y) = -log|D^u F|$.

Let $\mathscr{E}_i \subset E_i$ be an element of the Markov partition $\mathscr{MP}$. For each $\mathscr{E}_i$ we fix some unstable manifold W_0^u, and to any $(x,y) \in \mathscr{E}_i$ we let correspond $(x,y_0) = W^s(x,y) \cap W_0^u$. We define

$$u(x,y) = \sum_{k=0}^{\infty} \Phi(F^k(x,y)) - \Phi(F^k(x,y_0)).$$

Then we can construct a Hölder function on one-sided sequences cohomologous to $\Phi(x,y)$ in the following way,

$$\phi(x) = \Phi(x,y) + u(F(x,y)) - u(x,y). \tag{52}$$

We call ϕ the potential.

The transfer operator, L_ϕ is defined as

$$(L_\phi f)(x) = \sum_{F(y)=x} e^{\phi(y)} f(y). \tag{53}$$

In the next several sections we consider the space of functions on one-sided sequences for which we can prove the spectral gap property for L_ϕ. We denote by X the space of one-sided sequences. From this point forward, points x, y will be one-sided sequences belonging to X.

3. **Induced system**.
Just as in general case, we get SGP as a result of a particular induction procedure, see [10], [28], and references to earlier works in [28]. In our setting we induce on C_1; here our $[a]$ is $[1]$.

The induced system on $[1]$ is $F_1 : X_1 \to X_1$ where

$$X_1 := \{x \in X \mid x_0 = 1, x_i = 1 \text{ infinitely often}\}$$

and

$$F_1(x) := F^{\varphi_1}(x),$$

for

$$\varphi_1(x) := 1_{[1]}(x) \min\{n \geq 1 : x_n = 1\}.$$

The resulting transformation can be given the structure of a Markov Shift as follows. Let

$$\overline{S} := \{[1, \xi_2, ..., \xi_{n-2}, 1] : 2 \leq i \leq n-2,\ \xi_i \neq 1\}$$

and let $F^{\varphi_1} : \overline{X} \to \overline{X}$ denote the left shift on $\overline{X} = (\overline{S})^{\mathbf{N}}$. Then F^{φ_1} is topologically conjugate to F_1. The conjugacy $\overline{\pi} : \overline{X} \to X_1$ is given by

$$\overline{\pi}([1, \underline{\xi}^0, 1], [1, \underline{\xi}^1, 1], ...) := (1, \underline{\xi}^0, 1, \underline{\xi}^1, ...).$$

Let

$$\overline{\phi} := \left(\sum_{i=0}^{\varphi_1 - 1} \phi \circ F^i \right) \circ \overline{\pi}.$$

We call $\overline{\phi} : \overline{X} \to \mathbf{R}$ the induced potential.

4. **Gurevich Pressure.**

We introduce a few preliminary definitions and results.

Let $\phi_n(x) = \sum_{k=0}^{n-1} \phi \circ F^k(x)$ where $x = (.x_0 x_1 \ldots)$. Let

$$Z_n(\phi) = \sum_{\{x: T^n x = x; x_0 = 1\}} e^{\phi_n(x)}. \tag{54}$$

Then the limit, called the *Gurevich Pressure*,

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi) \tag{55}$$

exists, see [26], and we can calculate it explicitly in our setting.

Let T_1 be the first return map to C_1. Each periodic orbit of T_1 is contained in some admissible cylinder of F, also periodic but which has, in general, a larger period. Moreover there are F-strings of arbitrary large periods which correspond to a given T_1 period. Admissible cylinders of the same T_1 periods but with different T_1 labels do not intersect.

Proposition 3.1 and uniformly bounded distortions of $D^u F$ imply that the contribution to $P_G(\phi)$ from each periodic T_1 orbit differs from the length of the horizontal cross section of the respective two-dimensional cylinder by a uniformly bounded factor. That implies that the quantities $Z_n(\phi)$ are uniformly bounded from above.

As the measure of the Cantor set C_1 is positive, we get that $Z_n(\phi)$ are uniformly bounded from below. Thus,

$$P_G(\phi) = 0.$$

Remark: *This calculation works for any C_i, so as in the general case (see [28]) the Gurevich pressure is independent of the choice of partition set $[a]$.*

5. **Spectral gap.**

Following [10], [28] we would like to show that L_ϕ has spectral gap on the appropriate Banach space $\mathscr{L}$, defined below in (64). The implications of such a property are as follows.

If L_ϕ has spectral gap then it can be written as

$$L_\phi = \lambda P + N$$

where

$$\lambda = e^{P_G(\phi)},\ PN = NP = 0,\ P^2 = P$$

and the spectral radius, ρ, of N is less than λ. Since $\rho < \lambda$,

$$\| \lambda^{-n} L_\phi^n - P \|_{\mathscr{L}} = \lambda^{-n} \| N^n \|_{\mathscr{L}} \to 0 \tag{56}$$

exponentially fast as $n \to \infty$.

In our setting,

$$\lambda = e^{P_G(\phi)} = 1$$

and

$$Pf = h \int f\, d\nu \tag{57}$$

where h is the eigenfunction of L_ϕ and ν is the eigenmeasure of L_ϕ^*.

Following [10], [28] we introduce the a-discriminant

$$\Delta_a[\phi] := \sup_{p \in \mathbf{R}} \{ P_G(\overline{\phi + p}) \mid P_G(\overline{\phi + p}) < \infty \}.$$

The Discriminant Theorem 6.7, from [28], gives necessary and sufficient conditions for the spectral gap based on certain properties of the a-discriminant.

Specifically, it links the strict positivity of the discriminant to the spectral gap property. It involves the Gurevich pressure evaluated with respect to the induced system. We state one of the properties relevant to our setting.

Proposition 4.1 *Let X be a topologically mixing TMS and suppose $\phi : X \to \mathbf{R}$ is a weakly Hölder continuous function such that $P_G(\phi) < \infty$. If for some state a,*

$$\Delta_a[\phi] > 0, \tag{58}$$

then ϕ has the SGP on the Banach space $\mathscr{L}$ defined below in (64).

6. The following results prove $\Delta_1[\phi] > 0$ by showing that for sufficiently small p,

$$0 < P_G(\overline{\phi + p}) < \infty.$$

Proposition 4.2 *For sufficiently small $p > 0$, $P_G(\overline{\phi + p}) < \infty$.*

Recall that

$$P_G(\overline{\phi + p}) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\overline{\phi + p}). \tag{59}$$

We begin by calculating $Z_1(\overline{\phi + p})$:

$$\begin{aligned} Z_1(\overline{\phi + p}) &= \sum_{\{x:\, Tx=x;\, x_0=1\}} e^{\overline{\phi+p}} = \sum_{n=1}^{\infty} \sum_{\{x:\, Tx=F^n x=x;\, x_0=1\}} e^{\overline{\phi}} e^{pn} \\ &= \sum_{n=1}^{\infty} e^{pn} \sum_{\{x:\, Tx=F^n x=x;\, x_0=1\}} e^{\overline{\phi}} \\ &\le C_\phi \sum_{n=1}^{\infty} e^{pn} \beta^n = C_\phi M_1 < \infty \end{aligned} \tag{60}$$

Here β comes from the estimate in Proposition 3.6 and we use $p < \log\left(\frac{1}{\beta}\right)$. Constant C_ϕ depends on constant C from the same proposition, on the uniform distortion bounds and on the uniform bound of measures of crossections of the Cantor set C_1 by unstable manifords.

We define a sum similar to (54),

$$Z'_n(\phi) = \sum_{\{x \in \bar{a}_n\}} e^{\sup_{x \in \bar{a}_n} \phi_n(x)}, \tag{61}$$

where $\bar{a}_n = [1 i_1 ... i_{n-1}]$.

The definition of $Z'_n(\phi)$ implies

$$Z'_{n+m}(\phi) \le Z'_n(\phi) Z'_m(\phi), \tag{62}$$

Also from definitions of $Z_n(\phi)$ and of $Z'_n(\phi)$ and from uniformly bounded distortions we get

$$Z_n(\phi) \le Z'_n(\phi) \le d Z_n(\phi). \tag{63}$$

for some constant d. Combining (60), (62), and (63),

$$Z_n(\overline{\phi + p}) \le Z'_n(\overline{\phi + p}) \le \left(Z'_1(\overline{\phi + p})\right)^n \le \left(C\, Z_1(\overline{\phi + p})\right)^n \le d^n C_\phi^n M_1^n,$$

and thus,

$$P_G(\overline{\phi + p}) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\overline{\phi + p}) \le \lim_{n \to \infty} \frac{1}{n} \log(d^n C_\phi^n M_1^n)$$

$$= \log d + \log C_\phi + \log M_1 < \infty.$$

Proposition 4.3 *For $p > 0$ as in Proposition 4.2, $P_G(\overline{\phi + p}) > 0$.*

We combine the following properties from [28] with the fact that in our setting $P_G(\phi) = 0$ for $\phi(x)$ cohomologous to $\Phi(x, y)$.

i. $\overline{\phi + p} \ge \overline{\phi} + p, \forall p \in \mathbf{R}^+$
ii. If $\phi \le \psi$, then $P_G(\phi) \le P_G(\psi)$
iii. $P_G(\phi + p) = P_G(\phi) + p, \forall p \in \mathbf{R}$
iv. $P_G(\phi) = 0 \iff P_G(\overline{\phi}) = 0$

For our $p > 0$,

$$P_G(\overline{\phi + p}) \geq P_G(\overline{\phi} + p) = P_G(\overline{\phi}) + p = p > 0.$$

7. **Banach Space.**

   For all $x, y \in X$, let

   $$t(x,y) = min\{n : x_n \neq y_n\}$$

   $$s_1(x,y) = \#\{0 \leq i \leq t(x,y) - 1 : x_i = y_i = 1\}.$$

   Let [1] be the collection of one-sided sequences, $x = .x_0 x_1 ...$, such that $x_0 = 1$.

   As proved in [10] there is a positive function $h_0 : \mathbf{Z}_+ \to \mathbf{R}$ with the following properties.

   Consider the set of continuous functions $\{f : \| f \|_{\mathscr{L}} < \infty\}$ where

   $$\| f \|_{\mathscr{L}} = \sup_{b \in \mathbf{Z}_+} \frac{1}{h_0(b)} \left[\sup_{x \in [b]} |f(x)| + \sup_{(x,y) \in [b]; x \neq y} \frac{| f(x) - f(y) |}{\theta^{s_1(x,y)}} \right]. \quad (64)$$

   Then $\mathscr{L}$ is an L_ϕ-invariant Banach space, and L_ϕ on $\mathscr{L}$ is a bounded operator with spectral gap. Additionally the eigenfunction h of Ruelle operator belongs to $\mathscr{L}$ and for any bounded Hölder function ψ it holds that

   $$\psi h \in \mathscr{L} \quad (65)$$

   Note that bounded Hölder functions belong to $\mathscr{L}$.

8. It follows from Propositions 4.2 and 4.3 that the discriminant is strictly positive, and thus, by Proposition 4.1, L_ϕ has spectral gap on the Banach Space $\mathscr{L}$. As in [10], [28] this implies that (σ, μ_ϕ) has exponential decay of correlations.

Theorem 4.4 *For σ a one-sided full shift, consider ϕ the potential defined in (52), and let μ_ϕ be the respective invariant measure. Then (σ, μ_ϕ) has exponential decay of correlations for bounded Hölder functions f and $g \in L^\infty(\mu_\phi)$. Namely there exists $0 < \eta_1 < 1$ such that*

$$\left| \int f\,(g \circ \sigma^n)\, d\mu_\phi - \int f\, d\mu_\phi \int g\, d\mu_\phi \right| < C \parallel g \parallel_\infty \parallel fh \parallel_{\mathscr{L}} \eta_1^n. \qquad (66)$$

Note that $fh \in \mathscr{L}$ because of 65. As in [10], [28], the subsequent result follows.

Theorem 4.5 *(***Central Limit Theorem for one-sided shift***)*
Let (σ, μ_ϕ) satisfy the assumptions of Theorem 4.4 and suppose that $f \in \mathscr{L}$. If $\int f d\mu_\phi = 0$ and f cannot be expressed as $g - g \circ \sigma$ for g continuous, then there is a positive, finite constant $\mathsf{d} = \mathsf{d}(f)$ such that for every $t \in \mathbf{R}$,

$$\lim_{n \to \infty} \mu_\phi \left\{ x : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ \sigma^k(x) \, < \, t \right\} = \frac{1}{\sqrt{2\pi \mathsf{d}^2}} \int_{-t}^{\infty} e^{\frac{-s^2}{2\mathsf{d}^2}}\, ds.$$

Remark: *It follows from Theorem 1.1 part d in [10] that for g, bounded and Hölder continuous, $P(\phi + tg)$ is real analytic in a neighborhood of 0.*

9. **Exponential Decay of Correlations and Central Limit Theorem for functions of two variables.**

One can follow arguments of Section 4 of [33] to reduce the estimate of the n-th correlation for functions defined on the two-sided shift to the following estimate for functions defined on the one-sided shift,

$$\mid L_\phi^{n-2k}(L_\phi^{2k}(f_k h)) - (\textstyle\int L_\phi^{2k}(f_k h) dm) h \mid, \qquad (67)$$

where $h\, dm = d\nu$ and ν is the eigenmeasure of L_ϕ^* as in (57). Here $2k < n$, and f_k is a piecewise constant approximation of f on cylinders of length k. Thus f_k is a Hölder function bounded by $\max|f|$. From (65) we get $f_k h \in \mathscr{L}$.

As in [33] Section 4 we get that norms $\parallel L_\phi^{2k}(f_k h) \parallel_{\mathscr{L}}$ are uniformly bounded by a constant which only depends on $\max|f|$. From (67) we get an estimate similar to (66) but with a different constant and a different $0 < \eta < 1$. That proves Theorem 2.4.

Theorem 2.5 follows from arguments in Section 5 of [33] regarding a result referred to as Theorem [G] from [12]. Using the spectral gap property, one obtains the following estimate,

$$\int | L_\phi^j(f_0 h) | dm \leq C_0' \| L_\phi^j(f_0 h) \|_{\mathscr{L}} \leq C_0 \eta_0^j \| f_0 h \|_{\mathscr{L}} \tag{68}$$

where, as above, f_0 is a bounded Hölder approximation of f and $0 < \eta_0 < 1$. Theorem 2.5 relies on showing that the key assumption in Theorem [G] holds - finiteness of the sum of the L^2-norms of the relevant conditional expectations. Showing that this assumptions holds reduces to showing that the sum of estimate (68) is bounded, and thus, we just need that $f_0 h \in \mathscr{L}$. This again follows from (65).

10. **Concluding Remarks.**

The study of countable Markov partitions in the 1980's originated in particular from the work of Roy Adler [4]. His work motivated the use of countable Markov partitions as a tool for studying one-dimensional dynamics with critical points, and subsequently, Henon-like systems.

The first author keeps warmest memories of his visit in 1990 to IBM Thomas J Watson Research Center, when he worked within the wonderful group directed by Roy Adler.

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