Computing Global Characters

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$\pi$: irreducible admissible representation of $G$
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**Problem:** Compute the (distribution) character \( \theta_\pi \) of \( \pi \)
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Harish-Chandra: function on the regular semisimple set
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Roughly: fix $H$,

$$\theta_\pi(g) = \frac{\sum a(\pi, w)e^{w\lambda}(g)}{\Delta(g)}$$
Computing Global Characters

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**Problem:** Compute \( a(\pi, w) \)
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$\theta_{\pi}$ determines $\pi$
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Applications: the Langlands program, lifting, base change,...
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Application: Given a unipotent Arthur parameter $\Psi$, compute the Arthur packet $\Pi_\psi$. 
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Not known . . . use character theory to get some information

(see www.liegroups.org/tables/unipotent)
Theme: When can you encapsulate very complicated objects with surprisingly little data?
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\text{pair of } m \times n \text{ integral matrices } (A, B)
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$(A, B) \sim (g^t A, B g^{-1})$ for $g \in GL(m, \mathbb{Z})$
Example: Here is complete information about representations of $SL(2, \mathbb{R})$, including their characters.

block: block
0(0,1): 0 [i1] 1 (2,*) 0 e
1(1,1): 0 [i1] 0 (2,*) 0 e
2(2,0): 1 [r1] 2 (0,1) 1 1

block: klbasis
0: 0: 1
1: 1: 1
2: 0: 1
   1: 1
   2: 1

5 nonzero polynomials, and 0 zero polynomials, at 5 Bruhat-comparable pairs.
History

Inducted character formula, focus on the discrete series
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Harish-Chandra: compact Cartan subgroup $T$,
A character of $T_\rho$ (later)
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Harish-Chandra: compact Cartan subgroup $\mathcal{T}$, 
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$\exists$ unique irreducible representation $\pi = \pi(\Lambda)$ satisfying:

$$\theta_{\pi}(g) = \frac{\sum \text{sgn}(w)(w\Lambda)(\tilde{g})}{D(\tilde{g})}$$

Question: Formula for $\theta_{\pi}$ on other Cartans?
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Other approaches (Schmid, Goresky-Kottwitz-MacPherson, Zuckerman, . . .)
Alternative Approach:
All representations at once, using KLV polynomials
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**Theorem:**

$$\Pi(G, \lambda) = \{(H, \Lambda) \mid \Lambda \in \hat{H}(\mathbb{R})_{\rho}, d\Lambda \sim \lambda\}/G(\mathbb{R})$$
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Theorem:

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$$(H, \Lambda) \rightarrow \begin{cases} I(H, \Lambda) & \text{standard (induced) module} \\
\pi(H, \Lambda) & \text{irreducible Langlands quotient} \end{cases}$$
Fix $H, \Delta^+$,
Fix $H, \Delta^+, \rho = \frac{1}{2} \sum_{\Delta^+} \alpha, H \rho$
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(drop \( \Delta^+ \))
Harish-Chandra’s character formula for discrete series + induced character formula ⇒:
Harish-Chandra’s character formula for discrete series + induced character formula $\Rightarrow$:

**Proposition:** Formula for $\theta_{I(H,\Lambda)}$ on $H(\mathbb{R})$:

$$\theta_{I(H,\Lambda)}(h) = \sum_{w \in W_{\mathbb{R}}} \frac{\text{sgn}(w)(w\Lambda)(h)}{D(h)} \quad (h \in H(\mathbb{R})_+)$$

$W_{\mathbb{R}} = W(G(\mathbb{R}), H(\mathbb{R})) \subset W(G, H)$
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\]

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**Corollary:** \(\Gamma \in \overline{H(\mathbb{R})}_\rho\):

\[
a(I(H, \Lambda), \Gamma) = \begin{cases} 
\pm 1 & \Gamma = w\Lambda \\
0 & \text{otherwise}
\end{cases}
\]
Question: Formula for $\theta_{I(H,\Lambda)}$ on other Cartan subgroups?
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Theory of leading terms (growth of matrix coefficients):
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If

\[(*) \quad \text{Re}\langle d\Lambda, \alpha^\vee \rangle \geq 0 \quad \text{for all } \alpha \in \Delta^+\]

then $\Lambda$ occurs in $I(H, \Lambda)$ and the character formula for no other standard module:
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Theorem: Fix $(H, \Lambda)$ satisfying $(*)$:

$$a(I(H', \Lambda'), \Lambda) = \begin{cases} \pm 1 & (H, \Lambda) \sim (H', \Lambda') \\ 0 & \text{otherwise} \end{cases}$$
\{\pi(H, \Lambda)\} \text{ and } \{I(H, \Lambda)\} \text{ are both bases of the Grothendieck group}
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\[ \pi = \sum \text{M}(I, \pi)I \text{ (character formula)} \]

This is \textit{precisely} what is computed by the Kazhdan-Lustig-Vogan polynomials (the \texttt{klbasis} command)
Corollary: Assuming (*),

\[ a(\pi, \Lambda) = \pm M(I(H, \Lambda), \pi) \]
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\[ a(\pi, w \times \Lambda) = \pm M(I(H, \Lambda), w^{-1} \cdot \pi) \]
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**Conclusion:** KLV-polynomials \( \Rightarrow \)
explicit formulas for all \( a(\pi, \Lambda) \)
Example: $Sp(4, \mathbb{R})$

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Node labels: e, r1, r2, C-, C+, i1, i2, r1, rn
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