1 9.7, 9.8 WS solutions

1.1 Problem 1

1. (a) In order to use the Alternating Series Test for a given series $\sum_{n=1}^{\infty} c_n$, what properties must the terms c_n of the series have?

(b) It is a fact that $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \pi/4$. Find the smallest positive integer $j > 0$ for which the Alternating Series Test guarantees that $\sum_{n=0}^{j} \frac{(-1)^n}{2n+1}$ differs from $\pi/4$ by less than 0.001.

(a) In order to use the Alternating Series Test for a given series $\sum_{n=1}^{\infty} c_n$, what properties must the terms c_n of the series have?

The alternating series test needs the terms c_n to be positive, decreasing, and have $\lim_{n\to\infty} c_n = 0$.

(b) It is a fact that $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \pi/4$. Find the smallest positive integer $j > 0$ for which the Alternating Series Test guarantees that $\sum_{n=0}^{j} \frac{(-1)^n}{2n+1}$ differs from $\pi/4$ by less than 0.001.

The jth truncation error of any series $\sum_{n\geq 0} b_n$ is $E_j = |\sum_{n=0}^{\infty} b_n - \sum_{n=0}^{j} b_n|$.

We know that for a convergent alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ or $\sum_{n=0}^{\infty}(-1)^{n+1} a_n$, that $E_j \leq a_{j+1}$, and $E_j = |\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} - \sum_{n=0}^{j} \frac{(-1)^n}{2n+1}| = |\pi/4 - \sum_{n=0}^{j} \frac{(-1)^n}{2n+1}|$ since $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \pi/4$.

In this case, $a_{j+1} = \frac{1}{2(j+1)+1} = \frac{1}{2j+3}$, so we need to find the smallest j making this less than $0.001 = \frac{1}{10^3}$. Thus $\frac{1}{2j+3} \leq \frac{1}{10^3}$ or $1000 = 10^3 \leq 2j+3$, so $1000 - 3 \leq 2j \implies 997/2 \leq j$ so $498 \leq j$.

Thus $j = 498$ is the smallest positive integer for which the Alternating Series Test guarantees that $\sum_{n=1}^{j} \frac{(-1)^n}{2n+1}$ differs from $\pi/4$ by no more than 0.001.

1.2 Problem 2

2. Suppose $\sum_{n=1}^{\infty} a_n 3^n$ converges.

(a) Find $\lim_{n\to\infty}(a_n 3^n)$, giving reasons.

(b) Prove that $\sqrt[n]{a_n} \leq 1/3$ for all large values of n. (First show that $\sqrt[n]{a_n} \leq 1/3$ if and only if $\sqrt[n]{a_n 3^n} \leq 1$, and then use the Ratio or Root Test, whichever applies.)

(a) Find $\lim_{n\to\infty}(a_n 3^n)$, giving reasons.

Well, by the divergence test, a convergent series $\sum_{n\geq 0} b_n$ must have $\lim_{n\to\infty} b_n = 0$, so $\lim_{n\to\infty} a_n 3^n = 0$.

(b) Prove that $\sqrt[n]{a_n} \le 1/3$ for all large values of n. (First show that $\sqrt[n]{a_n} \le 1/3$ if and only if $\sqrt[n]{a_n 3^n} \le 1$, and then use the Ratio or Root Test, whichever applies.)

Well, $\sqrt[n]{a_n} \le 1/3 \iff 3\sqrt[n]{a_n} \le 1 \iff \sqrt[n]{3^n}\sqrt[n]{a_n} = \sqrt[n]{a_n 3^n} \le 1$.

Note that since $\lim_{n\to\infty} a_n 3^n = 0$, then using the definition of limit, given ϵ with $0 < \epsilon < 1$, there's some N so for all $n \ge N$ we have $a_n 3^n \le \epsilon$.

Then $\sqrt[n]{a_n 3^n} \le \sqrt[n]{\epsilon} < 1$ for all $n \ge N$ since $x^{1/n}$ is an increasing function for all positive integers n.

Thus, for all $n \ge N$, we have $\sqrt[n]{a_n} \le 1/3$.

1.3 Problem 3

3. Suppose the radius of convergence of $\sum_{n\ge1} a_n x^n$ is precisely 4. Which of the following numbers is necessarily in the interval of convergence, and why?

(a) 3.9 (b) 4.1 (c) -3 (d) -5

The radius of convergence R about 0 guarantees that all values of x in $(-R, R)$ make the series $\sum_{n\ge1} a_n x^n$ converge. Thus if this value is 4, we have at least an interval of convergence of $(-4, 4)$ (the only other possible intervals of convergence for the series are $(-4, 4], [-4, 4]$, and $[-4, 4)$, so the series does not necessarily converge for $x = \pm 4$).

The only numbers falling in the interval $(-4, 4)$ are (a) 3.9 and (c) -3.

1.4 Problem 4

4. (a) Write down the power series for $\frac{1}{1-x}$, and tell why the radius of convergence is 1.

(b) Use the fact that $\frac{d}{dx}(\frac{1}{1-x})$ is $\frac{1}{(1-x)^2}$ to find the power series for $\frac{1}{(1-x)^2}$. Then determinethe radius of convergence both by using the Ratio Test and by citing the Differentiation Theorem for power series.

(c) Evaluate$\sum_{n=1}^{\infty} n(\frac{1}{2})^{n-1}$. (Hint: The series found in (b) might be helpful.)

(a) Write down the power series for $\frac{1}{1-x}$, and tell why the radius of convergence is 1.

Recall that our geometric series $\frac{1}{1-r} = \sum_{n\ge0} r^n$ converges only when $|r| < 1$, so the interval of convergence of the geometric series is $(-1, 1)$. Thus the radius of convergence of $\frac{1}{1-x} = \sum_{n\ge0} x^n$ is 1.

(b) Use the fact that $\frac{d}{dx}(\frac{1}{1-x})$ is $\frac{1}{(1-x)^2}$ to find the power series for $\frac{1}{(1-x)^2}$. Then determine the radius of convergence both by using the Ratio Test and by citing the Differentiation Theorem for power series.

The power series for $\frac{1}{(1-x)^2}$ is the derivative of the series for $\frac{1}{1-x} = \sum_{n\ge0} x^n$. This derivative is $\sum_{n\ge0} nx^{n-1}$.

To find the radius of convergence we first use the Ratio test: $\lim_{n\to\infty}|a_{n+1}/a_n| = \lim_{n\to\infty}|\frac{(n+1)x^n}{nx^{n-1}}| = \lim_{n\to\infty}|\frac{(n+1)}{n}x| = |x|$. This is less than 1 for $|x|<1$, telling us that the radius of convergence is 1.

We could also cite the Differentiation Theorem for power series to say that differentiation/integration of power series doesn't change the radius of convergence.

(c) Evaluate $\sum_{n=1}^{\infty} n(\frac{1}{2})^{n1}$. (Hint: The series found in (b) might be helpful.)

Note that since 1/2 is in (-1,1), the interval of convergence of $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ by part (b), we have $\sum_{n=1}^{\infty} n(\frac{1}{2})^{n-1} = \frac{1}{(1-1/2)^2} = \frac{1}{1/4} = 4$.

1.5 Problem 5

5.(a) Write the power series for e^x, and then explain why $xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$.

(b) By integrating the power series for xe^x, show that $\sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \int_0^1 xe^x\, dx = 1$.

(a) Write the power series for e^x, and then explain why $xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$.

The power series for e^x is $\sum_{n\geq 0} \frac{x^n}{n!}$, convergent for all x. Note that $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x\cdot\sum_{n\geq 0}\frac{x^n}{n!}$ is a power series for f (a series of the form $\sum_{n\geq 0} a_n x^n$) that agrees with f on its domain. Note $f(x) = xe^x$ has derivatives of all orders at 0, so any such power series of f at 0 is the only one (i.e. a power series expansion for f is unique).

We could also check derivatives:
$f'(x) = e^x + xe^x$, $f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$, $f'''(x) = 3e^x + xe^x$, ... ,$f^{(n)}(x) = ne^x + xe^x$.

Then $f'(0) = 1$, $f''(0) = 2, f'''(0) = 3$, ..., $f^{(n)}(0) = n$ for $n \geq 1$. Thus the power series for f would be

$$\sum_{n\geq 0}\frac{f^{(n)}(0)}{n!}x^n = f(0) + \sum_{n\geq 1}\frac{n}{n!}x^n = 0 + \sum_{n\geq 1}\frac{1}{(n-1)!}x^n = \sum_{n\geq 0}\frac{1}{n!}x^{n+1}$$

which agrees with our result.

(b) By integrating the power series for xe^x, show that $\sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \int_0^1 xe^x\, dx = 1$.

Note $\int xe^x\, dx$ can be solved by integration by parts $u = x, = e^x\, dx$, so $du = dx$, $v = e^x$, yielding $\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C$. Then

$$\int_0^1 xe^x\, dx = [xe^x - e^x]_0^1 = (e-e) - (0-1) = 1.$$

Since $xe^x = \sum_{n\geq 0}\frac{1}{n!}x^{n+1}$, $\int_0^1 xe^x\, dx = \int_0^1 \sum_{n\geq 0}\frac{1}{n!}x^{n+1}\, dx$:

$$\sum_{n\geq 0}\int_0^1 \frac{1}{n!}x^{n+1}\,dx = \sum_{n\geq 0}[\frac{1}{(n+2)n!}x^{n+2}]_0^1 = \sum_{n\geq 0}\frac{1}{(n+2)n!}.$$

Thus $\sum_{n\geq 0}\frac{1}{(n+2)n!} = \int_0^1 xe^x\,dx = 1$.

2 9.7, 9.8 WS/Quiz solutions

2.1 Problem 1

1. (a) Define carefully: "The series $\sum_{n\geq 0} b_n$ converges absolutely," and write down a series that converges absolutely, and one that converges conditionally.

(b) Define the radius of convergence R of a given power series, and write down a power series whose radius of convergence is $\sqrt{3}$.

(c) Tell whether the interval of convergence I of a power series necessarily stays the same when the power series is differentiated. Explain your answer by quoting an appropriate theorem.

(d) Find the radius of convergence of the power series for:

$$\text{(a) } e^{3x} \qquad \text{(b) } \sin(2x) \qquad \text{(c) } \frac{2}{1-x^2}$$

(a) Define carefully: "The series $\sum_{n\geq 0} b_n$ converges absolutely," and write down a series that converges absolutely, and one that converges conditionally.

The convergent series series $\sum_{n\geq 0} b_n$ converges absolutely if $\sum_{n\geq 0}|b_n|$ converges.

An example of a series that converges absolutely is $\sum_{n\geq 1}\frac{(-1)^n}{n^2}$ since $\sum_{n\geq 1}|\frac{(-1)^n}{n^2}| = \sum_{n\geq 1}\frac{1}{n^2} = \frac{\pi^2}{6}$.

An example of a series that converges conditionally is $\sum_{n\geq 1}\frac{(-1)^n}{n}$: note $\sum_{n\geq 1}|\frac{(-1)^n}{n}| = \sum_{n\geq 1}\frac{1}{n}$, the harmonic series, diverges.
However, the terms $(-1)^n a_n = (-1)^n \cdot 1/n$ satisfy the hypotheses of the Alternating Series Test (since $1/n$ is positive, decreasing, and $\lim_{n\to\infty} 1/n = 0$), so $\sum_{n\geq 1}\frac{(-1)^n}{n}$ converges.

(b) Define the radius of convergence R of a given power series, and write down a power series whose radius of convergence is $\sqrt{3}$.

The radius of convergence R of a power series $\sum a_n x^n$ is the number $R > 0$ such that the series converges for all $|x| < R$ and diverges for all $|x| > R$ (if no R exists, then we say $R = 0$ when the series converges only for $x = 0$, and $R = \infty$ if the series converges for all x).

The series $\sum_{n=0}^{\infty}(\frac{x}{\sqrt{3}})^n$ is geometric, so it converges (to $\frac{1}{1-\frac{x}{\sqrt{3}}}$) if and only if $|\frac{x}{\sqrt{3}}| < 1 \iff |x| < \sqrt{3}$. Thus the series $\sum_{n=0}^{\infty}(\frac{x}{\sqrt{3}})^n$ has radius of convergence $\sqrt{3}$.

(c) Tell whether the interval of convergence I of a power series necessarily stays the same when the power series is differentiated. Explain your answer by quoting an appropriate theorem.

The interval of convergence of a power series does not necessarily stay the same when differentiated: note $\sum_{n=1}^{\infty} x^n/n$ has interval of convergence $[-1,1)$ but its derivative $\sum_{n=1}^{\infty} x^{n-1}$ has interval of convergence $(-1,1)$.

(d) Find the radius of convergence of the power series for:

$$\text{(A) } e^{3x} \quad \text{(B) } \sin(2x) \quad \text{(C) } \tfrac{2}{1-x^2}$$

(A) The power series for e^x is $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$ which converges for all x, in particular, it converges for $y = 3x$:

$$e^y = e^{3x} = \sum_{n\geq 0} \frac{y^n}{n!} = \sum_{n\geq 0} \frac{3^n}{n!} x^n$$

so the power series for e^{3x} is $e^{3x} = \sum_{n\geq 0} \frac{(3x)^n}{n!} = \sum_{n\geq 0} \frac{3^n}{n!} x^n$, which converges for all x, so $R = \infty$.

We could also do a ratio test to find the radius of convergence:

$$\lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)!} x^{n+1} / (\frac{3^n}{n!} x^n)\right| = \lim_{n\to\infty} \left|\frac{3\cdot n!}{(n+1)!} x\right| = \lim_{n\to\infty} \left|\frac{3x}{n+1}\right| = 0 < 1 \text{ for all } x$$

Thus $e^{3x} = \sum_{n\geq 0} \frac{3^n}{n!} x^n$ converges for all x, so $R = \infty$ for the power series $\sum_{n\geq 0} \frac{3^n}{n!} x^n$ for e^{3x}.

(B) The power series for $\sin(x)$ is $\sin(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ which converges for all x. We could say the same thing that we did in part (A), subbing in $y = 2x$ so that the power series

$$\sin(y) = \sin(2x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} y^{2n+1}$$

converges for all y, so $\sin(2x) = \sum_{n\geq 0} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$ converges for all x. Thus $R = \infty$ for the power series $\sum_{n\geq 0} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$.

We could also do a root test to find the radius of convergence of the power series

$$\sin(2x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} = \sum_{n\geq 0} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$$

A root test on the terms yields:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(2n+1)!} x^{2n+1}\right|} = |x| \lim_{n\to\infty} \sqrt[n]{\left|\frac{x^{2n}\cdot x}{(2n+1)!}\right|} = |x| \cdot \lim_{n\to\infty} \sqrt[n]{|x|} \cdot \lim_{n\to\infty} \sqrt[n]{\frac{1}{(2n+1)!}} = |x| \cdot \lim_{n\to\infty} \sqrt[n]{\frac{1}{(2n+1)!}}$$

and note $\lim_{n\to\infty} \sqrt[n]{(2n+1)!} = \infty$ since $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$, and $n! < (2n+1)!$ (by Stirling's Approximation $n! \approx \sqrt{2\pi n}(n/e)^n$ as $n \to \infty$). Thus, for any x, the limit is 0 (less than 1), so our power series converges for all x, so $R = \infty$ for the power series $\sum_{n\geq 0} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$ for $\sin(2x)$.

(C) Note that $\frac{1}{1-x} = 1 + x + x^2 + ... = \sum_{n\geq 0} x^n$ if and only if $|x| < 1$, so $\frac{2}{1-x^2} = 2(1 + (x^2) + (x^2)^2 + ...) = \sum_{n\geq 0} 2(x^2)^n = \sum_{n\geq 0} 2x^{2n}$ if and only if $|x^2| < 1 \iff |x| < 1$.

Thus the radius of convergence of the power series $\sum_{n\geq 0} 2x^{2n}$ for $\frac{2}{1-x^2}$ is $R = 1$.

2.2 Problem 2

Let $f(x) = \cos(2x)$. This problem focuses on the Taylor series of f about 0.

(a) The formula for the Taylor series of $\cos x$ about 0 is $\sum_{n\geq 0} \frac{(-1)^n}{(2n)!}x^{2n}$. Use it to find the Taylor series of $\cos(2x)$ about 0.

(b) Write down the 26th and the 27th Taylor polynomials of f. Are they the same? Explain your answer.

(c) Show that $|f^{(n+1)}(x)| \leq 2^{n+1}$ for all x and all positive integers n, and then use the Lagrange Remainder Formula ((11) in Section 9.9) to show that $\lim_{n\to\infty} r_n(x) = 0$ for all x. (Hint: You will need to use Corollary 9.21 for the limit, i.e. if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = r < 1$ or $\lim_{n\to\infty} |\sqrt[n]{a_n}| = r < 1$ then $\lim_{n\to\infty} a_n = 0$).

(a) The formula for the Taylor series of $\cos x$ about 0 is $\sum_{n\geq 0} \frac{(-1)^n}{(2n)!}x^{2n}$. Use it to find the Taylor series of $\cos(2x)$ about 0.

Since $\sum_{n\geq 0} \frac{(-1)^n}{(2n)!}x^{2n}$ is the Taylor series of $\cos x$ about 0, the Taylor series of $\cos 2x$ about 0 is

$$\sum_{n\geq 0} \frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n\geq 0} \frac{(-1)^n 4^n}{(2n)!}x^{2n}$$

(b) Write down the 26th and the 27th Taylor polynomials of f. Are they the same? Explain your answer.

The 26th Taylor polynomial of f is

$$\sum_{n\geq 0}^{13} \frac{(-1)^n 4^n}{(2n)!}x^{2n} = 1 - \frac{4}{2!}x^2 + \frac{4^2}{4!}x^4 + ... + \frac{4^{12}}{24!}x^{24} - \frac{4^{13}}{26!}x^{26}$$

The 27th Taylor polynomial is the same since odd derivatives of $f(x) = \cos(2x)$ evidently have $f^{(2n+1)}(0) = 0$ by uniqueness of the Taylor series about 0: if n is even, the nth coefficient is $a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n/2}4^{n/2}}{n!}$, so $f^{(n)}(0) = (-1)^{n/2}4^{n/2}$. If n is odd, $a_n = 0 = \frac{f^{(n)}(0)}{n!}$, so $f^{(n)}(0) = 0$ for odd n. You can verify this with derivatives but it is unnecessary.

(c) Show that $|f^{(n+1)}(x)| \leq 2^{n+1}$ for all x and all positive integers n, and then use the Lagrange Remainder Formula ((11) in Section 9.9) to show that $\lim_{n\to\infty} r_n(x) = 0$ for all x. (Hint: You will need to use Corollary 9.21 for the limit, i.e. if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = r < 1$ or $\lim_{n\to\infty} |\sqrt[n]{a_n}| = r < 1$ then $\lim_{n\to\infty} a_n = 0$).

Well, note $f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$, $f'''(x) = 8\sin(2x)$, $f^{(4)}(x) = 16\cos(2x)$, so we have

$$f^{(n)}(x) = \pm 2^n \cos(2x) \text{ or } f^{(n)}(x) = \pm 2^n \sin(2x).$$

Either way $|f^{(n)}(x)| \leq 2^n$ since both $|\pm\sin(2x)| \leq 1$, $|\pm\cos(2x)| \leq 1$. Thus $|f^{(n+1)}(x)| \leq 2^{n+1}$ for all x and all positive integers n.

Then, for each $x \neq 0$, $r_n(x) = \frac{f^{(n+1)}(t_x)}{(n+1)!}x^{n+1}$ for some t_x between 0 and x by the definition of the Lagrange Remainder. Then

$$|r_n(x)| = |\frac{f^{(n+1)}(t_x)}{(n+1)!}x^{n+1}| \leq \frac{2^{n+1}x^{n+1}}{(n+1)!}$$

since $f^{(n+1)}(t_x) \leq 2^{n+1}$. We'll now use corollary 9.21 to show

$$\lim_{n\to\infty} \sqrt[n]{|r_n(x)|} < 1$$

and conclude that $\lim_{n\to\infty} r_n(x) = 0$.
Well, for fixed x,

$$\lim_{n\to\infty} \sqrt[n]{|\frac{f^{(n+1)}(t_x)}{n+1!}x^{n+1}|} \leq \lim_{n\to\infty} \sqrt[n]{|\frac{2^{(n+2)}}{(n+2)!}x^{n+1}} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n \cdot 2^2}{(n+2)!}|x|^n \cdot |x|} \tag{1}$$

$$= 2|x| \cdot \lim_{n\to\infty} \sqrt[n]{4|x|} \cdot \sqrt[n]{\frac{1}{(n+2)!}} \tag{2}$$

since $|f^{(n+1)}(t_{n,x})| \leq 2^{n+1}$. Note

$$\lim_{n\to\infty} \sqrt[n]{4|x|} = 1$$

since $4|x|$ is finite (we fixed an x), and

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{(n+2)!}} = 0$$

since $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$ (by Stirling's approximation $n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$ as $n \to \infty$).

Thus $2|x| \cdot \lim_{n\to\infty} \sqrt[n]{4|x|} \cdot \sqrt[n]{\frac{1}{(n+2)!}} = 2|x| \cdot 1 \cdot 0 = 0$, so

$$\lim_{n\to\infty} \sqrt[n]{|r_n(x)|} = 0.$$

Thus $\lim_{n\to\infty} r_n(x) = 0$ for all x by Corollary 9.21.

3 Quiz

3.1 Quiz Problem 1

1. Find an upper bound for the 10th truncation error E_{10} of $\sum_{n\geq 1}(-1)^{n+1}\frac{1}{n}$

Note $E_{10} = |\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} - \sum_{n=1}^{10}(-1)^{n+1}\frac{1}{n}|$. Since $\sum_{n\geq 1}(-1)^{n+1}\frac{1}{n}$ is a convergent series by the Alternating Series Test, we have the formula $E_j < a_{j+1}$ for all j, or $E_{10} < a_{11}$. Since $a_n = \frac{1}{n}$, we have

$$E_{10} < \frac{1}{11}.$$

3.2 Quiz Problem 2

2. Determine whether the series diverges, converges conditionally, or converges absolutely: $\sum_{n\geq 3}(-1)^{n+1}\frac{1}{n(n-2)}$.

The series converges absolutely: first note $\frac{1}{n(n-2)} < \frac{1}{(n-2)^2}$. Then, by the comparison test,

$$\sum_{n=3} \frac{1}{n(n-2)} < \sum_{n=3} \frac{1}{(n-2)^2} = \sum_{n=1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Thus the series is absolutely convergent, since $\sum |a_n|$ is convergent for $a_n = (-1)^{n+1}\frac{1}{n(n-2)}$.

3.3 Quiz Problem 3

3. Find the interval of convergence of the given series: $\sum_{n\geq 1} \frac{2^n}{n^n} x^n$.

We'll use the root test to find the interval of convergence:

$$\lim_{n\to\infty} \sqrt[n]{|\frac{2^n}{n^n} x^n|} = \lim_{n\to\infty} \frac{2|x|}{n} = 0 < 1$$

since $|x| < \infty$. Thus, by the root test, the series $\sum_{n\geq 1} \frac{2^n}{n^n} x^n$ is convergent for all x since for $a_n = \frac{2^n}{n^n} x^n$ we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

Hence the interval of convergence is $(-\infty, \infty)$.