Examples and applications of noncommutative geometry and $K$-theory

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Plan of the Lectures

1. Introduction to Kasparov’s $KK$-theory.
3. The universal coefficient theorem for $KK$ and some of its applications.
4. A fundamental example in noncommutative geometry: topology and geometry of the irrational rotation algebra.
5. Applications of the irrational rotation algebra in number theory and physics.

Notes available at www.math.umd.edu/~jmr/BuenosAires/
Part I

Introduction to Kasparov’s $KK$-theory
What is $KK$?

$KK$-theory is a bivariant version of topological $K$-theory, due to Gennadi Kasparov, defined for $C^*$-algebras, with or without a group action. It can be defined for either real or complex algebras, but in this course we will stick to separable complex algebras for simplicity. For such algebras $A$ and $B$, an abelian group $KK(A, B)$ is defined, with the property that $KK(\mathbb{C}, B) = K(B) = K_0(B)$ if the first algebra $A$ is just the scalars.
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A class in $KK(A, B)$ gives rise to a map $K(A) \to K(B)$, but also to a natural family of maps $K(A \otimes C) \to K(B \otimes C)$ for all *C*. I.e., it gives a natural transformation from the functor $K(A \otimes \_)$ to the functor $K(B \otimes \_)$. Here $\otimes$ is the completed (minimal) tensor product.
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This is almost the definition — for $A$ and $B$ nice enough, any such natural transformation comes from a $KK$ element.
Why $KK$?

Let’s take $A$ and $B$ are commutative. Thus $A = C_0(X)$ and $B = C_0(Y)$, where $X$ and $Y$ are locally compact Hausdorff. We will abbreviate $KK(C_0(X), C_0(Y))$ to $KK(X, Y)$. We want $KK(\mathbb{C}, C_0(Y)) = KK(pt, Y) = K(Y)$, the $K$-theory of $Y$ with compact support, the Grothendieck group of complexes of vector bundles over $Y$ that are exact off a compact set, or the reduced $K$-theory $\tilde{K}(Y_+) \text{ of the one-point compactification } Y_+ \text{ of } Y.$
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The **Thom isomorphism theorem** asserts that if $p : E \to X$ is a complex vector bundle, there is a natural isomorphism $\beta_E : K(X) \to K(E)$. The map $\beta_E$ can be described by the formula $\beta_E(a) = p^*(a) \cdot \tau_E$. Here $p^*(a)$ is the pull-back of $a \in K(X)$ to $E$, and $\tau_E$ is the Thom class, which has compact support in the fiber directions. $\beta_E$ can be described by a class in $KK(X, E)$, though one can also just use simple vector bundle theory to define it.
Bu how do we prove that $\beta_E$ is an isomorphism? The simplest way would be to construct an inverse map $\alpha_E : K(E) \to K(X)$. As Atiyah recognized, $\alpha_E$ uses elliptic operators, in fact the family of Dolbeault operators along the fibers of $E$. We want a class $\alpha_E$ in $KK(E, X)$ corresponding to this family of operators, and the Thom isomorphism theorem is a Kasparov product calculation, the fact that $\alpha_E$ is a $KK$ inverse to the class $\beta_E \in KK(X, E)$. Atiyah also noticed it’s enough to prove that $\alpha_E$ is a one-way inverse to $\beta_E$, or in other words, in the language of Kasparov theory, that $\beta_E \otimes_E \alpha_E = 1_X$. This comes down to an index calculation, which because of naturality comes down to the single calculation $\beta \otimes \mathbb{C} \alpha = 1 \in KK(pt, pt)$ when $X$ is a point and $E = \mathbb{C}$, which amounts to the Riemann-Roch theorem for $\mathbb{C}\mathbb{P}^1$. 
The example of Atiyah’s class $\alpha_E \in KK(E, X)$, based on a family of elliptic operators over $E$ parametrized by $X$, shows that one gets an element of the bivariant $K$-group $KK(X, Y)$ from a family of elliptic operators over $X$ parametrized by $Y$. The element that one gets should be invariant under homotopies of such operators. Hence Kasparov’s definition of $KK(A, B)$ is based on a notion of homotopy classes of generalized elliptic operators for the first algebra $A$, “parametrized” by the second algebra $B$ (and thus commuting with a $B$-module structure).
Kasparov bimodules

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- a $\mathbb{Z}/2$-graded (right) Hilbert $B$-module $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, 
- a $\mathbb{Z}/2$-grading preserved $^\ast$-representation $\phi$ of $A$ on $\mathcal{H}$, and
- a self-adjoint bounded $B$-linear operator $T \in L(\mathcal{H})$ of the form
  \begin{equation}
  T = T^\ast = \begin{pmatrix} 0 & F \ast\ F^0 \\ F^0 \ast F & 0 \end{pmatrix},
  \end{equation}
  with
  \[ \phi(a)(T^2 - 1) \in K(\mathcal{H}) \quad \forall a \in A \text{ (ellipticity)}, \]
  \[ [\phi(a), T] \in K(\mathcal{H}) \quad \forall a \in A \text{ (pseudolocality)}. \]
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  \[ T = T^* = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}, \]

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- $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H}) \ \forall a \in A$ (ellipticity),
- $[\phi(a), T] \in \mathcal{K}(\mathcal{H}) \ \forall a \in A$ (pseudolocality).
Comments on the definition

If $B = C_0(Y)$, a Hilbert $B$-module is equivalent to a continuous field of Hilbert spaces over $Y$. In this case, $\mathcal{K}(\mathcal{H})$ is the continuous fields of compact operators, while $\mathcal{L}(\mathcal{H})$ consists of strong-$^*$ continuous fields of bounded operators. In general, a Hilbert $B$-module means a right $B$-module equipped with a $B$-valued inner product $\langle \_ , \_ \rangle_B$, right $B$-linear in the second variable, satisfying $\langle \xi, \eta \rangle_B = \langle \eta, \xi \rangle_B^*$ and $\langle \xi, \xi \rangle_B \geq 0$, with equality only if $\xi = 0$. Such an inner product gives rise to a norm on $\mathcal{H}$: $\|\xi\| = \|\langle \xi, \xi \rangle_B\|_B^{1/2}$, and we require $\mathcal{H}$ to be complete with respect to this norm. The $C^*$-algebra $\mathcal{L}(\mathcal{H})$, consists of bounded adjointable $B$-linear operators $a$ on $\mathcal{H}$, i.e., with an adjoint $a^*$ such that $\langle a\xi, \eta \rangle_B = \langle \xi, a^*\eta \rangle_B$ for all $\xi, \eta \in \mathcal{H}$. Inside $\mathcal{L}(\mathcal{H})$ is the ideal of $B$-compact operators $\mathcal{K}(\mathcal{H})$. This is the closed linear span of the "rank-one operators" $T_{\xi, \eta}$ defined by $T_{\xi, \eta}(\nu) = \xi \langle \eta, \nu \rangle_B$. 
The simplest kind of Kasparov bimodule is associated to a homomorphism $\phi: A \to B$. In this case, we simply take $\mathcal{H} = \mathcal{H}_0 = B$, viewed as a right $B$-module, with the $B$-valued inner product $\langle b_1, b_2 \rangle_B = b_1^* b_2$, and take $\mathcal{H}_1 = 0$ and $T = 0$. In this case, $\mathcal{L}(\mathcal{H}) = M(B)$ (the multiplier algebra of $B$, the largest $C^*$-algebra containing $B$ as an essential ideal), and $\mathcal{K}(\mathcal{H}) = B$. So $\phi$ maps $A$ into $\mathcal{K}(\mathcal{H})$, and even though $T = 0$, the condition that $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ is satisfied for any $a \in A$.
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In applications to index theory, Kasparov \( A-B \)-bimodules typically arise from elliptic (or hypoelliptic) pseudodifferential operators. Kasparov bimodules also arise from quasihomomorphisms.
There is a natural associative addition on Kasparov bimodules, obtained by taking the direct sum of Hilbert $B$-modules and the block direct sum of homomorphisms and operators. Then we divide out by the equivalence relation generated by addition of degenerate Kasparov bimodules (those for which for all $a \in A$, $\phi(a)(T^2 - 1) = 0$ and $[\phi(a), T] = 0$) and by homotopy. (A homotopy of Kasparov $A$-$B$-bimodules is just a Kasparov $A$-$C([0, 1], B)$-bimodule.) Then it turns out that $KK(A, B)$ is actually an abelian group, with inversion given by reversing the grading, i.e., reversing the roles of $H_0$ and $H_1$, and interchanging $F$ and $F^*$. It is not really necessary to divide out by degenerate bimodules, since if $(H, \phi, T)$ is degenerate, then $C_0((0, 1], H)$ (along with the action of $A$ and the operator which are given by $\phi$ and $T$ at each point of $(0, 1]$) is a homotopy from $(H, \phi, T)$ to the 0-module.
Relation with $K$-theory

An interesting exercise is to consider what happens when $A = \mathbb{C}$ and $B$ is a unital $C^*$-algebra. Then if $\mathcal{H}_0$ and $\mathcal{H}_1$ are finitely generated projective (right) $B$-modules and we take $T = 0$ and $\phi$ to be the usual action of $\mathbb{C}$ by scalar multiplication, we get a Kasparov $\mathbb{C}$-$B$-bimodule corresponding to the element $[\mathcal{H}_0] - [\mathcal{H}_1]$ of $K_0(B)$. With some work one can show that this gives an isomorphism between the Grothendieck group $K_0(B)$ of usual $K$-theory and $KK(\mathbb{C}, B)$. By considering what happens when one adjoins a unit, one can then show that there is still a natural isomorphism between $K_0(B)$ and $KK(\mathbb{C}, B)$, even if $B$ is nonunital.
Suppose $A$ and $B$ are **Morita equivalent** in the sense of Rieffel. That means we have an $A$-$B$-bimodule $X$ with the following special properties:

1. $X$ is a right Hilbert $B$-module and a left Hilbert $A$-module.
2. The left action of $A$ is by bounded adjointable operators for the $B$-valued inner product, and the right action of $B$ is by bounded adjointable operators for the $A$-valued inner product.
3. The $A$- and $B$-valued inner products on $X$ are compatible in the sense that if $\xi, \eta, \nu \in X$, then $A\langle \xi, \eta \rangle \nu = \xi \langle \eta, \nu \rangle_B$.
4. The inner products are “full,” in the sense that the image of $A\langle \_ , \_ \rangle$ is dense in $A$, and the image of $\langle \_ , \_ \rangle_B$ is dense in $B$.

Under these circumstances, $X$ defines classes in $[X] \in KK(A, B)$ and $[\tilde{X}] \in KK(B, A)$ which are inverses to each other (with respect to the product discussed below).
The product

The hardest aspect of Kasparov’s approach to $KK$ is to prove that there is a well-defined, functorial, bilinear, and associative product

$$\otimes_B : KK(A, B) \times KK(B, C) \to KK(A, C).$$

There is also an external product

$$\boxdot : KK(A, B) \times KK(C, D) \to KK(A \otimes C, B \otimes D),$$

where $\otimes$ denotes the completed minimal or spatial $C^*$-tensor product.
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There is also an external product
\[ \boxtimes : \text{KK}(A, B) \times \text{KK}(C, D) \to \text{KK}(A \otimes C, B \otimes D), \]
where $\otimes$ denotes the completed minimal or spatial $C^*$-tensor product. The external product is built from the usual product using dilation (external product with 1). We can dilate a class $a \in \text{KK}(A, B)$ to a class $a \boxtimes 1_C \in \text{KK}(A \otimes C, B \otimes C)$, by taking a representative $(\mathcal{H}, \phi, T)$ for $a$ to the bimodule $(\mathcal{H} \otimes C, \phi \otimes 1_C, T \otimes 1)$. Similarly, we can dilate a class $b \in \text{KK}(C, D)$ (on the other side) to a class $1_B \boxtimes b \in \text{KK}(B \otimes C, B \otimes D)$. Then
\[
\begin{align*}
a \boxtimes b &= (a \boxtimes 1_C) \otimes_B \otimes_C (1_B \boxtimes b) \in \text{KK}(A \otimes C, B \otimes D),
\end{align*}
\]
and this is the same as $(1_A \boxtimes b) \otimes_A \otimes_D (a \boxtimes 1_D)$. 
More on the products

The **Kasparov products** include all the usual cup and cap products relating $K$-theory and $K$-homology. For example, the cup product in ordinary topological $K$-theory for a compact space $X$,

$\cup: K(X) \times K(X) \to K(X)$, is a composite of two products:

$$a \cup b = (a \boxtimes b) \otimes_{C(X \times X)} \Delta,$$

where $\Delta \in KK(C(X \times X), C(X))$ is the class of the diagonal map $X \to X \times X$. 
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where $\Delta \in KK(C(X \times X), C(X))$ is the class of the diagonal map $X \to X \times X$. Suppose we have classes represented by $(\mathcal{E}_1, \phi_1, T_1)$ and $(\mathcal{E}_2, \phi_2, T_2)$, where $\mathcal{E}_1$ is a right Hilbert $B$-module, $\mathcal{E}_2$ is a right Hilbert $C$-module, $\phi_1: A \to \mathcal{L}(\mathcal{E}_1)$, $\phi_2: B \to \mathcal{L}(\mathcal{E}_2)$, $T_1$ essentially commutes with the image of $\phi_1$, and $T_2$ essentially commutes with the image of $\phi_2$. It is clear that we want to construct the product using $\mathcal{H} = \mathcal{E}_1 \otimes_{B, \phi_2} \mathcal{E}_2$ and $\phi = \phi_1 \otimes 1: A \to \mathcal{L}(\mathcal{H})$. The main difficulty is getting the correct operator $T$. In fact there is no canonical choice; the choice is only unique up to homotopy, and is defined using the Connes-Skandalis notion of a connection.
Cuntz’s approach

Joachim Cuntz noticed that all Kasparov bimodules come from a \textbf{quasihomomorphism} \( A \cong D \triangleright B \), a formal difference of two homomorphisms \( f_\pm : A \to D \) which agree modulo an ideal isomorphic to \( B \). Thus \( a \mapsto f_+(a) - f_-(a) \) is a linear map \( A \to B \). Suppose for simplicity (one can always reduce to this case) that \( D/B \cong A \), so that \( f_\pm \) are two splittings for an extension \( 0 \to B \to D \to A \to 0 \). Then for any \textit{split-exact} functor \( F \) from \( C^* \)-algebras to abelian groups (meaning it sends split extensions to short exact sequences — an example would be \( F(A) = K(A \otimes C) \) for some coefficient algebra \( C \)), we get an exact sequence

\[
0 \longrightarrow F(B) \longrightarrow F(D) \xrightarrow{(f_+)_*} F(A) \xrightarrow{(f_-)_*} 0.
\]

Thus \((f_+)_* - (f_-)_*\) gives a well-defined homomorphism \( F(A) \to F(B) \), which we might well imagine should come from a class in \( KK(A, B) \).
Cuntz’s universal construction

A quasihomomorphism $A \Rightarrow D \triangleright B$ factors through a universal algebra $qA$. Start with the free product $C^*$-algebra $QA = A \ast A$, the completion of linear combinations of words in two copies of $A$. There is an obvious morphism $QA \rightarrow A$ obtained by identifying the two copies of $A$. The kernel of $QA \rightarrow A$ is called $qA$, and if $0 \rightarrow B \rightarrow D \rightarrow A \rightarrow 0$ is a quasihomomorphism, we get a commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & qA & \rightarrow & QA & \rightarrow & A & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & \\
0 & \rightarrow & B & \rightarrow & D & \rightarrow & A & \rightarrow & 0,
\end{array}
\]

with the first copy of $A$ in $QA$ mapping to $D$ via $f_+$, and the second copy of $A$ in $QA$ mapping to $D$ via $f_-$. In this way $KK(A, B)$ turns out to be simply the set of homotopy classes of $\ast$-homomorphisms from $qA$ to $B \otimes K$.
Higson’s approach

Higson proposed making an additive category $\mathbf{KK}$ whose objects are the separable $C^*$-algebras, and where the morphisms from $A$ to $B$ are given by $KK(A, B)$. Associativity and bilinearity of the Kasparov product, along with properties of the special elements $1_A \in KK(A, A)$, ensure that this is indeed an additive category.
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1. **Matrix stability.** If $A$ is an object in $\mathbf{KK}$ (that is, a separable $C^*$-algebra) and if $e$ is a rank-one projection in $\mathcal{K} = \mathcal{K}(\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space, then the homomorphism $a \mapsto a \otimes e$, viewed as an element of $\text{Hom}(A, A \otimes \mathcal{K})$, is an equivalence in $\mathbf{KK}$, i.e., has an inverse in $KK(A \otimes \mathcal{K}, A)$. 
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2. **Split exactness.** $KK$ takes splits short exact sequences to split short exact sequences (in either variable).
Part II

$K$-theory and $KK$-theory of crossed products
Equivariant Kasparov theory

$G$ will be a second-countable locally compact group. A $G$-$C^*$-algebra will mean a $C^*$-algebra $A$ with a jointly continuous action of $G$ on $A$ by $*$-automorphisms. If $G$ is compact, making $KK$-theory equivariant is straightforward. We just require all algebras and Hilbert modules to be equipped with $G$-actions, we require $\phi: A \to \mathcal{L}(\mathcal{H})$ to be $G$-equivariant, and we require the operator $T \in \mathcal{L}(\mathcal{H})$ to be $G$-invariant. We get groups $KK^G(A, B)$ for (separable, say) $G$-$C^*$-algebras $A$ and $B$, and the same argument as before shows that $KK^G(\mathbb{C}, B) \cong K_0^G(B)$, equivariant $K$-theory. In particular, $KK^G(\mathbb{C}, \mathbb{C}) \cong R(G)$, the representation ring of $G$. For example, if $G$ is compact and abelian, $R(G) \cong \mathbb{Z}[\hat{G}]$, the group ring of the Pontrjagin dual. If $G$ is a compact connected Lie group with maximal torus $T$ and Weyl group $W = N_G(T)/T$, then $R(G) \cong R(T)^W \cong \mathbb{Z}[\hat{T}]^W$. The properties of the Kasparov product all go through, and product with $KK^G(\mathbb{C}, \mathbb{C})$ makes all $KK^G$-groups into modules over the ground ring $R(G)$. 

Jonathan Rosenberg
Applications of noncommutative geometry
The case of noncompact groups

When $G$ is noncompact, the definition and properties of $KK^G$ are considerably more subtle, and were worked out by Kasparov. The problem is that in this case, topological vector spaces with a continuous $G$-action are very rarely completely decomposable, and there are rarely enough $G$-equivariant operators to give anything useful. Kasparov’s solution was to work with $G$-continuous rather than $G$-equivariant Hilbert modules and operators; rather remarkably, these still give a useful theory with all the same formal properties as before. The $KK^G$-groups are again modules over the commutative ring $R(G) = KK^G(\mathbb{C}, \mathbb{C})$, though this ring no longer has such a simple interpretation as before, and in fact, is not known for most connected semisimple Lie groups.
Functorial properties

A few functorial properties of the $KK^G$-groups will be needed below, so we just mention a few of them. First of all, if $H$ is a closed subgroup of $G$, then any $G$-$C^*$-algebra is by restriction also an $H$-$C^*$-algebra, and we have restriction maps

$$KK^G(A, B) \rightarrow KK^H(A, B).$$

To go the other way, we can “induce” an $H$-$C^*$-algebra $A$ to get a $G$-$C^*$-algebra $\text{Ind}_H^G(A)$, defined by

$$\text{Ind}_H^G(A) = \{ f \in C(G, A) \mid f(gh) = h \cdot f(g) \quad \forall g \in G, h \in H,$$

$$\|f(g)\| \rightarrow 0 \text{ as } g \rightarrow \infty \mod H \}.\)

The induced action of $G$ on $\text{Ind}_H^G(A)$ is just left translation. An imprimitivity theorem due to Green shows that $\text{Ind}_H^G(A) \rtimes G$ and $A \rtimes H$ are Morita equivalent. If $A$ and $B$ are $H$-$C^*$-algebras, we then have an induction homomorphism

$$KK^H(A, B) \rightarrow KK^G(\text{Ind}_H^G(A), \text{Ind}_H^G(B)).$$
Basic properties of crossed products

If $A$ is a $G$-$C^*$-algebra, one can define two new $C^*$-algebras, called the full and reduced crossed products of $A$ by $G$, which capture the essence of the group action. These are easiest to define when $G$ is discrete and $A$ is unital. The full crossed product $A \rtimes_\alpha G$ (we often omit the $\alpha$ if there is no possibility of confusion) is the universal $C^*$-algebra generated by a copy of $A$ and unitaries $u_g$, $g \in G$, subject to the commutation condition $u_g a u_g^* = \alpha_g(a)$, where $\alpha$ denotes the action of $G$ on $A$. The reduced crossed product $A \rtimes_{\alpha,r} G$ is the image of $A \rtimes_\alpha G$ in its “regular representation” $\pi$ on $L^2(G, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space on which $A$ acts faithfully, say by a representation $\rho$. Here $A$ acts by $(\pi(a)f)(g) = \rho(\alpha_{g^{-1}}(a))f(g)$ and $G$ acts by left translation.
More general crossed products

In general, the full crossed product is defined as the universal $C^*$-algebra for covariant pairs of a $*$-representation $\rho$ of $A$ and a unitary representation $\pi$ of $G$, satisfying the compatibility condition \[ \pi(g)\rho(a)\pi(g^{-1}) = \rho(\alpha_g(a)). \] It may be constructed by defining a convolution multiplication on $C_c(G, A)$ and then completing in the greatest $C^*$-algebra norm. The reduced crossed product $A \rtimes_{\alpha, r} G$ is again the image of $A \rtimes_{\alpha} G$ in its “regular representation” on $L^2(G, \mathcal{H})$. 
More general crossed products

In general, the full crossed product is defined as the universal \( C^* \)-algebra for **covariant pairs** of a \(*\)-representation \( \rho \) of \( A \) and a unitary representation \( \pi \) of \( G \), satisfying the compatibility condition 
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It may be constructed by defining a convolution multiplication on \( C_c(G, A) \) and then completing in the greatest \( C^* \)-algebra norm. The reduced crossed product \( A \rtimes_{\alpha, r} G \) is again the image of \( A \rtimes_\alpha G \) in its “regular representation” on \( L^2(G, \mathcal{H}) \).

For example, \( \mathbb{C} \rtimes G \) is the group \( C^* \)-algebra \( C^*(G) \), and \( \mathbb{C} \rtimes_r G \) is \( C_r^*(G) \), the image of \( C^*(G) \) in the left regular representation on \( L^2(G) \).

The natural map \( C^*(G) \to C_r^*(G) \) is an isomorphism if and only if \( G \) is **amenable**. When the action \( \alpha \) is trivial, then \( A \rtimes G \) is the maximal tensor product \( A \otimes_{\text{max}} C^*(G) \) while \( A \rtimes_r G \) is the minimal tensor product \( A \otimes C_r^*(G) \). Again, \( A \otimes_{\text{max}} C^*(G) \) to \( A \otimes C_r^*(G) \) is an isomorphism if and only if \( G \) is amenable.
More about crossed products

When $A$ and the action $\alpha$ are arbitrary, the natural map

$$A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$$

is an isomorphism if $G$ is amenable, but also more generally if the action $\alpha$ is amenable in a certain sense. For example, if $X$ is a locally compact $G$-space, the action is automatically amenable if it is proper, whether or not $G$ is amenable.
More about crossed products

When $A$ and the action $\alpha$ are arbitrary, the natural map $A \rtimes_{\alpha} G \to A \rtimes_{\alpha,r} G$ is an isomorphism if $G$ is amenable, but also more generally if the action $\alpha$ is amenable in a certain sense. For example, if $X$ is a locally compact $G$-space, the action is automatically amenable if it is proper, whether or not $G$ is amenable.

When $X$ is a locally compact $G$-space, the crossed product $C_0(G) \rtimes G$ often serves as a good substitute for the “quotient space” $X/G$ in cases where the latter is badly behaved. Indeed, if $G$ acts freely and properly on $X$, then $C_0(X) \rtimes G$ is Morita equivalent to $C_0(X/G)$. But if the $G$-action is not proper, $X/G$ may be highly non-Hausdorff, while $C_0(X) \rtimes G$ may be a perfectly well-behaved noncommutative algebra. A key case later on will the one where $X = \mathbb{T}$ is the circle group, $G = \mathbb{Z}$, and the generator of $G$ acts by multiplication by $e^{2\pi i \theta}$. When $\theta$ is irrational, every orbit is dense, so $X/G$ is an indiscrete space, and $C(\mathbb{T}) \rtimes \mathbb{Z}$ is what’s usually denoted $A_{\theta}$, an irrational rotation algebra or noncommutative 2-torus.
Now we can explain the relationships between equivariant $KK$-theory and crossed products. One connection is that if $G$ is discrete and $A$ is a $G$-$C^*$-algebra, there is a natural isomorphism $KK^G(A, \mathbb{C}) \cong KK(A \rtimes G, \mathbb{C})$. Dually, if $G$ is compact, there is a natural Green-Julg isomorphism $KK^G(\mathbb{C}, A) \cong KK(\mathbb{C}, A \rtimes G)$. 
Now we can explain the relationships between equivariant $KK$-theory and crossed products. One connection is that if $G$ is discrete and $A$ is a $G$-$C^*$-algebra, there is a natural isomorphism $KK^G(A, \mathbb{C}) \cong KK(A \rtimes G, \mathbb{C})$. Dually, if $G$ is compact, there is a natural Green-Julg isomorphism $KK^G(\mathbb{C}, A) \cong KK(\mathbb{C}, A \rtimes G)$.

Still another connection is that there are (for arbitrary $G$) functorial homomorphisms

$$j, j_r : KK^G(A, B) \to KK(A \rtimes G, B \rtimes G), \ KK(A \rtimes_r G, B \rtimes_r G)$$

sending (when $B = A$) $1_A$ to $1_{A \rtimes G}$. (In fact, $j, j_r$ can be viewed as functors from the equivariant Kasparov category $KK^G$ to the non-equivariant Kasparov category $KK$. Later we will study how close they are to being faithful.) If $B = \mathbb{C}$ and $G$ is discrete, then $j : KK^G(A, \mathbb{C}) \to KK(A \rtimes G, C^*(G))$ is split injective, and if $G$ is compact, then $j : KK^G(\mathbb{C}, A) \to KK(C^*(G), A \rtimes G)$ is split injective.
The dual action and Takai duality

When the group $G$ is not just locally compact but also abelian, then it has a Pontrjagin dual group $\hat{G}$. In this case, given any $G$-$C^*$-algebra algebra $A$, say with $\alpha$ denoting the action of $G$ on $A$, there is a dual action $\hat{\alpha}$ of $\hat{G}$ on the crossed product $A \rtimes G$. When $A$ is unital and $G$ is discrete, so that $A \rtimes G$ is generated by a copy of $A$ and unitaries $u_g$, $g \in G$, the dual action is given simply by

$$\hat{\alpha}_{\gamma}(au_g) = au_g\langle g, \gamma \rangle.$$ 

The same formula still applies in general, except that the elements $a$ and $u_g$ don’t quite live in the crossed product but in a larger algebra. The key fact about the dual action is the Takai duality theorem: $(A \rtimes_{\alpha} G) \rtimes_{\hat{\alpha}} \hat{G} \cong A \otimes \mathcal{K}(L^2(G))$, and the double dual action $\hat{\hat{\alpha}}$ of $\hat{\hat{G}} \cong G$ on this algebra can be identified with $\alpha \otimes \text{Ad} \lambda$, where $\lambda$ is the left regular representation of $G$ on $L^2(G)$. 

Jonathan Rosenberg

Applications of noncommutative geometry
If $\mathbb{C}^n$ (or $\mathbb{R}^{2n}$) acts on $X$ by a trivial action $\alpha$, then
$$C_0(X) \rtimes_\alpha \mathbb{C}^n \cong C_0(X) \otimes C^*(\mathbb{C}^n) \cong C_0(X) \otimes C_0(\hat{\mathbb{C}}^n) \cong C_0(E),$$
where $E$ is a trivial rank-$n$ complex vector bundle over $X$. (We have used Pontrjagin duality and the fact that abelian groups are amenable.) It follows that $K(C_0(X)) \cong K(C_0(X) \rtimes_\alpha \mathbb{C}^n)$. Since any action $\alpha$ of $\mathbb{C}^n$ is homotopic to the trivial action and "$K$-theory is supposed to be homotopy invariant," that suggests that perhaps $KK(A) \cong KK(A \rtimes_\alpha \mathbb{C}^n)$ for any $C^*$-algebra $A$ and for any action $\alpha$ of $\mathbb{C}^n$. This is indeed true and the isomorphism is implemented by classes (which are inverse to one another) in $KK(A, A \rtimes_\alpha \mathbb{C}^n)$ and $KK(A \rtimes_\alpha \mathbb{C}^n, A)$. It is clearly enough to prove this in the case $n = 1$, since we can always break a crossed product by $\mathbb{C}^n$ up as an $n$-fold iterated crossed product.
Connes’ Theorem

That $A$ and $A \rtimes_\alpha \mathbb{C}$ are always $KK$-equivalent or that they at least have the same $K$-theory, or (this is equivalent since one can always suspend on both sides) that $A \otimes C_0(\mathbb{R})$ and $A \rtimes_\alpha \mathbb{R}$ are always $KK$-equivalent or that they at least have the same $K$-theory for any action of $\mathbb{R}$, is called Connes’ “Thom isomorphism”. Connes’ original proof is relatively elementary, but only gives an isomorphism of $K$-groups, not a $KK$-equivalence.
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To illustrate Connes’ idea, let’s suppose $A$ is unital and we have a class in $K_0(A)$ represented by a projection $p \in A$. (One can always reduce to this special case.) If $\alpha$ were to fix $p$, then $1 \mapsto p$ gives an equivariant map from $\mathbb{C}$ to $A$ and thus would induce a map of crossed products $\mathbb{C} \rtimes \mathbb{R} \cong C_0(\hat{\mathbb{R}}) \to A \rtimes_\alpha \mathbb{R}$ or $\mathbb{C} \rtimes \mathbb{C} \cong C_0(\hat{\mathbb{C}}) \to A \rtimes_\alpha \mathbb{C}$ giving a map on $K$-theory $\beta: \mathbb{Z} \to K_0(A \rtimes \mathbb{C})$. The image of $[p]$ under the isomorphism $K_0(A) \to K_0(A \rtimes \mathbb{C})$ will be $\beta(1)$. So the idea is to show that one can modify the action to one fixing $p$ (using a cocycle conjugacy) without changing the isomorphism class of the crossed product.
There are now quite a number of proofs of Connes’ theorem available, each using somewhat different techniques. We just mention a few of them. A proof using $K$-theory of Wiener-Hopf extensions was given by Rieffel. There are also fancier proofs using $KK$-theory. If $\alpha$ is a given action of $\mathbb{R}$ on $A$ and if $\beta$ is the trivial action, one can try to construct $KK^\mathbb{R}$ elements $c \in KK^\mathbb{R}((A, \alpha), (A, \beta))$ and $d \in KK^\mathbb{R}((A, \beta), (A, \alpha))$ which are inverses of each other in $KK^\mathbb{R}$. Then the morphism $j$ of Section 1 sends these to $KK$-equivalences $j(c)$ and $j(d)$ between $A \rtimes_\alpha \mathbb{R}$ and $A \rtimes_\beta \mathbb{R} \cong A \otimes C_0(\mathbb{R})$. 
Proofs of Connes’ Theorem

There are now quite a number of proofs of Connes’ theorem available, each using somewhat different techniques. We just mention a few of them. A proof using $K$-theory of Wiener-Hopf extensions was given by Rieffel. There are also fancier proofs using $KK$-theory. If $\alpha$ is a given action of $\mathbb{R}$ on $A$ and if $\beta$ is the trivial action, one can try to construct $KK^\mathbb{R}$ elements $c \in KK^\mathbb{R}((A, \alpha), (A, \beta))$ and $d \in KK^\mathbb{R}((A, \beta), (A, \alpha))$ which are inverses of each other in $KK^\mathbb{R}$. Then the morphism $j$ of Section 1 sends these to $KK$-equivalences $j(c)$ and $j(d)$ between $A \rtimes \alpha \mathbb{R}$ and $A \rtimes \beta \mathbb{R} \cong A \otimes C_0(\mathbb{R})$.

Fack and Skandalis give another proof using the group $KK^1(A, B)$. This is defined with triples $(\mathcal{H}, \phi, T)$ like those used for $KK(A, B)$, but with two modifications.
The proof of Fack and Skandalis

Conditions for $KK^1$:

1. $\mathcal{H}$ is no longer graded, and there is no grading condition on $\phi$.

2. $T$ is self-adjoint but with no grading condition, and $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ and $[\phi(a), T] \in \mathcal{K}(\mathcal{H})$ for all $a \in A$.

It turns out that $KK^1(A, B) \cong KK(A \otimes C_0(\mathbb{R}), B)$, and that the Kasparov product can be extended to a graded commutative product on the direct sum of $KK = KK^0$ and $KK^1$. The product of two classes in $KK^1$ can by Bott periodicity be taken to land in $KK^0$. 
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We can now explain the proof of Fack and Skandalis as follows. They show that for each separable $C^*$-algebra $A$ with an action $\alpha$ of $\mathbb{R}$, there is a special element $t_\alpha \in KK^1(A, A \rtimes_\alpha \mathbb{R})$ (constructed using a singular integral operator). Note by the way that doing the construction with the dual action and applying Takai duality gives $t_\widehat{\alpha} \in KK^1(A \rtimes_\alpha \mathbb{R}, A)$, since $(A \rtimes_\alpha \mathbb{R}) \rtimes_\widehat{\alpha} \mathbb{R} \cong A \otimes \mathcal{K}$, which is Morita equivalent to $A$. 
The elements $t_\alpha$

These elements have the following properties:

1. **Normalization** If $A = \mathbb{C}$ (so that necessarily $\alpha = 1$ is trivial), then $t_1 \in KK^1(\mathbb{C}, C_0(\mathbb{R}))$ is the usual generator of this group (which is isomorphic to $\mathbb{Z}$).

2. **Naturality** The elements are natural with respect to equivariant homomorphisms $\rho : (A, \alpha) \to (C, \gamma)$, in that if $\bar{\rho}$ denotes the induced map on crossed products, then
   
   \[ \bar{\rho}^*(t_\alpha) = \rho^*(t_\gamma) \in KK(A, C \rtimes_\gamma \mathbb{R}), \]
   
   and similarly,
   
   \[ \bar{\rho}^*(t_{\bar{\alpha}}) = \rho^*(t_{\bar{\gamma}}) \in KK(A \rtimes_\alpha \mathbb{R}, C). \]

3. **Compatibility with external products** Given $x \in KK(A, B)$ and $y \in KK(C, D)$,

   \[ (t_{\bar{\alpha}} \otimes_A x) \boxtimes y = t_{\alpha \otimes 1_C} \otimes_A C \otimes (x \boxtimes y). \]

   Similarly, given $x \in KK(B, A)$ and $y \in KK(D, C)$,

   \[ y \boxtimes (x \otimes_A t_\alpha) = (y \boxtimes x) \otimes_C A t_{1_C \otimes \alpha}. \]

\[ \square \]
Idea of the proof of Fack-Skandalis

**Theorem (Fack-Skandalis)**

*These properties completely determine \( t_\alpha \), and \( t_\alpha \) is a KK-equivalence (of degree 1) between \( A \) and \( A \rtimes_\alpha \mathbb{R} \).*
The Pimsner-Voiculescu Theorem

Now suppose \( A \) is a \( C^* \)-algebra equipped with an action \( \alpha \) of \( \mathbb{Z} \) (or equivalently, a single \(*\)-automorphism \( \theta \), the image of \( 1 \in \mathbb{Z} \) under the action). Then \( A \rtimes_{\alpha} \mathbb{Z} \) is Morita equivalent to \( \left( \text{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha) \right) \rtimes \mathbb{R} \). The algebra \( T_{\theta} = \text{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha) \) is often called the mapping torus of \((A, \theta)\); it can be identified with the algebra of continuous functions \( f : [0, 1] \to A \) with \( f(1) = \theta(f(0)) \). It comes with an obvious short exact sequence

\[
0 \rightarrow C_0((0, 1), A) \rightarrow T_\theta \rightarrow A \rightarrow 0,
\]

for which the associated exact sequence in \( K \)-theory has the form

\[
\cdots \rightarrow K_1(A) \xrightarrow{1-\theta_*} K_1(A) \rightarrow K_0(T_\theta) \rightarrow K_0(A) \xrightarrow{1-\theta_*} K_0(A) \rightarrow \cdots.
\]

Since \( K_0(A \rtimes_{\alpha} \mathbb{Z}) \cong K_0(T_\theta \rtimes_{\text{Ind}_{\alpha}} \mathbb{R}) \cong K_1(T_\theta) \), and similarly for \( K_0 \), we obtain the Pimsner-Voiculescu exact sequence

\[
\cdots \rightarrow K_1(A) \xrightarrow{1-\theta_*} K_1(A) \xrightarrow{\iota_*} K_1(A \rtimes_{\alpha} \mathbb{Z}) \rightarrow K_0(A) \xrightarrow{1-\theta_*} K_0(A) \xrightarrow{\iota_*} K_0(A \rtimes_{\alpha} \mathbb{Z}) \rightarrow \cdots. \tag{2}
\]
The Baum-Connes Conjecture (without coefficients)

Let $G$ be a locally compact group, and let $EG$ be the universal proper $G$-space. (This is a contractible space on which $G$ acts properly, characterized up to $G$-homotopy equivalence by two properties: that every compact subgroup of $G$ has a fixed point in $EG$, and that the two projections $EG \times EG \to EG$ are $G$-homotopic. If $G$ has no compact subgroups, then $EG$ is the usual universal free $G$-space $EG$.)
The Baum-Connes Conjecture (without coefficients)

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Conjecture (Baum-Connes)

Let $G$ be a locally compact group, second-countable for convenience. There is an assembly map

$$\lim_{X \subseteq EG, X/G \text{ compact}} K^G_*(X) \to K_*(C^*_r(G))$$

defined by taking $G$-indices of $G$-invariant elliptic operators, and this map is an isomorphism.
The Baum-Connes Conjecture with coefficients

Conjecture (Baum-Connes with coefficients)

With notation as in the previous Conjecture, if $A$ is any separable $G$-$C^*$-algebra, the assembly map

$$
\lim_{\substack{\text{X compact} \\ X \subseteq EG}} KK^G_*(C_0(X), A) \rightarrow K_*(A \rtimes_r G)
$$

is an isomorphism.
Special cases

If $G$ is compact, $EG$ can be taken to be a single point. The conjecture then asserts that the *assembly map* $KK^*_G(pt, A) \to K_*(A \rtimes G)$ is an isomorphism. This is true by the Green-Julg theorem.
Special cases

If $G$ is compact, $E_G$ can be taken to be a single point. The conjecture then asserts that the assembly map $KK_*^G(\text{pt}, A) \to K_*(A \times G)$ is an isomorphism. This is true by the the Green-Julg theorem.

If $G = \mathbb{R}$, we can take $E_G = G = \mathbb{R}$. If $A$ is an $\mathbb{R}$-$C^*$-algebra, the assembly map is a map $KK_*^\mathbb{R}(C_0(\mathbb{R}), A) \to K_*(A \rtimes \mathbb{R})$. This map turns out to be Kasparov’s morphism $j : KK_*^\mathbb{R}(C_0(\mathbb{R}), A) \to KK_*^\mathbb{R}(C_0(\mathbb{R}) \rtimes \mathbb{R}, A \rtimes \mathbb{R}) = KK_*^\mathbb{K}(\mathcal{K}, A \rtimes \mathbb{R}) \cong K_*(A \rtimes \mathbb{R})$, which is the isomorphism of Connes’ Theorem.
Special cases

If $G$ is compact, $EG$ can be taken to be a single point. The conjecture then asserts that the assembly map $KK^G_*(\text{pt}, A) \to K_*(A \rtimes G)$ is an isomorphism. This is true by the Green-Julg theorem.

If $G = \mathbb{R}$, we can take $EG = G = \mathbb{R}$. If $A$ is an $\mathbb{R}$-$\mathrm{C}^*$-algebra, the assembly map is a map $KK^\mathbb{R}_*(C_0(\mathbb{R}), A) \to K_*(A \rtimes \mathbb{R})$. This map turns out to be Kasparov’s morphism

$$j: KK^\mathbb{R}_*(C_0(\mathbb{R}), A) \to KK^\mathbb{R}_*(C_0(\mathbb{R}) \rtimes \mathbb{R}, A \rtimes \mathbb{R}) = KK^\mathbb{R}_*(\mathcal{K}, A \rtimes \mathbb{R}) \cong K_*(A \rtimes \mathbb{R}),$$

which is the isomorphism of Connes’ Theorem.

Now suppose $G$ is discrete and torsion-free. Then $EG = EG$, and the quotient space $EG/G$ is the usual classifying space $BG$. The assembly map $K_{\text{cmpct}}^*(BG) \to K_*(C_r^*(G))$ can be viewed as an index map, since classes in the $K$-homology group on the left are represented by generalized Dirac operators $D$ over Spin$^c$ manifolds $M$ with a $G$-covering, and the assembly map takes such an operator to its “Mishchenko-Fomenko index”. The conjecture (without coefficients) implies a strong form of the Novikov Conjecture for $G$. 
Meyer and Nest gave an alternative approach. They observe that the equivariant $KK$-category, $\text{KK}^G$, is a triangulated category. It has a distinguished class $\mathcal{E}$ of weak equivalences, morphisms $f \in \text{KK}^G(A, B)$ which restrict to equivalences in $\text{KK}^H(A, B)$ for every compact subgroup $H$ of $G$. The Baum-Connes Conjecture with coefficients basically amounts to the assertion that if $f \in \text{KK}^G(A, B)$ is in $\mathcal{E}$, then $j_r(f) \in \text{KK}(A \rtimes_r G, B \rtimes_r G)$ is a $KK$-equivalence. In particular, suppose $G$ has no nontrivial compact subgroups and satisfies B-C with coefficients. Then if $A$ is a $G$-$C^*$-algebra which, forgetting the $G$-action, is contractible, then the unique morphism in $\text{KK}^G(0, A)$ is a weak equivalence, and so (applying $j_r$), the unique morphism in $\text{KK}(0, A \rtimes_r G)$ is a $KK$-equivalence. Thus $A \rtimes_r G$ is $K$-contractible, i.e., all of its topological $K$-groups must vanish. When $G = \mathbb{R}$, this follows from Connes’ Theorem, and when $G = \mathbb{Z}$, this follows from the Pimsner-Voiculescu exact sequence.

Jonathan Rosenberg

Applications of noncommutative geometry
There is no known counterexample to Baum-Connes for groups, without coefficients. Counterexamples are now known to Baum-Connes with coefficients (Higson-Lafforgue-Skandalis).
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Baum-Connes with coefficients is true if $G$ is amenable, or more generally, if it is $a$-$T$-menable (Higson-Kasparov), that is, if it has an affine, isometric and metrically proper action on a Hilbert space. Such groups include $SO(n,1)$ or $SU(n,1)$. 
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Baum-Connes without coefficients is true for connected reductive Lie groups, connected reductive $p$-adic groups, for hyperbolic discrete groups, and for cocompact lattice subgroups of $Sp(n, 1)$ or $SL(3, \mathbb{C})$ (Lafforgue).
Current status of Baum-Connes

1. There is no known counterexample to Baum-Connes for groups, without coefficients. Counterexamples are now known to Baum-Connes with coefficients (Higson-Lafforgue-Skandalis).

2. Baum-Connes with coefficients is true if $G$ is amenable, or more generally, if it is $a$-$T$-menable (Higson-Kasparov), that is, if it has an affine, isometric and metrically proper action on a Hilbert space. Such groups include $SO(n, 1)$ or $SU(n, 1)$.

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4. There is a vast literature; this is just for starters.
Part III

The universal coefficient theorem for $KK$ and some of its applications
Introduction to the UCT

Now that we have discussed $KK$ and $KK^G$, a natural question arises: how computable are they? In particular, is $KK(A, B)$ determined by $K_*(A)$ and by $K_*(B)$? Is $KK^G(A, B)$ determined by $K^*_G(A)$ and by $K^*_G(B)$?
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A first step was taken by Kasparov: he pointed out that $KK(X, Y)$ is given by an explicit topological formula when $X$ and $Y$ are finite CW complexes.
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A first step was taken by Kasparov: he pointed out that $KK(X, Y)$ is given by an explicit topological formula when $X$ and $Y$ are finite CW complexes.

Let’s make a definition — we say the pair of $C^*$-algebras $(A, B)$ satisfies the Universal Coefficient Theorem for $KK$ (or UCT for short) if there is an exact sequence

$$0 \to \bigoplus_{* \in \mathbb{Z}/2} \text{Ext}^1_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \to KK(A, B) \xrightarrow{\varphi} \bigoplus_{* \in \mathbb{Z}/2} \text{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0.$$ 

Here $\varphi$ sends a $KK$-class to the induced map on $K$-groups.
The UCT

We need one more definition. Let $\mathcal{B}$ be the bootstrap category, the smallest full subcategory of the separable $C^*$-algebras containing all separable type I algebras, and closed under extensions, countable $C^*$-inductive limits, and $KK$-equivalences. Note that $KK$-equivalences include Morita equivalences, and type I algebras include commutative algebras.

Theorem (Rosenberg-Schochet)
The UCT holds for all pairs $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}$ an object in $\mathcal{B}$ and $\mathcal{B}$ separable.

Unsolved problem: Is every separable nuclear $C^*$-algebra in $\mathcal{B}$?

Skandalis showed that there are non-nuclear algebras not in $\mathcal{B}$.
The universal coefficient theorem for $KK$

Applications of the UCT

The UCT

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Applications of noncommutative geometry
The proof of Rosenberg and Schochet

First suppose $K_*(B)$ is injective as a $\mathbb{Z}$-module, i.e., divisible as an abelian group. Then $\text{Hom}_\mathbb{Z}(\_ , K_*(B))$ is an exact functor, so $A \mapsto \text{Hom}_\mathbb{Z}(K_*(A), K_*(B))$ gives a cohomology theory on $C^*$-algebras. In particular, $\varphi$ is a natural transformation of homology theories

$$(X \mapsto KK_*(C_0(X), B)) \leadsto (X \mapsto \text{Hom}_\mathbb{Z}(K^*(X), K_*(B))).$$
The proof of Rosenberg and Schochet

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$$(X \mapsto KK_*(C_0(X), B)) \sim (X \mapsto \text{Hom}_{\mathbb{Z}}(K^*(X), K_*(B))).$$

Since $\varphi$ is an isomorphism for $X = \mathbb{R}^n$ by Bott periodicity, it is an isomorphism whenever $X_+$ is a finite CW complex.
The proof of Rosenberg and Schochet

First suppose $K_\ast(B)$ is injective as a $\mathbb{Z}$-module, i.e., divisible as an abelian group. Then $\text{Hom}_{\mathbb{Z}}(\_, K_\ast(B))$ is an exact functor, so $A \mapsto \text{Hom}_{\mathbb{Z}}(K_\ast(A), K_\ast(B))$ gives a cohomology theory on $C^\ast$-algebras. In particular, $\varphi$ is a natural transformation of homology theories

$$(X \mapsto KK_\ast(C_0(X), B)) \rightsquigarrow (X \mapsto \text{Hom}_{\mathbb{Z}}(K_\ast(X), K_\ast(B))).$$

Since $\varphi$ is an isomorphism for $X = \mathbb{R}^n$ by Bott periodicity, it is an isomorphism whenever $X_+$ is a finite CW complex. We extend to arbitrary locally compact $X$ by taking limits, and then to the rest of $\mathcal{B}$. (Type I $C^\ast$-algebras are colimits of iterated extensions of stably commutative algebras.) So the theorem holds when $K_\ast(B)$ is injective.
Geometric resolutions

The rest of the proof uses an idea due to Atiyah, of geometric resolutions. The idea is that given arbitrary $B$, we can change it up to $KK$-equivalence so that it fits into a short exact sequence

$$0 \rightarrow C \rightarrow B \rightarrow D \rightarrow 0$$

for which the induced $K$-theory sequence is short exact: $K_*(B) \hookrightarrow K_*(D) \rightarrow K_{*-1}(C)$ and $K_*(D), K_*(C)$ are $\mathbb{Z}$-injective. Then we use the theorem for $KK_*(A, D)$ and $KK_*(A, C)$, along with the long exact sequence in $KK$ in the second variable, to get the UCT for $(A, B)$. 
The equivariant case

If one asks about the UCT in the equivariant case, then the homological algebra of the ground ring $R(G)$ becomes relevant. This is not always well behaved, so as noticed by Hodgkin, one needs restrictions on $G$ to get anywhere. But for $G$ a connected compact Lie group with $\pi_1(G)$ torsion-free, $R(G)$ has finite global dimension.
The equivariant case

If one asks about the UCT in the equivariant case, then the homological algebra of the ground ring $R(G)$ becomes relevant. This is not always well behaved, so as noticed by Hodgkin, one needs restrictions on $G$ to get anywhere. But for $G$ a connected compact Lie group with $\pi_1(G)$ torsion-free, $R(G)$ has finite global dimension.

**Theorem (Rosenberg-Schochet)**

If $G$ is a connected compact Lie group with $\pi_1(G)$ torsion-free, and if $A$, $B$ are separable $G$-$C^*$-algebras with $A$ in a suitable bootstrap category containing all commutative $G$-$C^*$-algebras, then there is a convergent spectral sequence

$$\text{Ext}^p_{R(G)}(K_*^G(A), K_*^{G+}(A)) \Rightarrow KK_*^G(A, B).$$

The proof is more complicated than in the non-equivariant case, but in the same spirit.
The UCT implies a lot of interesting facts about the bootstrap category $\mathcal{B}$. Here are a few examples.
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**Theorem (Rosenberg-Schochet)**

Let $A, B$ be $C^*$-algebras in $\mathcal{B}$. Then $A$ and $B$ are $KK$-equivalent if and only if they have the isomorphic topological $K$-groups.
The UCT implies a lot of interesting facts about the bootstrap category $\mathcal{B}$. Here are a few examples.

**Theorem (Rosenberg-Schochet)**

Let $A$, $B$ be $C^*$-algebras in $\mathcal{B}$. Then $A$ and $B$ are $KK$-equivalent if and only if they have the isomorphic topological $K$-groups.

**Proof.**

$\Rightarrow$ is trivial. So suppose $K_*(A) \cong K_*(B)$. Choose an isomorphism $\psi: K_*(A) \rightarrow K_*(B)$. Since the map $\varphi$ in the UCT is surjective, $\psi$ is realized by a class $x \in KK(A, B)$. 

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Applications of noncommutative geometry
The universal coefficient theorem for $KK$

Applications of the UCT

The $KK$-equivalence theorem (cont’d)

Proof (cont’d).

Now consider the commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext}^1(K_{*+1}(B),K_*(A)) & \longrightarrow & KK_*(B,A) & \overset{\varphi}{\longrightarrow} & \text{Hom}(K_*(B),K_*(A)) & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & \text{Ext}^1(K_{*+1}(A),K_*(A)) & \longrightarrow & KK_*(A,A) & \overset{\varphi}{\longrightarrow} & \text{Hom}(K_*(A),K_*(A)) & \longrightarrow & 0
\end{array}
$$

By the 5-Lemma, Kasparov product with $x$ is an isomorphism $KK_*(B,A) \to KK_*(A,A)$. In particular, there exists $y \in KK(B,A)$ with $x \otimes_B y = 1_A$. Similarly, there exists $z \in KK(B,A)$ with $z \otimes_A x = 1_B$. Then by associativity

$$
z = z \otimes_A (x \otimes_B y) = (z \otimes_A x) \otimes_B y = y
$$

and we have a $KK$-inverse to $x$. □
The \textbf{KK} ring

Recall that $\text{KK}(A, A) = \text{End}_{\text{KK}}(A)$ is a ring under Kasparov product.

\textbf{Theorem (Rosenberg-Schochet)}

Suppose $A$ is in $\mathcal{B}$. In the UCT sequence

$$0 \rightarrow \bigoplus_{i \in \mathbb{Z}/2} \text{Ext}^1_\mathbb{Z}(K_{i+1}(A), K_i(A)) \rightarrow \text{KK}(A, A) \xrightarrow{\varphi} \bigoplus_{i \in \mathbb{Z}/2} \text{End}(K_i(A)) \rightarrow 0,$$

$\varphi$ is a split surjective homomorphism of rings, and $J = \ker \varphi$ (the Ext term) is an ideal with $J^2 = 0$.

\textbf{Proof.}

Choose $A_0$ and $A_1$ commutative with $K_0(A_0) \cong K_0(A)$, $K_1(A_0) = 0$, $K_0(A_1) = 0$, $K_1(A_1) \cong K_1(A)$. Then by the last theorem, $A_0 \oplus A_1$ is \text{KK}-equivalent to $A$, and we may assume $A = A_0 \oplus A_1$. By the UCT, $\text{KK}(A_0, A_0) \cong \text{End} K_0(A)$ and $\text{KK}(A_1, A_1) \cong \text{End} K_1(A)$. 

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Applications of noncommutative geometry
The KK-ring (cont’d)

Proof.

So $KK(A_0, A_0) \oplus KK(A_1, A_1)$ is a subring of $KK(A, A)$ mapping isomorphically under $\varphi$. This shows $\varphi$ is split surjective. We also have $J = KK(A_0, A_1) \oplus KK(A_1, A_0)$. If, say, $x$ lies in the first summand and $y$ in the second, then $x \otimes_{A_1} y$ induces the 0-map on $K_0(A)$ and so is 0 in $KK(A_0, A_0)$. Similarly, $y \otimes_{A_0} x$ induces the 0-map on $K_1(A)$ and so is 0 in $KK(A_1, A_1)$. □
There is a homotopy-theoretic approach to the UCT that topologists might find attractive; it seems to have been discovered independently by several people. Let $A$ and $B$ be $C^*$-algebras and let $\mathbb{K}(A)$ and $\mathbb{K}(B)$ be their topological $K$-theory spectra. These are module spectra over $\mathbb{K} = \mathbb{K}(\mathbb{C})$, the usual spectrum of complex $K$-theory. Then we can define

$$KK^{\text{top}}(A, B) = \pi_0(\text{Hom}_\mathbb{K}(\mathbb{K}(A), \mathbb{K}(B))).$$

**Theorem**

There is a natural map $KK(A, B) \to KK^{\text{top}}(A, B)$, and it’s an isomorphism if and only if the UCT holds for the pair $(A, B)$.

Observe that $KK^{\text{top}}(A, B)$ even makes sense for Banach algebras, and always comes with a UCT.
An application of \( KK^{\text{top}} \)

We promised in the first lecture to show that defining \( KK(X, Y) \) to be the set of natural transformations

\[
(Z \mapsto K(X \times Z)) \rightsquigarrow (Z \mapsto K(Y \times Z))
\]

indeed agrees with Kasparov’s \( KK(C_0(X), C_0(Y)) \). Indeed, \( Z \mapsto K(X \times Z) \) is basically the cohomology theory defined by \( K(X) \), and \( Z \mapsto K(Y \times Z) \) is similarly the cohomology theory defined by \( K(Y) \). So the natural transformations (commuting with Bott periodicity) are basically a model for \( KK^{\text{top}}(C_0(X), C_0(Y)) \).
The UCT can be used to prove facts about topological $K$-theory which on their face have nothing to do with $C^*$-algebras or $KK$. For example, we have the following purely topological fact:

**Theorem**

Let $X$ and $Y$ be locally compact spaces such that $K^*(X) \cong K^*(Y)$ just as abelian groups. Then the associated $K$-theory spectra $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ are homotopy equivalent.

**Proof.**

We have seen that the hypothesis implies $C_0(X)$ and $C_0(Y)$ are $KK$-equivalent, which gives the desired conclusion.

Note that this theorem is quite special to complex $K$-theory; it fails even for ordinary cohomology (since one needs to consider the action of the Steenrod algebra).
Similarly, the UCT implies facts about cohomology operations in complex $K$-theory and $K$-theory mod $p$. For example, one has:

**Theorem (Rosenberg-Schochet)**

The $\mathbb{Z}/2$-graded ring of homology operations for $K(\_; \mathbb{Z}/n)$ on the category of separable $C^*$-algebras is the exterior algebra over $\mathbb{Z}/n$ on a single generator, the Bockstein $\beta$.

**Theorem (Araki-Toda, new proof by Rosenberg-Schochet)**

There are exactly $n$ admissible multiplications on $K$-theory mod $n$. When $n$ is odd, exactly one is commutative. When $n = 2$, neither is commutative.
Applications to $C^*$-algebras

Probably the most interesting applications of the UCT for $KK$ are to the classification problem for nuclear $C^*$-algebras. The Elliott program (to quote M. Rørdam) is to classify “all separable, nuclear $C^*$-algebras in terms of an invariant that has $K$-theory as an important ingredient.” Kirchberg and Phillips have shown how to do this for Kirchberg algebras, that is simple, purely infinite, separable and nuclear $C^*$-algebras. The UCT for $KK$ is a key ingredient.
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**Theorem (Kirchberg-Phillips)**

Two stable Kirchberg algebras $A$ and $B$ are isomorphic if and only if they are $KK$-equivalent; and moreover every invertible element in $KK(A, B)$ lifts to an isomorphism $A \to B$. Similarly in the unital case if one keeps track of $[1_A] \in K_0(A)$. 

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Applications of noncommutative geometry
We will not attempt to explain the proof of Kirchberg-Phillips, but it’s based on the idea that a $KK$-class is given by a quasihomomorphism, which under the specific hypotheses can be lifted to a true homomorphism.
We will not attempt to explain the proof of Kirchberg-Phillips, but it’s based on the idea that a $KK$-class is given by a quasihomomorphism, which under the specific hypotheses can be lifted to a true homomorphism. Given the Kirchberg-Phillips result, one is still left with the question of determining when two Kirchberg algebras are $KK$-equivalent. But those of “Cuntz type” (like $O_n$) lie in $B$, and Kirchberg and Phillips show that $\forall$ abelian groups $G_0$ and $G_1$ and $\forall g \in G_0$, there is a nonunital Kirchberg algebra $A \in B$ with these $K$-groups, and there is a unital Kirchberg algebra $A \in B$ with these $K$-groups and with $[1_A] = g$. By the UCT, these algebras are classified by their $K$-groups.
The original work on the Elliott program dealt with the opposite extreme: stably finite algebras. Here again, $KK$ can play a useful role. Here is a typical result from the vast literature:

**Theorem (Elliott)**

If $A$ and $B$ are $C^*$-algebras of real rank 0 which are inductive limits of certain “basic building blocks”, then any $x \in KK(A, B)$ preserving the “graded dimension range” can be lifted to a $*$-homomorphism. If $x$ is a $KK$-equivalence, it can be lifted to an isomorphism.

This theorem applies for example to the irrational rotation algebras $A_\theta$. 