Transverse geometry

The ‘space of leaves’ of a foliation \((V, \mathcal{F})\) can be described in terms of \((M, \Gamma)\), with \(M = \text{complete transversal}\) and \(\Gamma = \text{holonomy pseudogroup}\). The ‘natural’ ‘transverse coordinates’ form the crossed product algebra

\[
\mathcal{A}^\Gamma_M := C_c^\infty(M) \rtimes \Gamma,
\]

consisting of finite sums of monomials of the form

\[
\sum f U^*_\phi, \quad f \in C_c^\infty(FM), \phi \in \Gamma,
\]

with the product

\[
f U^*_\phi \cdot g U^*_\psi = (f \cdot g|\phi) U^*_\psi\phi.
\]

How to find a geometric structure = spectral triple that is ‘invariant’ under the holonomy? \(D\) cannot be taken elliptic, unless the foliation admits a transverse Riemannian structure.
Foliation

Transversals

**Diff\(^+\)(M)-invariant structure**

First, one replaces \(M\) by \(PM = F^+M/\text{SO}(n)\), where \(F^+M = J^1(M) = \text{GL}^+(n, \mathbb{R})\)-principal bundle of oriented frames on \(M\). The sections of \(\pi : PM \to M\) are precisely the Riemannian metrics on \(M\).

**Canonical structure on \(PM\):** the vertical sub-bundle \(\mathcal{V} \subset T(PM), \mathcal{V} = \text{Ker} \pi_*\), has \(\text{GL}^+(n, \mathbb{R})\)-invariant Riemannian metric, since its fibers \(\cong \text{GL}^+(n, \mathbb{R})/\text{SO}(n)\). The bundle \(\mathcal{N} = T(PM)/\mathcal{V}\) has tautological Riemannian structure: every point \(q \in PM\) is an Euclidean structure on \(T_{\pi(q)}(M) \cong \mathcal{N}_q\) via \(\pi_*\).
Hypoelliptic signature operator

The hypoelliptic signature operator $D$ on $PM$ is uniquely determined by $Q = D|D|$, 

$$Q = (d^*_{V} d_{V} - d_{V} d^*_{V}) \oplus \gamma_{V} (d_{H} + d^*_{H}),$$

acting on $\mathcal{H}_{PM} = L^2(\wedge V^* \otimes \wedge N^*, \text{vol}_{PM});$

$d_{V} =$ vertical exterior derivative,

$\gamma_{V} =$ grading for the vertical signature,

$d_{H} =$ horizontal exterior differentiation with respect to a torsion-free connection,

$\text{vol}_{PM} = \text{Diff}^{+}(M)$-invariant volume form.

*If $n \equiv 1$ or $2 \pmod{4}$, one takes $PM \times S^1$ so that the dimension of the vertical fiber be even.
Theorem 1. The operator $Q$ is selfadjoint and so is $D$ defined by $Q = D|D|$. Moreover, $(\mathcal{A}_P^\Gamma, \mathcal{H}_P, D)$ is a (nonunital) spectral triple with simple dimension spectrum $\Sigma_P = \{k \in \mathbb{Z}^+, \ k \leq p := \frac{n(n+1)}{2} + 2n\}$.

Proof – By means of adapted pseudodifferential calculus = a version of $\Psi DO$ for Heisenberg manifolds:

$$\lambda \cdot \xi = (\lambda \xi_v, \lambda^2 \xi_n), \quad \xi = (\xi_v, \xi_n), \quad \lambda \in \mathbb{R}^*_+,$$

$$\|\xi\|' = (\|\xi_v\|^4 + \|\xi_n\|^2)^{1/4},$$

$$\sigma'(x, \lambda \cdot \xi) = \lambda^q \sigma'(x, \xi), \quad \sigma' = q\text{-homogeneous}.$$ 

In particular, the residue density of $R \in \Psi' DO$

$$= \frac{1}{(2\pi)^{p-n}} \int_{\|\xi\|'=1} \sigma'_{-p}(R)(q, \xi) \, d\xi \, dq .$$
Example (codimension 1): $S^1 / \text{Diff}(S^1)$

$$\mathcal{H} = L^2(FS^1 \times S^1, ds \, d\theta \, d\alpha) \otimes \mathbb{C}^2$$

$$Q = -2\partial_s \partial_\alpha \gamma_1 + \frac{1}{i} e^{-s} \partial_\theta \gamma_2 + \left( \partial_s^2 - \partial_\alpha^2 - \frac{1}{4} \right) \gamma_3,$$

where $\gamma_1, \gamma_2, \gamma_3$ are the Pauli matrices

$$\gamma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

the dimension spectrum is $\Sigma = \{0, 1, 2, 3, 4\}$. The components of the Chern character are $\{\varphi_1, \varphi_3\}$ and are given by:

$$\varphi_1(a^0, a^1) = \Gamma \left( \frac{1}{2} \right) \int (a^0 [Q, a^1](Q^2)^{-1/2}) - \frac{1}{2!} \Gamma \left( \frac{3}{2} \right) \int (a^0 \nabla [Q, a^1](Q^2)^{-3/2}) + \frac{1}{3!} \Gamma \left( \frac{5}{2} \right) \int (a^0 \nabla^2 [Q, a^1](Q^2)^{-5/2}) - \frac{1}{4!} \Gamma \left( \frac{7}{2} \right) \int (a^0 \nabla^3 [Q, a^1](Q^2)^{-7/2})$$
\[ \varphi_3(a^0, a^1, a^2, a^3) = \]
\[ \frac{1}{3i} \Gamma \left( \frac{3}{2} \right) \int (a^0[Q, a^1][Q, a] \cdots [Q, a^3](Q^2)^{-3/2}) \]
\[ - \frac{1}{4!} \Gamma \left( \frac{5}{2} \right) \int (a^0\nabla[Q, a^1][Q, a^2] \cdots [Q, a^3](Q^2)^{-5/2}) \]
\[ - \frac{1}{3 \cdot 4} \Gamma \left( \frac{5}{2} \right) \int (a^0[Q, a^1]\nabla([Q, a^2][Q, a^3](Q^2)^{-5/2}) \]
\[ - \frac{1}{2 \cdot 4} \Gamma \left( \frac{5}{2} \right) \int (a^0[Q, a^1][Q, a^2]\nabla[Q, a^3](Q^2)^{-5/2}). \]

The computation is purely symbolical, but requires the symbol \( \sigma'_{-4} \), hence about \( 10^3 \) terms!

It eventually yields the following result:

\[ (\varphi_1)_{(1)}(a^1, a^1) = 0, \quad \forall a^0, a^1 \in A; \]

in fact, each of the 4 terms turns out to be 0;

on the other hand

\[ (\varphi_3)_{(1)} = \frac{1}{12\pi^{3/2}} (\tilde{\mu} + b\psi), \]

where

\[ \tilde{\mu}(f^0U\varphi_0, f^1U\varphi_1, \ldots, f^3U\varphi_3) = 0, \quad \varphi_0\varphi_1\varphi_2\varphi_3 \neq 1 \]

\[ = \int f^0\varphi_0^*(df^1) \wedge (\varphi_0\varphi_1)^*(df^2) \wedge (\varphi_0\varphi_1\varphi_2)^*(df^3). \]
Underlying algebraic structure

W.l.o.g. can assume $M = \mathbb{R}^n$, with the flat connection; \{$X_k; 1 \leq k \leq n$\}, \{$Y^j_i; 1 \leq i,j \leq n$\} horizontal, resp. vertical vector fields. The operator $Q$ is built of these vector fields, and the cocycle involves iterated commutators of them acting on $A^\Gamma_{FM}$.

E.g. in case $n = 1$,

\[ Y = y \frac{\partial}{\partial y} \quad \text{and} \quad X = y \frac{\partial}{\partial x}, \]

acting as

\[ Y(f U_\varphi) = Y(f) U_\varphi, \quad X(f U_\varphi) = X(f) U_\varphi. \]

However, while $Y$ acts as derivation

\[ Y(ab) = Y(a) b + a Y(b), \quad a,b \in A^\Gamma. \]

$X$ satisfies instead

\[ X(ab) = X(a) b + a X(b) + \delta_1(a) Y(b). \]
\[ \delta_1(f U_{\varphi^{-1}}) = y \frac{d}{dx} \left( \log \frac{d\varphi}{dx} \right) f U_{\varphi^{-1}}. \]

\( \delta_1 \) is a derivation,

\[ \delta_1(ab) = \delta_1(a) b + a \delta_1(b), \]

but its higher commutators with \( X \)

\[ \delta_n(f U_{\varphi^{-1}}) = y^n \frac{d^n}{dx^n} \left( \log \frac{d\varphi}{dx} \right) f U_{\varphi^{-1}}, \quad \forall n \geq 1, \]

satisfy more complicated Leibniz rules.

All this information can be encoded in a Hopf algebra \( \mathcal{H}_1 \). As algebra = universal enveloping algebra of the Lie algebra with presentation

\[
\begin{align*}
[Y, X] &= X, \\
[Y, \delta_n] &= n \delta_n, \\
[X, \delta_n] &= \delta_{n+1}, \\
[\delta_k, \delta_\ell] &= 0, \quad n, k, \ell \geq 1.
\end{align*}
\]
The coproduct is determined by
\[
\Delta Y = Y \otimes 1 + 1 \otimes Y, \\
\Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \\
\Delta \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1, \\
\Delta(\delta_3) = \delta_3 \otimes 1 + 1 \otimes \delta_3 + \\
+ \delta_2 \otimes \delta_1 + 3\delta_1 \otimes \delta_2 + \delta_1^2 \otimes \delta_1;
\]
the antipode is determined by
\[
S(Y) = -Y, \ S(X) = -X + \delta_1 Y, \ S(\delta_1) = -\delta_1
\]
and the counit is
\[
\varepsilon(h) = \text{constant term of} \quad h \in \mathcal{H}_1.
\]
The canonical trace \(\tau_\Gamma\) on \(A^\Gamma\) satisfies
\[
\tau_\Gamma(h(a)) = \delta(h) \tau_\Gamma(a), \quad \forall h \in \mathcal{H}_1, a \in A.
\]
where \(\delta \in \mathcal{H}_1^*\) is the character
\[
\delta(Y) = 1, \quad \delta(X) = 0, \quad \delta(\delta_n) = 0.
\]
While \( S^2 \neq \text{Id} \), the \( \delta \)-twisted antipode,

\[
\tilde{S}(h) = \delta(h_{(1)}) S(h_{(2)}) ,
\]

is involutive: \( \tilde{S}^2 = \text{Id} \).

Finally, the cochains \( \{\varphi_1, \varphi_3\} \) can be recognized as belonging to the range of a certain cohomological characteristic map.

More precisely, requiring the assignment

\[
\chi_\Gamma(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau_\Gamma(a^0 h^1(a^1) \ldots h^n(a^n)),
\]

to induce a characteristic homomorphism

\[
\chi_\Gamma^*: HC^*_\text{Hopf}(H_1) \to HC^*(A_\Gamma),
\]

practically dictates the definition of the Hopf cyclic cohomology.

$\mathcal{H}$ = Hopf algebra over a field $k$ containing $\mathbb{Q}$, $(\delta, \sigma) =$modular pair: $\delta \in \mathcal{H}^*$ character , and $\sigma \in \mathcal{H}$, $\Delta(\sigma) = \sigma \otimes \sigma$, $\varepsilon(\sigma) = 1$, with $\delta(\sigma) = 1$. One also requires $\tilde{S}^2 = \text{Id}$.

Then the following is a (co)cyclic structure:

$$\mathcal{H}^\bullet_{(\delta, \sigma)} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{H}^\otimes n :$$

$$\delta_0(h^1 \otimes ... \otimes h^{n-1}) = 1 \otimes h^1 \otimes ... \otimes h^{n-1}$$
$$\delta_j(h^1 \otimes ... \otimes h^{n-1}) = h^1 \otimes ... \otimes \Delta h^j \otimes ... \otimes h^{n-1}$$
$$1 \leq j \leq n - 1$$
$$\delta_n(h^1 \otimes ... \otimes h^{n-1}) = h^1 \otimes ... \otimes h^{n-1} \otimes \sigma$$
$$\sigma_i(h^1 \otimes ... \otimes h^{n+1}) = h^1 \otimes ... \otimes \varepsilon(h^{i+1}) \otimes ... \otimes h^{n+1}$$
$$0 \leq i \leq n$$
$$\tau_n(h^1 \otimes ... \otimes h^n) = \tilde{S}(h^1) \cdot (h^2 \otimes ... \otimes h^n \otimes \sigma).$$
Equivalence of characteristic maps

[Gelfand-Fuchs-Bott-Haefliger] $\Rightarrow$ Hopf

$J^\infty M := \{j_0^\infty(\psi); \psi : \mathbb{R}^n \to M \}$,

$
\pi_1 : J^\infty M \to J^1 M = FM$ projection with cross-section

$\sigma_\nabla(u) = j_0^\infty(\exp_x^\nabla \circ u), \quad u \in F_xM$
given by connection $\nabla$; \forall a \in \text{GL}_n(\mathbb{R}), \forall \varphi \in \Gamma$

$\sigma_\nabla \circ R_a = R_a \circ \sigma_\nabla$ and $\sigma_\nabla \varphi = \bar{\varphi}^{-1} \circ \sigma_\nabla \circ \bar{\varphi}$.

Define $\sigma_\nabla(\varphi_0, \ldots, \varphi_p) : \Delta^p \times FM \to J^\infty M$
by

$\sigma_\nabla(\varphi_0, \ldots, \varphi_p)(t, u) = \sigma_\nabla(\varphi_0, \ldots, \varphi_p; t)(u),$

where $\nabla(\varphi_0, \ldots, \varphi_p; t) = \sum_0^p t_i \nabla \phi_i$;

$\sigma_\nabla(\varphi_0 \varphi, \ldots, \varphi_p \varphi)(t, u) = \bar{\varphi}^{-1} \sigma_\nabla(\varphi_0, \ldots, \varphi_p)(t, \bar{\varphi}(u)).$
\( C^*(\mathfrak{a}_n) = \text{Gelfand-Fuchs} \) Lie algebra cohomology complex of \( \mathfrak{a}_n = \text{Lie algebra of formal vector fields on } \mathbb{R}^n \).

For \( \varpi \in C^q(\mathfrak{a}_n) \), define \( \forall \eta \in \Omega^m_c(FM) \),

\[
\langle C_{p,m}(\varpi)(\varphi_0, \ldots, \varphi_p), \eta \rangle = (-1)^{m(m+1)/2} \int_{\Delta_p \times FM} \eta \wedge \sigma \nabla(\varphi_0, \ldots, \varphi_p)^*(\varpi)
\]

\( C_{\nabla}(\varpi) = \sum C_{p,m}(\varpi) : C^*(\mathfrak{a}_n) \to C^*(\Gamma; \Omega^*_c(FM)) \); defines a map of (total) complexes,

\[
C_{\nabla}(d\varpi) = (\delta + \partial)C_{\nabla}(\varpi).
\]

For the relative (to \( SO_n \)) cohomology, one constructs similarly a homomorphism

\[
H^*(\mathfrak{a}_n, SO_n) \to H^*(\Gamma; \Omega^*_c(PM)),
\]

which can be followed by Connes’ map \( \Phi^*_{\Gamma} : H^*_\Gamma(PM) \to HC^*(A^\Gamma_{PM}) \), yielding

\[
\chi^\Gamma_{GF} : H^*(\mathfrak{a}_n, SO_n) \to HC^*(A^\Gamma_{PM}).
\]
Composing $\chi_{GF}^\Gamma$ with the natural restriction

$$PHC^*(A_{PM}^\Gamma) \to PHC^*(C_c^\infty(PM))$$

one recovers the Pontryagin classes of $M$ as images of the universal Chern classes

$$c_{2i_1} \cdots c_{2i_k} \in H^*(a_n, SO_n), \quad 2i_1 + \ldots + 2i_k \leq n.$$ 

From Hopf cyclic to cyclic: \quad $M = \mathbb{R}^n$

$$\chi_\tau(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau(a^0 h^1(a^1) \ldots h^n(a^n)),$$

inducing characteristic homomorphism

$$\chi_{Hopf}^\Gamma : HC^*_\text{Hopf}(\mathcal{H}_n, SO_n) \to HC^*(A_{PM}^\Gamma)(1).$$

**Theorem 2.** There is a canonical isomorphism

$$\kappa_n^* : H^*(a_n, SO_n) \overset{\sim}{\to} PHC^*_\text{Hopf}(\mathcal{H}_n, SO_n),$$

such that \quad $\chi_{Hopf}^\Gamma \circ \kappa_n^* = \chi_{GF}^\Gamma$. 
Summary: Transverse Index Theorem

Theorem 3. There are canonical constructions for the following entities:

- a Hopf algebra $H_n$ with modular character $\delta$, and with $(\delta,1)$ modular pair in involution;
- a co-cyclic structure for any Hopf algebra with a modular pair in involution $(\delta,\sigma)$;
- an isomorphism $\kappa_n^*$ between the Gelfand-Fuks cohomology $H^*_\text{GF}(a_n)$, resp. $H^*_\text{GF}(a_n, SO_n)$, and $HP^*(H_n; \mathbb{C}_\delta)$, resp. $HP^*(H_n, SO_n; \mathbb{C}_\delta)$;
- an action of $H_n$ on $A_\Gamma(F \mathbb{R}^n)$, inducing a characteristic map $\chi_\Gamma^* : HP^*(H_n, SO_n; \mathbb{C}_\delta) \to HP^*_\Gamma(A_\Gamma(P \mathbb{R}^n)) \cong H_*(P \mathbb{R}^n \times_\Gamma EG)$;
- a class $L_n \in H^*_\text{GF}(a_n, SO_n)$, such that $\chi_\Gamma^*(A_\Gamma(P \mathbb{R}^n), \mathcal{H}(P \mathbb{R}^n), D)_{(1)} = (\chi_\Gamma^* \circ \kappa_n^*)(L_n)$. 