A Selective History of the Stone-von Neumann Theorem

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Two Giants of 20th Century Mathematics

Marshall Stone, 1903–1989

John von Neumann, 1903–1957

Stone's Original Statement
Proc. Nat. Acad. USA **16** (1930), 172–175.

LINEAR TRANSFORMATIONS IN HILBERT SPACE. III. OPERATIONAL METHODS AND GROUP THEORY

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A second question of group theory, to which we can apply the operational calculus, is raised by the Heisenberg permutation relations connecting the self-adjoint transformations P_k, Q_k, $k = 1, \ldots, n$. For convenience, we write these relations in the form

$$P_kQ_l - Q_lP_k = i\delta_{kl}I,\ P_kP_l - P_lP_k = 0,\ Q_kQ_l - Q_lQ_k = 0,\ k, l = 1, \ldots, n.$$

In quantum mechanics, these transformations refer essentially to the coördinates and momenta of a dynamical system of n degrees of freedom. The content of these permutation relations must be made precise by expressing them in terms of the one parameter groups of unitary transformations $U_\tau^{(k)}$ and $V_\tau^{(k)}$ generated by iP_k and iQ_k, respectively. We have $U_\sigma^{(k)} V_\tau^{(k)} = e^{-i\sigma\tau} V_\tau^{(k)} U_\sigma^{(k)}$, and $U_\sigma^{(k)} V_\sigma^{(l)} = V_\tau^{(l)} U_\sigma^{(k)}$ when k and l are different, for $k, l = 1, \ldots, n$; we have also $U_\sigma^{(k)} U_\tau^{(l)} = U_\tau^{(l)} U_\sigma^{(k)}$ and $V_\sigma^{(k)} V_\tau^{(l)} = V_\tau^{(l)} V_\sigma^{(k)}$ for $k, l = 1, \ldots, n$. We prove the following theorem:

Theorem. If the family of transformations $U_{\sigma_1}^{(1)} \ldots U_{\sigma_n}^{(n)} V_{\tau_1}^{(1)} \ldots V_{\tau_n}^{(n)}$ is irreducible in $\mathfrak{H}$, then there exists a one-to-one linear isometric correspondence or transformation S which takes $\mathfrak{H}$ into $\mathfrak{H}_n$, the space of all complex-valued Lebesgue-measurable functions $f(x_1, \ldots, x_n)$, $-\infty < x_1 < +\infty, \ldots, -\infty < x_n < +\infty$, for which the integral $\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} |f(x_1, \ldots, x_n)|^2 dx_1 \ldots dx_n$ exists, such that

$$SP_kS^{-1}f(x_1, \ldots, x_n) = i\frac{\partial}{\partial x_k}f,\ SQ_kS^{-1}f(x_1, \ldots, x_n) = x_kf.$$

Von Neumann's Contribution

Math. Ann. **104** (1931), 570–578.

Die Eindeutigkeit der Schrödingerschen Operatoren

1. Die sogenannte Vertauschungsrelation

$$PQ - QP = \frac{h}{2\pi i} 1$$

ist in der neuen Quantentheorie von fundamentaler Bedeutung, sie ist es, die den „Koordinaten-Operator" R und den „Impuls-Operator" P im wesentlichen definiert[1]). Mathematisch gesprochen, liegt darin die folgende Annahme: Seien P, Q zwei Hermitesche Funktionaloperatoren des Hilbertschen Raumes, dann werden sie durch die Vertauschungsrelation bis auf eine Drehung des Hilbertschen Raumes, d. i. eine unitäre Transformation U, eindeutig festgelegt[2]). Es liegt im Wesen der Sache, daß noch der Zusatz gemacht werden muß: vorausgesetzt, daß P, Q ein irreduzibles System bilden (vgl. weiter unten Anm. [6])). Wird nun, wie es sich durch die Schrödingersche Fassung der Quantentheorie als besonders günstig erwies, der Hilbertsche Raum als Funktionenraum interpretiert — der Einfachheit halber etwa als Raum aller komplexen Funktionen $f(q)$ $(-\infty < q < +\infty)$ mit endlichem $\int_{-\infty}^{+\infty} |f(q)|^2 dq$ —, so gibt es nach Schrödinger ein besonders einfaches Lösungssystem der Vertauschungsrelation

$$Q\colon\ f(q) \to q f(q), \qquad P\colon\ f(q) \to \frac{h}{2\pi i} \frac{d}{dq} f(q) \ ^{3}).$$

Sind nun dies die im wesentlichen einzigen (irreduziblen) Lösungen der Vertauschungsrelation?

Es bliebe daher zu zeigen, daß die einzigen irreduziblen Lösungen[6]) der Weylschen Gleichungen die Schrödingerschen (d. h. die aus Anm. [5])) sind. Beweisansätze hierfür gab Stone (vgl. Anm. [5])) an, jedoch ist bisher ein Beweis auf dieser Grundlage, wie mir Herr Stone freundlichst mitteilte, nicht erbracht worden.

Im folgenden soll der genannte Eindeutigkeitssatz bewiesen werden.

Analysis of What Stone and Von Neumann Did

Canonical Commutation Relations:

$$[P_j, P_k] = 0, \quad [Q_j, Q_k] = 0, \quad [P_j, Q_k] = -i\delta_{jk}\hbar,$$

P_j, Q_k self-adjoint, $1 \leq j, k \leq n$. Unbounded operator problems are bypassed by going to the Weyl integrated form. In other words, assume we have unitary representations U, V of $\mathbb{R}^n$, obtained by "integrating" the P's and Q's, respectively. Thus result can be phrased as follows:

Theorem 1 *Consider pairs (U, V) of unitary representations of $\mathbb{R}^n$ on a Hilbert space $\mathcal{H}$, satisfying the commutation rule*

$$U(x)V(y) = \exp(i\omega(x, y))V(y)U(x),$$

$\omega\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *bilinear and non-degenerate. Such pairs are all equivalent to multiples of the standard Schrödinger representation.*

Note that we can combine U and V into a projective unitary representation of $\mathbb{R}^{2n}$, or equivalently into a unitary representation of the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & x_1 & \cdots & x_n & z \\ 0 & 1 & \cdots & 0 & y_n \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & y_1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} : x_j, y_k, z \in \mathbb{R} \right\}$$

with specified central character $z \mapsto e^{i\hbar z}$.

Mackey's Version

A theorem of Stone and von Neumann, Duke Math. J. **16** (1949), 313–326.

Theorem 2 *Let G be a locally compact group. Then any covariant pair of representations of G and $C_0(G)$ on a Hilbert space $\mathcal{H}$ is a multiple of the standard representation on $L^2(G)$. (A covariant pair is (π, α), where π is a unitary representation of G on $\mathcal{H}$, α is a $*$-representation of $C_0(G)$ on $\mathcal{H}$, and $\pi(g)\alpha(f)\pi(g)^* = \alpha(g \cdot f)$, $g \cdot f(h) = f(g^{-1}h)$.)*

Remark 3 The situation of Stone and von Neumann is the case of $G = \mathbb{R}^n$. For G abelian, a covariant pair is the same as a pair (π, σ) with π a rep. of G, σ a rep. of $\widehat{G}$, and $\pi(g)\sigma(\widehat{g})\pi(g)^* = \langle g, \widehat{g} \rangle \sigma(\widehat{g})$.

The Shale-Weil Representation

We've seen in particular that if $\mathbb{F}$ is a self-dual field (e.g., $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}_p$) with non-trivial character χ, and (V, B) is a symplectic vector space of dimension $2n$ over $\mathbb{F}$, the Stone-von Neumann Theorem says there is a unique irreducible representation π of the commutation relation

$$\pi(v)\pi(w) = \chi(B(v,w))\pi(w)\pi(v).$$

Since $Sp(V)$ preserves the relation, Schur's Lemma implies there is a projective representation ω of $Sp(V)$ on the Hilbert space of the Schrödinger representation, given by

$$\omega(g)\pi(v)\omega(g)^* = \pi(g \cdot v).$$

Theorem 4 *The representation ω lifts to a true representation of a double cover of $Sp(V)$.*

[D. Shale, Linear symmetries of free boson fields, Trans. Amer. Math. Soc. **103** (1962), 149–167.
A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math. **111** (1964), 143–211.]

As pointed out by Weil, the same theory also goes through not just locally but over the adèles $\mathbb{A}(K)$, K a number field. This has important consequences for number theory.

This Shale-Weil representation of the double cover of the symplectic group, also called (especially by Roger Howe) the oscillator representation, plays a fundamental role in representation theory and number theory. In fact, most of the correspondences in the theory of automorphic forms are derived from special cases of the theta correspondence based on this representation.

Rieffel's Version and the Notion of Morita Equivalence

On the uniqueness of the Heisenberg commutation relations, Duke Math. J. **39** (1972), 745–752.
Induced representations of C^*-algebras, Advances in Math. **13** (1974), 176–257.

The "modern" approach to the Stone-von Neumann Theorem, which is somewhat more algebraic, is due to Rieffel. The key observation is that the theorem is really about an equivalence of categories of representations between (Weyl integrated forms) of representations of the commutation relations and representations of $\mathbb{C}$, or in the language of ring theory, a Morita equivalence. For example, the rings $\mathbb{C}$ and $M_n(\mathbb{C})$ are Morita equivalent. Rieffel exploited this by showing that on any Hilbert space representation of the Heisenberg commutation relations, there is a natural action of $\varinjlim M_n(\mathbb{C})$.

Rieffel then went on to explain the Stone-von Neumann-Mackey Theorem, as well as the Mackey Imprimitivity Theorem, in similar terms. Covariant pairs of representations of G and of $C_0(G/H)$, or in Mackey's language, systems of imprimitivity based on G/H, can be identified with representations of a crossed product algebra $C_0(G/H) \rtimes G$ which is Morita equivalent to the group C^*-algebra $C^*(H)$. The associated correspondence of representations matches induced representations $\mathrm{Ind}\,\sigma$ with the inducing representations σ.

The Baggett-Kleppner-Pukánszky Theorem

Recall that when G is abelian, the Stone-von Neumann-Mackey Theorem deals with covariant pairs of representations of G and $\widehat{G}$ satisfying

$$\pi(g)\sigma(\widehat{g})\pi(g)^* = \langle g, \widehat{g}\rangle\sigma(\widehat{g}).$$

One can ask what happens if $\widehat{G}$ is replaced by some other abelian group H with a non-degenerate pairing

$$\langle\ ,\ \rangle\colon G \times H \to \mathbb{T},$$

$\mathbb{T}$ the circle group. The simplest example is $G = H = \mathbb{Z}$ with a dense embedding $H \hookrightarrow \mathbb{T}$ sending the generator to $e^{2\pi i\lambda}$, $\lambda \notin \mathbb{Q}$. The associated C^*-algebra is called an irrational rotation algebra or noncommutative torus.

It was discovered in part by Pukánszky and in part by Baggett and Kleppner that in the case where $\langle\ ,\ \rangle$ gives a dense embedding $H \hookrightarrow \widehat{G}$ with non-closed range, the Morita equivalence with $\mathbb{C}$ breaks down, but a vestige of it remains: the C^*-algebra generated by the commutation relation is simple with unique trace. (So while it has many non-equivalent irreducible representations, it has a unique quasi-equivalence class of traceable factor representations.)

Supersymmetry and Analogues of Stone-von Neumann

We conclude by mentioning some other analogues or generalizations of the Stone-von Neumann Theorem. The Heisenberg commutation relations are appropriate for free bosons. One can equally well consider the canonical anti-commutation relations (CAR), appropriate for free fermions, or a more realistic mixture of both. This is a necessary prerequisite for studying what physicists call supersymmetry. The CAR with finitely many degrees of freedom generate a Clifford algebra isomorphic to a matrix algebra over $\mathbb{C}$, so again one has a Morita equivalence with $\mathbb{C}$, i.e., the analogue of Stone-von Neumann holds for fermions. In the case of infinitely many degrees of freedom, Stone-von Neumann no longer holds for either bosons or fermions, but the analogue of the Baggett and Kleppner Theorem still holds; one gets a simple C^*-algebra with unique trace.