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T-Duality for Torus Bundles with H-Fluxes via Noncommutative Topology

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Abstract: It is known that the T-dual of a circle bundle with H-flux (given by a Neveu-Schwarz 3-form) is the T-dual circle bundle with dual H-flux. However, it is also known that torus bundles with H-flux do not necessarily have a T-dual which is a torus bundle. A big puzzle has been to explain these mysterious "missing T-duals." Here we show that this problem is resolved using noncommutative topology. It turns out that every principal T^2-bundle with H-flux does indeed have a T-dual, but in the missing cases (which we characterize), the T-dual is non-classical and is a bundle of noncommutative tori. The duality comes with an isomorphism of twisted K-theories, just as in the classical case. The isomorphism of twisted cohomology which one gets in the classical case is replaced by an isomorphism of twisted cyclic homology.

1. Introduction

An important symmetry of string theories is T-duality, which exchanges wrapping of fields over a torus with wrapping over the dual torus [8, 9, 1, 2]. (The exact mathematical meaning of "dual torus" is that if Λ is a lattice in $\mathbb{R}^n$ and Λ^* is the dual lattice in the dual vector space $(\mathbb{R}^n)^*$, then $(\mathbb{R}^n)^*/\Lambda^*$ is the dual torus to $\mathbb{R}^n/\Lambda$.) Many authors have tried to understand this duality from various points of view. Since Ramond-Ramond (RR) charges are expected to be represented by classes in K-theory (see, e.g., [39, 40, 27, 24]), T-duality should come with an isomorphism of K-theories (usually with a degree shift) between a theory and its dual. The type of K-theory appropriate for the situation (e.g., K, KO, or KSp) depends on the type of string theory being considered; here we deal with the type II situation, which leads to complex K-theory. (For a few comments on type I theories, see Sect. 6.)

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As pointed out in many contexts (e.g., [37, 17]), T-duality can apply not only to theories over spaces of the form $X \times T^n$, but also to non-trivial torus bundles, and even to spaces which are only "approximately" of this form, for example, spaces admitting a torus action which is generically free. (However, in this paper we only consider the case of free torus actions.) In addition, it should apply as well to situations with a non-trivial Neveu-Schwarz (NS) 3-form H. In these situations, the H-flux gives rise to a twisting of K-theory, so that one expects an isomorphism of *twisted* K-theories. In its general form, T-duality often involves a change of topology (see, e.g., [5, 6 and 7]).

Our initial interest was in trying to explain the T-duality of torus bundles, in the presence of twisting by an H-flux, from the perspective of noncommutative topology. An unexpected byproduct, which we will discuss in Sect. 5, is that we have found that several known cases of torus bundles with "missing" T-duals are in fact naturally T-dual to *noncommutative* torus bundles, in a sense we will make precise below. This suggests an unexpected link between classical string theories and the "noncommutative" ones, obtained by "compactifying" along noncommutative tori, as in [13] (cf. also [36, §§6–7]).

Just as a complete characterization of T-duality on circle bundles with H-flux is given in [5 and 6], in this paper, we give a complete characterization of T-duality on principal $\mathbb{T}^2$-bundles with H-flux, Theorem 4.13. We also describe partial results for T-duality on general principal torus bundles with H-flux. The main mathematical result is a detailed analysis of the equivariant Brauer group for principal $\mathbb{T}^2$-bundles, Theorem 4.10, which refines earlier results in [14 and 28]. This depends on some explicit calculations of Moore's "Borel cochain" cohomology groups.

2. Preliminaries on Noncommutative Tori

Here the definition of a (2-dimensional) noncommutative torus is recalled, cf. [34]. This algebra (stabilized by tensoring with the compact operators $\mathcal{K}$) occurs geometrically as the foliation algebra associated to Kronecker foliations on the torus [10]. It also occurs naturally in the matrix formulation of M-theory as the components of Yang-Mills connections in the classification of BPS states [13].

For each $\theta \in [0, 1]$, the *noncommutative torus* A_θ is defined abstractly as the C^*-algebra generated by two unitaries U and V in an infinite dimensional Hilbert space satisfying the relation $UV = \exp(2\pi i\theta)VU$. Elements in A_θ can be represented by infinite power series

$$f = \sum_{(m,n)\in\mathbb{Z}^2} a_{(n,m)}\, U^m V^n, \tag{1}$$

where the coefficients $a_{(m,n)} \in \mathbb{C}$ satisfy a decay condition (very hard to make precise) as $(m, n) \to \infty$ in $\mathbb{Z}^2$. There is a natural smooth subalgebra A_θ^∞ called the *smooth noncommutative torus*, which is defined as those elements in A_θ that can be represented by infinite power series (1) with $(a_{(m,n)}) \in \mathcal{S}(\mathbb{Z}^2)$, the Schwartz space of rapidly decreasing sequences on $\mathbb{Z}^2$.

A_θ can also be realized as the crossed product $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\mathbb{T}$ by rotation by the angle $2\pi\theta$. When θ is rational, A_θ is type I, and is even Morita equivalent to $C(\mathbb{T}^2)$. However, when θ is irrational, A_θ is a simple non-type I C^*-algebra. Because of the realization of A_θ as a crossed product by rotation by $2\pi\theta$, the algebra in this case is often called an *irrational rotation algebra*.

Consider the 2 dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For each $\theta \in [0, 1]$, the noncommutative torus A_θ is Morita equivalent to the foliation algebra associated to the foliation on $\mathbb{T}^2$ defined by the differential equation $dx = \theta\, dy$ on $\mathbb{T}^2$.

3. Mathematical Framework

We begin by explaining the precise mathematical framework in which we are working. We assume X (which will be the spacetime of a string theory) is a (second-countable) locally compact Hausdorff space. In practice it will usually be a compact manifold, though we do not need to assume this. However it is convenient to assume that X is finite-dimensional and has the homotopy type of a finite CW-complex. (This assumption can be weakened but some finiteness assumption is necessary to avoid some pathologies. This is not a problem as far as the physics is concerned.) We assume X comes with a free action of a torus T; thus (by the Gleason slice theorem [21]) the quotient map $p\colon\ X \to Z$ is a principal T-bundle.

A *continuous-trace algebra A over X* is a particular type of type I C^*-algebra with spectrum X and good local structure (the "Fell condition" [20]).[1] We will always assume A is separable; then a basic structure theorem of Dixmier and Douady [16] says that after stabilization (i.e., tensoring by $\mathcal{K}$, the algebra of compact operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$), A becomes *locally* isomorphic to $C_0(X, \mathcal{K})$, the continuous $\mathcal{K}$-valued functions on X vanishing at infinity. However, A need not be *globally* isomorphic to $C_0(X, \mathcal{K})$, even after stabilization. The reason is that a stable continuous-trace algebra is the algebra of sections (vanishing at infinity) of a bundle of algebras over X, with fibers all isomorphic to $\mathcal{K}$. The structure group of the bundle is $\operatorname{Aut}\mathcal{K} \cong PU(\mathcal{H})$, the projective unitary group $U(\mathcal{H})/\mathbb{T}$. Since $U(\mathcal{H})$ is contractible and the circle group $\mathbb{T}$ acts freely on it, $PU(\mathcal{H})$ is an Eilenberg-MacLane $K(\mathbb{Z}, 2)$-space, and thus bundles of this type are classified by homotopy classes of continuous maps from X to $BPU(\mathcal{H})$, which is a $K(\mathbb{Z}, 3)$-space, or in other words by $H^3(X, \mathbb{Z})$. Alternatively, the bundles are classified by $H^1(X,\ \underline{PU(\mathcal{H})})$, the sheaf cohomology of the sheaf $\underline{PU(\mathcal{H})}$ of germs of continuous PU-valued functions on X, where the transition functions of the bundle naturally live. But because of the exact sequences in sheaf cohomology

$$0 = H^1(X,\ \underline{U(\mathcal{H})}) \to H^1(X,\ \underline{PU(\mathcal{H})}) \to H^2(X,\ \underline{\mathbb{T}}) \to 0$$

and

$$0 = H^2(X,\ \underline{\mathbb{R}}) \to H^2(X,\ \underline{\mathbb{T}}) \to H^3(X, \mathbb{Z}) \to H^3(X,\ \underline{\mathbb{R}}) = 0,$$

the bundles are classified by $H^2(X, \underline{\mathbb{T}}) \cong H^3(X, \mathbb{Z})$ [35, §1]. Hence stable isomorphism classes of continuous-trace algebras over X are classified by the *Dixmier-Douady class* in $H^3(X, \mathbb{Z})$. It turns out that continuous-trace algebras over X, modulo Morita equivalence over X, naturally form a group under the operation of tensor product over $C_0(X)$, called the *Brauer group* $\operatorname{Br}(X)$, and that this group is isomorphic to $H^3(X, \mathbb{Z})$ via the Dixmier-Douady class.

Given an element $\delta \in H^3(X, \mathbb{Z})$, we denote by $CT(X, \delta)$ the associated stable continuous-trace algebra. (Thus if $\delta = 0$, this is simply $C_0(X, \mathcal{K})$.) The (complex topological) K-theory $K_\bullet(CT(X, \delta))$ is called the *twisted K-theory* [35, §2] of X with twist

[1] Except in Sect. 6 below, all C^*-algebras and Hilbert spaces in this paper will be over $\mathbb{C}$.

δ, denoted $K^{-\bullet}(X,\delta)$. When δ is torsion, twisted K-theory had earlier been considered by Karoubi and Donovan [18]. When $\delta = 0$, twisted K-theory reduces to ordinary K-theory (with compact supports).

Now recall we are assuming X is equipped with a free T-action with quotient $X/T = Z$. (This means our theory is "compactified along tori" in a way reflecting a global symmetry group of X.) In general, a group action on X need not lift to an action on $CT(X,\delta)$ for any value of δ other than 0, and even when such a lift exists, it is not necessarily essentially unique. So one wants a way of keeping track of what lifts are possible and how unique they are. The correct generalization of $\mathrm{Br}(X)$ to the equivariant setting is the *equivariant Brauer group* defined in [14], consisting of equivariant Morita equivalence classes of continuous-trace algebras over X equipped with group actions lifting the action on X. By [14, Lemma 3.1], two group actions on the same stable continuous-trace algebra over X define the same element in the equivariant Brauer group if and only if they are outer conjugate. (This implies in particular that the crossed products are isomorphic.) Now let G be the universal cover of the torus T, a vector group. Then G also acts on X via the quotient map $G \twoheadrightarrow T$ (whose kernel N can be identified with the free abelian group $\pi_1(T)$). In our situation there are three Brauer groups to consider: $\mathrm{Br}(X) \cong H^3(X,\mathbb{Z})$, $\mathrm{Br}_T(X)$, and $\mathrm{Br}_G(X)$. It turns out, however, that $\mathrm{Br}_T(X)$ is rather uninteresting, as it is naturally isomorphic to $\mathrm{Br}(Z)$ [14, §6.2]. Again by [14, §6.2], the natural "forgetful map" (forgetting the T-action) $\mathrm{Br}_T(X) \to \mathrm{Br}(X)$ can simply be identified with $p^*\colon\ \mathrm{Br}(Z) \cong H^3(Z,\mathbb{Z}) \to H^3(X,\mathbb{Z}) \cong \mathrm{Br}(X)$.

Finally, we can summarize what we are interested in.

Basic Setup 3.1. *A spacetime X compactified over a torus T will correspond to a space X (locally compact, finite-dimensional homotopically finite) equipped with a free T-action. The quotient map $p\colon X \to Z$ is a principal T-bundle. The NS 3-form H on X has an integral cohomology class δ which corresponds to an element of* $\mathrm{Br}(X) \cong H^3(X,\mathbb{Z})$. *A pair (X,δ) will be a candidate for having a T-dual when the T-symmetry of X lifts to an action of the vector group G on $CT(X,\delta)$, or in other words, when δ lies in the image of the forgetful map $F\colon$* $\mathrm{Br}_G(X) \to \mathrm{Br}(X)$.

4. Structure of the Equivariant Brauer Group and T-Duality

Throughout this section, the above Basic Setup 3.1 will be in force. We let $n = \dim T$, the dimension of the tori involved.

4.1. Review of the case $n = 1$. The case $n = 1$ was treated in [31, Theorem 4.12], from a purely C^*-algebraic perspective, in [5], from a combined mathematical and physical perspective, and in [6] from a more physical point of view. In this case, $G = \mathbb{R}$, $T = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $N = \mathbb{Z}$. By [14, Cor. 6.1], the forgetful map $F\colon\ \mathrm{Br}_G(X) \to \mathrm{Br}(X)$ is an isomorphism, and thus *every* $\delta \in H^3(X,\mathbb{Z})$ is dualizable, in fact in a unique way. It is proven in [5] that the T-dual of the pair $(p\colon\ X \to Z,\ \delta)$ is a pair $(p^\#\colon\ X^\# \to Z,\ \delta^\#)$, where $X^\#$ is another principal circle bundle over Z and $\delta^\# \in H^3(X^\#,\mathbb{Z})$. Furthermore, there is a beautiful symmetry in this situation. Principal $\mathbb{T}$-bundles over Z are classified by their Euler class in $H^2(Z,\mathbb{Z})$, or equivalently by the first Chern class of the associated complex line bundle. So let $[p],\ [p^\#] \in H^2(Z,\mathbb{Z})$ be the characteristic classes of the two circle bundles. One has

$$p_!(\delta) = [p^\#], \quad (p^\#)_!(\delta^\#) = [p], \tag{1}$$

where $p_!$ and $(p^\#)_!$ are the push-forward maps in the Gysin sequences of the two bundles. At the level of forms, $p_!$ and $(p^\#)_!$ are simply "integration over the fiber," which reduces the degree of a form by one.

Furthermore, the crossed product $CT(X, \delta) \rtimes \mathbb{R}$ is isomorphic to $CT(X^\#, \delta^\#)$, and $CT(X^\#, \delta^\#) \rtimes \mathbb{R}$ is isomorphic to $CT(X, \delta)$. In fact, the $\mathbb{R}$-action on $CT(X^\#, \delta^\#)$ may be chosen to be the dual action on the crossed product. If one takes the crossed product $CT(X, \delta) \rtimes \mathbb{Z}$ by the $\mathbb{R}$-action restricted to $\mathbb{Z} = \ker(\mathbb{R} \to \mathbb{T})$, or the similar crossed product $CT(X^\#, \delta^\#) \rtimes \mathbb{Z}$, the result is

$$CT\big(X \times_Z X^\#,\ p^*(\delta^\#) = (p^\#)^*(\delta)\big).$$

Thus one obtains a commutative diagram of principal $\mathbb{T}$-bundles

$$\begin{tikzcd} & X \times_Z X^\# \arrow[dl, "p^*(p^\#)"'] \arrow[dr, "(p^\#)^*(p)"] & \\ X \arrow[dr, "p"'] & & X^\# \arrow[dl, "p^\#"] \\ & Z & \end{tikzcd} \tag{2}$$

Finally, we get the desired isomorphisms of twisted K-theory and of twisted homology by using the above results on crossed products and applying Connes' Thom isomorphism theorem [11] and its analogue in cyclic homology, due to Elliott, Natsume, and Nest [19]. The final result, found in [5], is a commutative diagram

$$\begin{tikzcd} K^{\bullet+1}(X, \delta) \arrow[r, "T_!", "\cong"'] \arrow[d, "\mathrm{Ch}"] & K^\bullet(X^\#, \delta^\#) \arrow[d, "\mathrm{Ch}"] \\ H^{\bullet+1}(X, \delta) \arrow[r, "T_*", "\cong"'] & H^\bullet(X^\#, \delta^\#). \end{tikzcd} \tag{3}$$

Here Ch is the Chern character, which is an isomorphism after tensoring with $\mathbb{R}$, and homology should be $\mathbb{Z}/2$-graded (i.e., we lump together all the even cohomology and all the odd cohomology). Since this duality interchanges even and odd K-theory, it also exchanges type IIa and type IIb string theories.

4.2. Features of the general case. We return again to the Basic Setup 3.1, but now with T a torus of arbitrary dimension n, so $G \cong \mathbb{R}^n$. When $n > 1$, it is no longer true that the forgetful map $F : \mathrm{Br}_G(X) \to \mathrm{Br}(X)$ is an isomorphism. However, some facts about this map are contained in [14] and in [28]. We briefly summarize a few of these results, specialized to the case where G is connected (which forces G to act trivially on the cohomology of X). So as to avoid confusion between cohomology of spaces and of topological groups, we have denoted by $H^\bullet_M(G, A)$ the cohomology of the topological group G with coefficients in the topological G-module A, as defined in [26]. This is sometimes called "Moore cohomology" or "cohomology with Borel cochains."

Theorem 4.1 ([14, Theorem 5.1]). *Suppose G is a connected Lie group and X is a locally compact G-space (satisfying our finiteness assumptions). Then there is an exact sequence*

$$\mathrm{Br}_G(X) \xrightarrow{F} \ker(d_2) \xrightarrow{d_3} H^3_M(G, C(X, \mathbb{T}))/\operatorname{im}(d_2') ,$$

where

$$d_2\colon\ H^3(X, \mathbb{Z}) \to H^2_M(G, H^2(X, \mathbb{Z}))$$

and

$$d_2'\colon\ H^1_M(G, H^2(X, \mathbb{Z})) \to H^3_M(G, C(X, \mathbb{T})).$$

In addition, there is an exact sequence

$$H^2(Z, \mathbb{Z}) \xrightarrow{d_2''} H^2_M(G, C(X, \mathbb{T})) \xrightarrow{\xi} \ker F \xrightarrow{\eta} H^1_M(G, H^2(X, \mathbb{Z})).$$

Fortunately, since in our situation G is a vector group and is thus contractible, $H^\bullet_M(G, A)$ vanishes when A is discrete, thanks to:

Theorem 4.2 ([38, Theorem 4]). *If G is a Lie group and A is a discrete G-module, then $H^\bullet_M(G, A)$ is canonically isomorphic to $H^\bullet(BG, \underline{A})$ (the sheaf cohomology of the classifying space BG with coefficients in the locally constant sheaf defined by A).*

Corollary 4.3. *If G is a vector group and if A is a discrete abelian group on which G acts trivially, then $H^\bullet_M(G, A) = 0$ for $\bullet > 0$.*

Proof. Since the action of G on A is trivial, the sheaf $\underline{A}$ is constant and can be replaced by A. Since BG is contractible, $H^\bullet(BG, A) = 0$.

Substituting Corollary 4.3 into Theorem 4.1, we obtain (since our finiteness assumption on X implies $H^2(X, \mathbb{Z})$ is countable and discrete):

Theorem 4.4. *Suppose $G \cong \mathbb{R}^n$ is a vector group and X is a locally compact G-space (satisfying our finiteness assumptions). Then there is an exact sequence:*

$$H^2(X, \mathbb{Z}) \xrightarrow{d_2''} H^2_M(G, C(X, \mathbb{T})) \xrightarrow{\xi} \mathrm{Br}_G(X) \xrightarrow{F} H^3(X, \mathbb{Z}) \xrightarrow{d_3} H^3_M(G, C(X, \mathbb{T})).$$

This still leaves one set of Moore cohomology groups to calculate, namely

$$H^\bullet_M(G, C(X, \mathbb{T})), \qquad \bullet = 2,\ 3.$$

For purposes of doing this calculation, it is convenient to use the exact sequence of G-modules:

$$0 \to H^0(X, \mathbb{Z}) \to C(X, \mathbb{R}) \to C(X, \mathbb{T}) \to H^1(X, \mathbb{Z}) \to 0. \tag{4}$$

This is just the start of the long exact cohomology sequence for the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \underline{\mathbb{R}} \to \underline{\mathbb{T}} \to 0.$$

Our finiteness assumption on X implies that the cohomology groups of X are countable and discrete. So by Corollary 4.3 again, $H^0(X, \mathbb{Z})$ and $H^1(Z, \mathbb{Z})$ are cohomologically trivial (for $H^\bullet_M(G, —)$), and thus

$$H^\bullet_M(G, C(X, \mathbb{T})) \cong H^\bullet_M(G, C(X, \mathbb{R})), \qquad \bullet > 1. \tag{5}$$

Finally, for computing the latter we can use another result from [38]:

Theorem 4.5 ([38, Theorem 3]). *If G is a Lie group and A is a G-module which is a topological vector space, then $H^\bullet_M(G, A)$ agrees with "continuous cohomology" $H^\bullet_{\mathrm{cont}}(G, A)$, the cohomology of the complex of continuous cochains.*

On the other hand, "continuous cohomology" for modules which are topological vector spaces is well studied, so we can apply:

Theorem 4.6 ("Generalized van Est" [23, Cor. III.7.5] or [29]). *If G is a connected Lie group and A is a G-module which is a complete metrizable topological vector space, then $H^\bullet_{\mathrm{cont}}(G, A)$ agrees with the relative Lie algebra cohomology $H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{k}; A_\infty)$, where $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K, and A_∞ is the set of smooth vectors in A (for the action of G).*

Corollary 4.7. *If G is a vector group with Lie algebra $\mathfrak{g}$, and if A is a G-module which is a complete metrizable topological vector space, then $H^\bullet_{\mathrm{cont}}(G, A) \cong H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, A_\infty)$. In particular, it vanishes for $\bullet > \dim G$.*

Proof. For a vector group, K is trivial. Lie algebra cohomology is computed from the complex $\mathrm{Hom}(\bigwedge^\bullet \mathfrak{g}, A_\infty)$, which vanishes for $\bullet > \dim G$.

4.3. Calculations for the case $n = 2$. We now specialize our Basic Setup 3.1 to the case where $n = 2$, i.e., $p\colon X \to Z$ is a principal $\mathbb{T}^2$-bundle, and now $G = \mathbb{R}^2$. We apply Theorem 4.4. But since $H^3_M(G, C(X, \mathbb{T})) \cong H^3_M(G, C(X, \mathbb{R}))$ (by Eq. (5)), to which we can apply Theorem 4.5 and Corollary 4.7, we obtain:

Proposition 4.8. *If $G = \mathbb{R}^2$ and X is a G-space as above, then $H^3_M(G, C(X, \mathbb{T}))$ vanishes and the forgetful map $F\colon \mathrm{Br}_G(X) \to H^3(X, \mathbb{Z})$ is surjective.*

Furthermore, we can also explicitly compute $H^2_M(G, C(X, \mathbb{T}))$, because of the following:

Lemma 4.9. *If $G = \mathbb{R}^2$ and X is a G-space as in the Basic Setup 3.1, then the maps $p^*\colon C(Z, \mathbb{R}) \to C(X, \mathbb{R})$ and "averaging along the fibers of p" $\int\colon C(X, \mathbb{R}) \to C(Z, \mathbb{R})$ (defined by $\int f(z) = \int_T f(g \cdot x)\, dg$, where dg is Haar measure on the torus T and we choose $x \in p^{-1}(z)$) induce isomorphisms*

$$H^2_M(G, C(X, \mathbb{R})) \leftrightarrows H^2_M(G, C(Z, \mathbb{R})) \cong C(Z, \mathbb{R})$$

which are inverses to one another.

Proof. We apply Theorem 4.6. Note that the G-action on $C(Z, \mathbb{R})$ is trivial, so every element of $C(Z, \mathbb{R})$ is smooth for the action of G. But since $\dim G = 2$, we have for any real vector space V with trivial G-action the isomorphisms

$$H^2_M(G, V) \cong H^2_{\mathrm{Lie}}(\mathfrak{g}, V) \cong H^2_{\mathrm{Lie}}(\mathfrak{g}, \mathbb{R}) \otimes V \cong V,$$

since $H^2_{\mathrm{Lie}}(\mathfrak{g}, \mathbb{R}) \cong (\bigwedge^2 \mathfrak{g})^* \cong \mathbb{R}$.

Clearly $\int \circ\, p^*$ is the identity on $C(Z, \mathbb{R})$, so we need to show $p^* \circ \int$ induces an isomorphism on $C(X, \mathbb{R})$. The calculation turns out to be local, so by a Mayer-Vietoris argument we can reduce to the case where p is a trivial bundle, i.e., $X = (G/N) \times Z$, with $N = \mathbb{Z}^2$ and G acting only on the first factor. The smooth vectors in $C(X, \mathbb{R})$ for the action of G can then be identified with $C(Z, C^\infty(G/N))$. So we obtain

$$H^2_M\big(G, C(X, \mathbb{R})\big) \cong H^2_{\mathrm{Lie}}\big(\mathfrak{g}, C(Z, C^\infty(G/N))\big) \cong C\Big(Z, H^2_{\mathrm{Lie}}\big(\mathfrak{g}, C^\infty(G/N)\big)\Big),$$

with the cohomology moving inside since G acts trivially on Z. However, by Poincaré duality for Lie algebra cohomology,

$$H^2_{\mathrm{Lie}}\big(\mathfrak{g}, C^\infty(G/N)\big) \cong H_0^{\mathrm{Lie}}\big(\mathfrak{g}, C^\infty(G/N)\big),$$

which is the quotient of $C^\infty(G/N)$ by all derivatives $X \cdot f$, $X \in \mathfrak{g}$ and $f \in C^\infty(G/N)$. This quotient is $\mathbb{R}$ by the de Rham theorem, since $f(g)\, d\mathrm{vol}(g)$ is exact on T exactly when f is constant. And it's easy to check that the isomorphism $H^2_M\big(G, C(X, \mathbb{R})\big) \cong C(Z, \mathbb{R})$ is induced by $\int$.

Theorem 4.10. *In Basic Setup 3.1 with $n = 2$, there is a commutative diagram of exact sequences:*

$$\begin{tikzcd}
 & H^0(Z, \mathbb{Z}) \arrow[d] & 0 \arrow[d] & \\
H^2(X, \mathbb{Z}) \arrow[r, "d_2''"] & H^2_M(G, C(X, \mathbb{T})) \arrow[r, "\xi"] \arrow[d, dotted, "a"'] & \ker F \arrow[r, "\eta"] \arrow[d] & 0 \\
 & C(Z, H^2_M(\mathbb{Z}^2, \mathbb{T})) \arrow[d, "h"'] & \mathrm{Br}_G(X) \arrow[l, "M"'] \arrow[d] & \\
 & H^1(Z, \mathbb{Z}) \arrow[d] & H^3(X, \mathbb{Z}) \arrow[l, dotted, "p_!"'] \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

Here $M\colon\ \mathrm{Br}_G(X) \to C(Z, H^2_M(\mathbb{Z}^2, \mathbb{T})) \cong C(Z, \mathbb{T})$ is the Mackey obstruction map defined in [28]*, and $h\colon\ C(Z, \mathbb{T}) \to H^1(X, \mathbb{Z})$ is the map sending a continuous function $Z \to S^1$ to its homotopy class. The definitions of the dotted arrows will be given in the course of the proof.*

Proof. Most of this is immediate from Theorem 4.4 together with Proposition 4.8. There are just a few more things to check. First we define the dotted arrows in the diagram. The arrow $p_!\colon\ H^3(X, \mathbb{Z}) \to H^1(Z, \mathbb{Z})$ is "integration over the fibers" of the bundle $T^2 \to X \xrightarrow{p} Z$; more specifically, it is the projection of $H^3(X, \mathbb{Z})$ onto $E_\infty^{1,2}$ in the Serre spectral sequence of p. Since $E_\infty^{1,2} \subseteq E_2^{1,2} = H^1(Z, H^2(T^2, \mathbb{Z}))$, we can think of the image as lying in $H^1(Z, \mathbb{Z})$. In fact,

$$E_\infty^{1,2} \subseteq E_3^{1,2} = \ker d_2\colon\ H^1(Z, H^2(T^2, \mathbb{Z})) \to H^3(Z, H^1(T^2, \mathbb{Z})) \cong H^3(Z, \mathbb{Z}^2),$$

and this map d_2 can be identified with the cup product with $[p] \in H^2(Z, \mathbb{Z}^2)$.

Next we define the downward dotted arrow a using Lemma 4.9. It is simply the following composite:

$$H^2_M(G, C(X, \mathbb{T})) \xrightarrow[\cong]{\text{eq. (5)}} H^2_M(G, C(X, \mathbb{R})) \xrightarrow[\cong]{\text{Lemma 4.9}} C(Z, \mathbb{R}) \xrightarrow{\exp} C(Z, \mathbb{T}).$$

Exactness of the middle downward sequence

$$H^0(Z, \mathbb{Z}) \to H^2_M(G, C(X, \mathbb{T})) \xrightarrow{a} C(Z, \mathbb{T}) \xrightarrow{h} H^1(Z, \mathbb{Z})$$

follows immediately from (4) with X replaced by Z.

We still need to check commutativity of the squares. As far as the upper square is concerned, the key fact is that the restriction map

$$\mathbb{R} \cong H^2_M(\mathbb{R}^2, \mathbb{T}) \to H^2_M(\mathbb{Z}^2, \mathbb{T}) \cong \mathbb{T}$$

is surjective and can be identified with the exponential map (see the Hochschild-Serre spectral sequence

$$H^p_M(\mathbb{R}^2/\mathbb{Z}^2, H^q_M(\mathbb{Z}^2, \mathbb{T})) \Rightarrow H^\bullet_M(\mathbb{R}^2, \mathbb{T})$$

of [25] for a method of calculation). To check commutativity for the upper square, choose a Borel cocycle $\omega \in Z^2_M(G, C(X, \mathbb{T}))$ representing a class in $H^2_M(G, C(X, \mathbb{T}))$. By Lemma 4.9, we may assume ω takes its values in functions constant on T-orbits, i.e., pulled back from $C(Z, \mathbb{T})$ via p^*. As in [14, Theorem 5.1(3)], choose a Borel map $u \to U\mathcal{M}(C_0(X, \mathcal{K}))$ satisfying

$$u_s \tau_s(u_t) = \omega(s, t) u_{s+t}, \quad s, t \in G.$$

(Here τ is the action of G on X.) Then by the prescription in [28], $\xi([\omega])$ is given by $C_0(X, \mathcal{K})$ with the G-action $s \mapsto (\operatorname{Ad} u_s)\tau_s$. We need to compute the Mackey obstruction for the restriction of the action to $N = \mathbb{Z}^2$. But this is just given by $z \mapsto M(u_z)$, the Mackey obstruction of the projective unitary representation of N defined by u over a point $z \in Z$. But as the cocycle of the representation is just ω restricted to z (this makes sense since we took ω to have values constant on G-orbits), we can use the above fact about restricting the Moore cohomology from G to N to deduce that $M(\xi([\omega])) = a([\omega])$.

Finally we need to check commutativity of the bottom square. This amounts to showing that if we have an action α of G on $CT(X, \delta)$ representing an element of $\mathrm{Br}_G(X)$, then $h \circ M(\alpha) = p_!(\delta)$. (In the case where $M(\alpha)$ is trivial, this is basically in [28].) First of all, we note that $h \circ M(\alpha)$ can only depend on δ, not on the choice of the action α on $CT(X, \delta)$. The reason is that any two different actions differ by an element of $\ker F$, which by the rest of the diagram is in the image of $H^2_M(G, C(X, \mathbb{T})) \cong C(Z, \mathbb{R})$. By commutativity of the upper square, this only changes $M(\alpha)$ within its homotopy class. Since we already know $\mathrm{Br}_G(X) \to H^3(X, \mathbb{Z})$ is surjective, it follows that $h \circ M$ induces a homomorphism from $H^3(X, \mathbb{Z}) \to H^1(Z, \mathbb{Z})$. This map is trivial on $p^*(H^3(Z, \mathbb{Z}))$, since this part of $H^3(X, \mathbb{Z})$ is represented by G-actions where $N = \mathbb{Z}^2$ acts trivially [14, §6.2]. And of course when N acts trivially, there is no Mackey obstruction.

Next we show that the map $H^3(X, \mathbb{Z}) \to H^1(Z, \mathbb{Z})$ induced by $h \circ M$ vanishes on the $E^{2,1}_\infty$ subquotient of the spectral sequence. This consists (modulo classes pulled back from $H^3(Z, \mathbb{Z})$) of classes pulled back from some intermediate space Y, where $X \xrightarrow{p_1} Y \xrightarrow{p_2} Z$ is some factorization of the T^2-bundle $p\colon X \to Z$ as a composite of two principal S^1-bundles. But given such a factorization and a class $\delta_Y \in Y$, there is an essentially unique action of $\mathbb{R}$ on $CT(Y, \delta_Y)$ compatible with the S^1-action on Y with quotient Z, because of the results of Sect. 4.1. Pulling back from Y to X, we get an action of $\mathbb{R} \times \mathbb{T}$ on $CT(X, p_1^*\delta_Y)$, or in other words an action of G factoring through $\mathbb{R} \times \mathbb{T}$. Such an action necessarily has trivial Mackey obstruction.

So it follows that the map induced by $h \circ M$ factors through the remaining subquotient of $H^3(Z, \mathbb{Z})$, i.e., $E_\infty^{1,2}$. That says exactly that the map factors through $p_!$. By naturality, it must be a multiple of $p_!$, and we just need to compute in the case of a trivial bundle to verify that the multiple is 1. Thus the proof is completed with the following Proposition 4.11.

Proposition 4.11. *Let $p\colon\ X = Z \times \mathbb{T}^2 \to Z$ be a trivial $\mathbb{T}^2$-bundle, let $\beta \in H^1(Z, \mathbb{Z})$, and let $\delta = \beta \times \gamma \in H^3(X, \mathbb{Z})$, where γ is the usual generator of $H^2(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}$. Then there is an action α of $G = \mathbb{R}^2$ on $CT(X, \delta)$, compatible with the free $\mathbb{T}^2$-action on X, for which $h \circ M(\alpha) = \beta$.*

Proof. Choose a function $f\colon\ Z \to \mathbb{T}$ with $h(f) = \beta$. Let $\mathcal{H} = L^2(\mathbb{T})$ and for $z \in Z$, consider the projective unitary representation $\rho_{f(z)}\colon\ \mathbb{Z}^2 \to PU(\mathcal{H})$ defined by sending the first generator of $\mathbb{Z}^2$ to multiplication by the identity map $\mathbb{T} \to \mathbb{T} \hookrightarrow \mathbb{C}$, and the second generator to translation by $f(z) \in \mathbb{T}$. Then the Mackey obstruction of $\rho_{f(z)}$ is $f(z) \in \mathbb{T} \cong H^2(\mathbb{Z}^2, \mathbb{T})$. We can view ρ as a spectrum-fixing automorphism of $\mathbb{Z}^2$ on $C(Z, \mathcal{K}(\mathcal{H}))$, which is given at the point $z \in Z$ by $\operatorname{Ad}\rho_{f(z)}$. We now let (A, α) be the C^*-dynamical system obtained by inducing up $\big(C(Z, \mathcal{K}(\mathcal{H})), \rho\big)$ from $\mathbb{Z}^2$ to $\mathbb{R}^2$. More precisely,

$$\begin{aligned} A &= \operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} (C(Z, \mathcal{K}(\mathcal{H})), \rho) \\ &= \left\{ f\colon\ \mathbb{R}^2 \to C(Z, \mathcal{K}(\mathcal{H})) : f(t+g) = \rho(g)(f(t)),\quad t \in \mathbb{R}^2, g \in \mathbb{Z}^2 \right\}. \end{aligned}$$

Since ρ acts trivially on the spectrum Z of the inducing algebra and A is an algebra of sections of a locally trivial bundle of C^*-algebras with fibers isomorphic to $\mathcal{K}$, A is a continuous-trace algebra having spectrum $Z \times \mathbb{T}^2$. There is a natural action α of $\mathbb{R}^2$ on A by translation, and by construction, $M(\alpha) = f$. We just need to compute the Dixmier-Douady invariant of A. We get it by "inducing in stages". Let $B = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} C(Z, \mathcal{K}(\mathcal{H}))$ be the result of inducing over the first copy of $\mathbb{R}$. Since the first generator of $\mathbb{Z}^2$ was always acting by conjugation by multiplication by the identity map $\mathbb{T} \to \mathbb{T}$ on $L^2(\mathbb{T})$, one can see that B is a trivial continuous-trace algebra, viz., $B \cong C_0(Z \times \mathbb{T}, \mathcal{K}(\mathcal{H}))$. We still have another action of $\mathbb{Z}$ on B coming from the second generator of $\mathbb{Z}^2$, and $A = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} B$, where we induce over the second copy of $\mathbb{R}$ to get A. The action of $\mathbb{Z}$ acts on B is by means of a map $\sigma\colon\ Z \times \mathbb{T} \to PU(\mathcal{H}) = \operatorname{Aut}\mathcal{K}(\mathcal{H})$, whose value at (z, t) is the product of multiplication by t with translation by $f(z)$. Thus the Dixmier-Douady invariant of A is then $[\sigma] \times c$, where $[\sigma] \in H^2(Z \times \mathbb{T}, \mathbb{Z})$ is the homotopy class of $\sigma\colon\ Z \times \mathbb{T} \to PU(\mathcal{H}) = K(\mathbb{Z}, 2)$ and c is the usual generator of $H^1(S^1, \mathbb{Z})$. But $[\sigma]$ is now $h(f) \times c$, so the Dixmier-Douady class of A is $\beta \times c \times c = \beta \times \gamma$.

4.4. Applications to T-duality. Now we are ready to apply Theorem 4.10 to T-duality in type II string theory. First we need a definition.

Definition 4.12. *Let $p\colon\ X \to Z$ be a principal T-bundle as in the Basic Setup 3.1, and let $\delta \in H^3(X, \mathbb{Z})$. We will say that the pair (p, δ) has a* **classical T-dual** *if there is an* *element* $[A, \alpha]$ *of* $\mathrm{Br}_G(X)$*, with* A *a continuous-trace algebra over* X *with Dixmier-Douady class* δ*, and with* α *an action of* G *on* A *inducing the given free action of* $T = G/N$ *on* X*, such that the crossed product* $A \rtimes G$ *is again a continuous-trace algebra over some other principal torus bundle over* Z*, with the dual action of* $\widehat{G}$ *inducing the bundle projection to* Z*.*

This definition is essentially equivalent to that in [7]; we will say more about this later in Remark 4.15.

The following is the main result of this paper.

Theorem 4.13. *Let* $p\colon\ X \to Z$ *be a principal* $\mathbb{T}^2$*-bundle as in the Basic Setup* 3.1. *Let* $\delta \in H^3(X, \mathbb{Z})$ *be an "H-flux" on* X*. Then:*

1. *If* $p_!\delta = 0 \in H^1(Z, \mathbb{Z})$*, then there is a* (*uniquely determined*) *classical T-dual to* (p, δ)*, consisting of* $p^\#\colon\ X^\# \to Z$*, which is a another principal* $\mathbb{T}^2$*-bundle over* Z*, and* $\delta^\# \in H^3(X^\#, \mathbb{Z})$*, the "T-dual H-flux" on* $X^\#$*. One obtains a picture exactly like Eq.* (2).
2. *If* $p_!\delta \neq 0 \in H^1(Z, \mathbb{Z})$*, then a classical T-dual as above does* not *exist. However, there is a "nonclassical" T-dual bundle of noncommutative tori over* Z*. It is not unique, but the non-uniqueness does not affect its* K*-theory.*

Proof. By Theorem 4.10, the map $F\colon\ \mathrm{Br}_G(X) \to H^3(X, \mathbb{Z})$ is always surjective. This will be the key to the proof.

First consider the case when $p_!\delta = 0 \in H^1(Z, \mathbb{Z})$. This case is considered in [7], but we will redo the results using Theorem 4.10. By commutativity of the lower square, we can lift $\delta \in H^3(X, \mathbb{Z})$ to an element $[CT(X, \delta), \alpha]$ of $\mathrm{Br}_G(X)$ with $M(\alpha)$ homotopically trivial. Then by using commutativity of the upper square in Theorem 4.10, we can perturb α, without changing δ, so that $M(\alpha)$ actually vanishes. Once this is done, the element we get in $\mathrm{Br}_G(X)$ is actually unique. On the one hand, this can be seen from [28, Lemma 1.3] and [28, Cor. 5.18]. Alternatively, it can be read off from Theorem 4.10, since any two classes in $\ker M$ mapping to the same $\delta \in H^3(X, \mathbb{Z})$ differ by the image under ξ of something in $\ker a$. Thus they differ by the image under ξ of an $\mathbb{Z}$-valued cocycle, which is trivial since such a cocycle exponentiates to the trivial cocycle with values in $\mathbb{T}$, and this is all that is used in the construction of ξ in [14]. Finally, if $[CT(X, \delta), \alpha]$ has trivial Mackey obstruction, then as explained in [28, §1], $CT(X, \delta) \rtimes_\alpha G$ has continuous trace and has spectrum which is another principal torus bundle over Z (for the dual torus, $\widehat{G}$ divided by the dual lattice).

Now consider the case when

$$p_!\delta \neq 0 \in H^1(Z, \mathbb{Z}). \tag{6}$$

It is still true as before that we can find an element $[CT(X, \delta), \alpha]$ in $\mathrm{Br}_G(X)$ corresponding to δ. But there is no classical T-dual in this situation since the Mackey obstruction *can't* be trivial, because of Theorem 4.10. In fact, since any representative $f\colon\ Z \to \mathbb{T}$ of a non-zero class in $H^1(Z, \mathbb{Z})$ must take on all values in $\mathbb{T}$, there are necessarily points $z \in Z$ for which the Mackey obstruction in $H^2(\mathbb{Z}^2, \mathbb{T}) \cong \mathbb{T}$ is irrational, and hence the crossed product $CT(X, \delta) \rtimes_\alpha G$ cannot be type I. Nevertheless, we can view this crossed product as a *non-classical* T-dual to (p, δ). The crossed product can be viewed as the algebra of sections of a bundle of algebras (not locally trivial) over Z, in the sense of [15]. The fiber of this bundle over $z \in Z$ will be $C(p^{-1}(z), \mathcal{K}(\mathcal{H})) \rtimes G \cong C(G/\mathbb{Z}^2, \mathcal{K}(\mathcal{H})) \rtimes G \cong A_{f(z)} \otimes \mathcal{K}(\mathcal{H})$, which is Morita equivalent to the twisted group C^*-algebra $A_{f(z)}$ of the stabilizer group $\mathbb{Z}^2$ for the

Mackey obstruction class $f(z)$ at that point. In other words, the T-dual will be realized by a bundle of (stabilized) *noncommutative tori* fibered over Z. (See Fig. 1.)

The bundle is not unique since there is no *canonical* representative f for a given non-zero class in $H^1(X, \mathbb{Z})$. However, any two choices are homotopic, and the resulting bundles will be in some sense homotopic to one another.

As expected, our notion of T-duality comes with isomorphisms in twisted K-theory and (periodic cyclic) homology:

Theorem 4.14. *In the situation of Theorem* 4.13*, if X is a manifold, H is an integral 3-form representing δ* (*in de Rham cohomology*)*, and we choose a smooth model for $CT(X, \delta)$* (*by taking a smooth bundle over X with fibers the smoothing operators*)*, we have a commutative diagram*

$$\begin{array}{ccc} K^\bullet(X, H) & \xrightarrow[\cong]{T_!} & K_\bullet(CT(X, \delta) \rtimes \mathbb{R}^2) \\ {\scriptstyle \mathrm{Ch}_H}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ch}} \\ H^\bullet(X, H) & \xrightarrow[\cong]{T_*} & HP_\bullet(CT(X, \delta)^\infty \rtimes \mathbb{R}^2) \end{array} \tag{7}$$

where the horizontal arrows are isomorphisms, Ch_H *is the twisted Chern character and* Ch *is the Connes-Chern character* [12].

When $p_!\delta = 0$ and there is a classical T-dual, this reduces to a diagram like Eq. (3)*, except that there is no degree shift since the tori are even-dimensional.*

Proof. This is done almost exactly as in [5], so we will be brief. We have the isomorphisms in K-theory

$$\begin{aligned} K^\bullet(X, H) &\cong K_\bullet(CT(X, \delta)) \\ &\cong K_\bullet(CT(X, \delta) \rtimes \mathbb{R}^2) \qquad \text{(Connes-Thom isomorphism [11]).} \end{aligned}$$

We can also consider the smooth subalgebra $CT(X, \delta)^\infty \rtimes G$. The fiber at $z \in Z$ is given by $C^\infty(p^{-1}(z), \mathcal{K}^\infty(\mathcal{H})) \rtimes G \cong C^\infty(G/\mathbb{Z}^2, \mathcal{K}^\infty(\mathcal{H})) \rtimes G \cong A^\infty_{f(z)} \otimes \mathcal{K}^\infty(\mathcal{H})$,

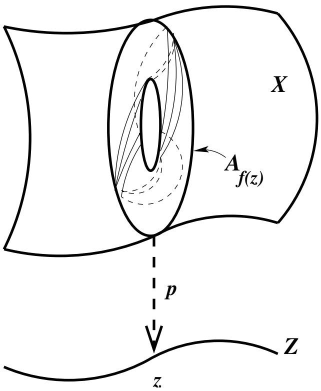

Fig. 1. In the diagram, the fiber over $z \in Z$ is the noncommutative torus $A_{f(z)}$, which is represented by a foliated torus, with foliation angle equal to $f(z)$

where $\mathcal{K}^\infty(\mathcal{H})$ is the algebra of smoothing operators on $\mathcal{H}$ and $A^\infty_{f(z)}$ is the smooth noncommutative torus with multiplier equal to $f(z)$.

Then we have the isomorphisms

$$\begin{aligned} H^\bullet(X, H) &\cong HP_\bullet(CT(X,\delta)^\infty) \\ &\cong HP_\bullet(CT(X,\delta)^\infty \rtimes \mathbb{R}^2) \qquad \text{(ENN-Thom isomorphism [19]).} \end{aligned}$$

It is well known that the Chern character is compatible with the isomorphisms in K-theory and cohomology, from which the commutativity of the diagram in (7) follows.

Remark 4.15. The reader might wonder what happened to the dual H-flux $H^\#$ in the context of Theorem 4.13(2). It doesn't really make sense as a cohomology class or differential form since the nonclassical T-dual is not a space; rather, it is subsumed in the noncommutative structure of the dual.

Now let us describe the relationship between our Definition 4.12 and Theorem 4.13 and the corresponding notions in [7]. If the pair $(p\colon\ X \to Z, \delta)$ is T-dualizable in the sense of [7], that means δ is represented by a closed 3-form H, such that $\iota_\Xi H = p^*\widehat{F}(\Xi)$, for some integral closed 2-form $\widehat{F}$ with values in the dual of $\mathfrak{g}$, the Lie algebra of T, and for all $\Xi \in \mathfrak{g}$. This essentially means that when we integrate H over the fibers of p_1, where $X \xrightarrow{p_1} Y \xrightarrow{p_1} Z$ is a factorization of p into two circle bundles, then the resulting 2-form is pulled back from Z. This implies in turn that integrating H over the fibers of p gives 0, which is the condition $p_![H] = 0$. (We do not need to worry about torsion in cohomology since $p_!\delta$ lies in $H^1(Z, \mathbb{Z})$, which is always torsion-free.) Thus the condition in our Theorem 4.13(1) is satisfied.

Conversely, suppose our condition $p_!\delta = 0$ is satisfied, so we have a classical T-dual $(p^\#\colon\ X^\# \to Z, \delta^\#)$. The condition of [7] that $\iota_\Xi H = p^*\widehat{F}(\Xi)$, for some closed integral 2-form $\widehat{F}$ with values in the dual of $\mathfrak{g}$ and for all $\Xi \in \mathfrak{g}$, will follow from the fact that since $p_!\delta = 0$ (and we can divide out by trivial cases where δ is pulled back from Z), δ comes from the $E_\infty^{2,1}$ subquotient of $H^3(X, \mathbb{Z})$.

5. Examples: Torus Bundles and Noncommutative Torus Bundles over the Circle

A famous example of a principal torus bundle with non T-dualizable H-flux is provided by $\mathbb{T}^3$, considered as the trivial $\mathbb{T}^2$-bundle over $\mathbb{T}$, with H given by k times the volume form on $\mathbb{T}^3$, $k \neq 0$. H is non T-dualizable in the classical sense since $p_![H] \neq 0$. Alternatively, there are no non-trivial $\mathbb{T}^2$-bundles over $\mathbb{T}$, since $H^1(\mathbb{T}, \underline{\mathbb{T}^2}) \cong H^2(\mathbb{T}, \mathbb{Z}^2) = 0$, that is, there is no way to dualize the H-flux by a (principal) torus bundle over $\mathbb{T}$.

This example is covered by Theorem 4.13(2) and by Theorem 4.14. The T-dual is realized by a bundle of stabilized *noncommutative tori* fibered over $\mathbb{T}$. In fact the construction of the non-classical T-dual in this case is a special case of the construction in the proof of Proposition 4.11, but we repeat the details since we can make things more explicit. Let $\mathcal{H} = L^2(\mathbb{T})$ and consider the projective unitary representation $\rho_\theta : \mathbb{Z}^2 \to PU(\mathcal{H})$ given by the first $\mathbb{Z}$ factor acting by multiplication by z^k (where $\mathbb{T}$ is thought of as the unit circle in $\mathbb{C}$) and the second $\mathbb{Z}$ factor acting by translation by $\theta \in \mathbb{T}$. Then the Mackey obstruction of ρ_θ is $\theta^k \in \mathbb{T} \cong H^2(\mathbb{Z}^2, \mathbb{T})$. Let $\mathbb{Z}^2$ act on $C(\mathbb{T}, \mathcal{K}(\mathcal{H}))$ by α, which is given at the point θ by ρ_θ. Define the C^*-algebra

$$\begin{aligned} B &= \operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} (C(\mathbb{T}, \mathcal{K}(\mathcal{H})), \alpha) \\ &= \left\{ f : \mathbb{R}^2 \to C(\mathbb{T}, \mathcal{K}(\mathcal{H})) : f(t+g) = \alpha(g)(f(t)),\ \ t \in \mathbb{R}^2, g \in \mathbb{Z}^2 \right\}. \end{aligned}$$

That is, B (with an implied action of $\mathbb{R}^2$) is the result of inducing a $\mathbb{Z}^2$-action on $C(\mathbb{T}, \mathcal{K}(\mathcal{H}))$ from $\mathbb{Z}^2$ up to $\mathbb{R}^2$. Then B is a continuous-trace C^*-algebra having spectrum $\mathbb{T}^3$, having an action of $\mathbb{R}^2$ whose induced action on the spectrum of B is the trivial bundle $\mathbb{T}^3 \to \mathbb{T}$. The crossed product algebra $B \rtimes \mathbb{R}^2 \cong C(\mathbb{T}, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^2$ has fiber over $\theta \in \mathbb{T}$ given by $\mathcal{K}(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta \otimes \mathcal{K}(\mathcal{H})$, where A_θ is the noncommutative 2-torus. In fact, the crossed product $B \rtimes \mathbb{R}^2$ is Morita equivalent to $C(\mathbb{T}, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^2$ and is even isomorphic to the stabilization of this algebra (by [22]). Thus $B \rtimes \mathbb{R}^2$ is isomorphic to $C^*(H_\mathbb{Z}) \otimes \mathcal{K}$, where $H_\mathbb{Z}$ is the integer Heisenberg-type group,

$$H_\mathbb{Z} = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

a lattice in the usual Heisenberg group $H_\mathbb{R}$ (consisting of matrices of the same form, but with $x,\ y,\ z \in \mathbb{R}$). Then we have the isomorphisms in K-theory

$$\begin{aligned}
K_\bullet(B) &= K^\bullet(\mathbb{T}^3, k\,d\mathrm{vol}) && \text{(definition)} \\
&\cong K_\bullet(B \rtimes \mathbb{R}^2) && \text{(Connes-Thom isomorphism)} \\
&\cong K_\bullet(C^*(H_\mathbb{Z})) && \text{(above identification)} \\
&\cong K_\bullet(H_\mathbb{R}/H_\mathbb{Z}) && \text{(special case of the Baum-Connes conjecture}^2\text{)} \\
&\cong K^{\bullet+1}(H_\mathbb{R}/H_\mathbb{Z}) && \text{(Poincaré duality for } H_\mathbb{R}/H_\mathbb{Z}\text{)}.
\end{aligned}$$

where we observe that the Heisenberg nilmanifold $H_\mathbb{R}/H_\mathbb{Z}$ (which happens to be the classifying space $BH_\mathbb{Z}$) is a circle bundle over $\mathbb{T}^2$ with first Chern class equal to $kdx \wedge dy$.

Notice that as far as K-theory is concerned, the T-dual of $(T^3, k\,d\mathrm{vol})$ can also be taken to be the nilmanifold $H_\mathbb{R}/H_\mathbb{Z}$ with the trivial H-field. This is a *non-principal* T^2-bundle over S^1. But a better model for a non-classical T-dual is simply the group C^*-algebra of $H_\mathbb{Z}$.

We can also consider the smooth subalgebra B^∞ of B defined by

$$\begin{aligned}
B^\infty &= \mathrm{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} (C^\infty(\mathbb{T}, \mathcal{K}^\infty(\mathcal{H})), \alpha) \\
&= \left\{ f : \mathbb{R}^2 \to C^\infty(\mathbb{T}, \mathcal{K}^\infty(\mathcal{H})) : f(t+g) = \alpha(g)(f(t)), \quad t \in \mathbb{R}^2, g \in \mathbb{Z}^2 \right\},
\end{aligned}$$

where $\mathcal{K}^\infty(\mathcal{H}))$ denotes the algebra of smoothing operators on $\mathbb{T}$. Note that $B^\infty \rtimes \mathbb{R}^2 \cong C^\infty(\mathbb{T}, \mathcal{K}^\infty(\mathcal{H})) \rtimes \mathbb{Z}^2$ has fiber over $\theta \in \mathbb{T}$ given by $\mathcal{K}^\infty(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta^\infty \otimes \mathcal{K}^\infty(\mathcal{H})$, where A_θ^∞ is the smooth noncommutative torus and the tensor product is the projective tensor product. In this case, the crossed product $B^\infty \rtimes \mathbb{R}^2 \cong \mathcal{S}(H_\mathbb{Z}) \otimes \mathcal{K}^\infty(\mathcal{H})$, where

[2] This is not as complicated as it sounds. The Baum-Connes conjecture (for torsion-free groups) says that the "index map" or "assembly map" $K_\bullet(B\Gamma) \to K_\bullet(C_r^*(\Gamma))$ should be an isomorphism for an arbitrary discrete torsion-free group Γ [4]. Here $B\Gamma$ is the classifying space of Γ, which if Γ is a torsion-free cocompact discrete subgroup of a connected Lie group G can be taken to be $K\backslash G/\Gamma$, K a maximal compact subgroup of G, and $C_r^*(\Gamma)$ denotes the reduced group C^*-algebra, i.e., the C^*-algebra generated by the left regular representation of Γ on $\ell^2(\Gamma)$. If Γ is amenable, this coincides with the full group C^*-algebra, or in other words the universal C^*-algebra whose $*$-representations correspond to unitary representations of Γ. When Γ, like $H_\mathbb{Z}$, is a poly-$\mathbb{Z}$ group, i.e., has a composition series with infinite cyclic composition factors, then this is easy to prove by induction on the length of the composition series, using the Pimsner-Voiculescu exact sequence [30] for the K-theory of a crossed product by an action of $\mathbb{Z}$. Finally, the Pimsner-Voiculescu sequence can be deduced from Connes' Thom isomorphism theorem (see [11]) by inducing the action of $\mathbb{Z}$ to an action of $\mathbb{R}$.

$\mathcal{S}(H_{\mathbb{Z}})$ is the rapid decrease algebra. Then we have the isomorphisms

$$\begin{aligned} HP_\bullet(B^\infty) &= H^\bullet(\mathbb{T}^3, k\,d\mathrm{vol}) && \text{(definition)} \\ &\cong HP_\bullet(B^\infty \rtimes \mathbb{R}^2) && \text{(ENN-Thom isomorphism)} \\ &\cong HP_\bullet(\mathcal{S}(H_{\mathbb{Z}})) && \text{(above identification)} \\ &\cong H_\bullet(H_{\mathbb{R}}/H_{\mathbb{Z}}) && \text{(Cyclic homology Baum-Connes conjecture)} \\ &\cong H^{\bullet+1}(H_{\mathbb{R}}/H_{\mathbb{Z}}) && \text{(Poincaré duality for } H_{\mathbb{R}}/H_{\mathbb{Z}}) \end{aligned}$$

where $HP_\bullet$ denotes periodic cyclic homology, which is stable under the (projective) tensor product with $\mathcal{K}^\infty(\mathcal{H})$ and $H_\bullet$, $H^\bullet$ denote the $\mathbb{Z}_2$-graded homology and cohomology respectively.

Finally, T-duality can be expressed in this case by the following commutative diagram,

$$\begin{tikzcd} K^\bullet(\mathbb{T}^3, k\,d\mathrm{vol}) \arrow[r, "T_!"] \arrow[d, "\mathrm{Ch}_H"'] & K_\bullet(C^*(H_{\mathbb{Z}})) \arrow[d, "\mathrm{Ch}"] \\ H^\bullet(\mathbb{T}^3, k\,d\mathrm{vol}) \arrow[r, "T_*"] & HP_\bullet(\mathcal{S}(H_{\mathbb{Z}})) \end{tikzcd} \tag{1}$$

where $H = k\,d\mathrm{vol}$, Ch_H is the twisted Chern character and Ch is the Connes-Chern character [12].

6. Concluding Remarks

In this paper, we have only dealt with complex C^*-algebras and complex K-theory, which are relevant for type II string theory. In principle, most of what we have done should also extend to the type I case, which involves real K-theory. However, one has to be careful. Since T-duality is related to the Fourier transform, and since the Fourier transform of a real function is not necessarily real, a theory of T-duality in type I string theory necessarily involves KR-theory, or Real K-theory in the sense of Atiyah [3]. The correct notion of twisted KR-theory is that of K-theory of real continuous-trace algebras in the sense of [35, §3]. What complicates things is that such algebras are built out of continuous-trace algebras of real, quaternionic, and complex type (locally isomorphic to $C(X, \mathcal{K}_{\mathbb{R}})$, $C(X, \mathcal{K}_{\mathbb{H}})$, and $C(X, \mathcal{K}_{\mathbb{C}})$, respectively). Even if one's original interest is in algebras of real type, passage to the T-dual will often involve algebras of the other types.

One possibility suggested by the example in Sect. 5 is that there is a good theory of T-duality for arbitrary torus bundles with H-fluxes, that doesn't require going to a category of noncommutative bundles, but that it is necessary to include the possibility of non-principal bundles. We have seen that there is a sense in which the Heisenberg nilmanifold (with trivial H-field) can be viewed as a T-dual to T^3 with a non-trivial H-field. (This is literally true in the sense of [5] if we think of both manifolds as $\mathbb{T}$-bundles over T^2, rather than as T^2-bundles over S^1.)

It is of course a little disappointing that our main theorem only applies when the fibers of the torus bundle are 2-dimensional. From Theorem 4.4, it is not even clear if the map $\mathrm{Br}_G(X) \to H^3(X, \mathbb{Z})$ is surjective when $n = \dim G > 2$. However, the methods of this paper should apply on the image of this map.

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