Additional Examples and Exercises for Math 461

Contents.

Instructions for MATLAB work

Present answers in vector form $(w, x, y, z) = \cdots$.

Check answers (when possible).

Use vpa(x,3) to rewrite (messy) fractions and other expressions in decimal form.

For this class, usually at most two decimal figures are enough, often one is enough.

For non-Matlab work, usually fractions are preferred.

Additional Examples for Ch. 1 Sec. 4.

Example I.4.9. *Given geometric vectors* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w} \in R^4$ *with* $\mathbf{u}$ *being perpendicular to both* $\mathbf{v}$ *and* $\mathbf{w}$*, prove that* $\mathbf{u}$ *is also perpendicular to the geometric vector* $(3\mathbf{v} + 4\mathbf{w})$*.*

What is special about the numbers 3 *and* 4*?*

State a general rule that descibes what has occured.

Translate into equations: Since $\mathbf{u} \perp \mathbf{v}$ is equivalent to $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$, $\forall_{geometric\ vectors,\ \mathbf{u}\ \text{and}\ \mathbf{v}}$, this example translates into:

Given: $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \cdot \mathbf{w} = 0$.

To show: $\mathbf{u} \cdot (3\mathbf{v} + 4\mathbf{w}) = 0$.

Proof. (by calculation) We use the distributive rule for dot products:

$$\mathbf{u} \cdot (3\mathbf{v} + 4\mathbf{w}) = 3\mathbf{u} \cdot \mathbf{v} + 4\mathbf{u} \cdot \mathbf{w} = 3 \times 0 + 4 \times 0 = 0.$$

Translating back from this equation: $\boxed{\mathbf{u} \perp (3\mathbf{v} + 4\mathbf{w})}$. $\surd$

Remark. There is nothing special about the numbers 3 and 4, this proof by calculation will work for any numbers. Replacing the numbers 3 and 4 by the letters A and B, in this calculation yields:

$$\mathbf{u} \cdot (A\mathbf{v} + B\mathbf{w}) = A\mathbf{u} \cdot \mathbf{v} + B\mathbf{u} \cdot \mathbf{w} = A \times 0 + B \times 0 = 0.$$

This proves:

General rule: If $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$, then $\mathbf{u} \perp (A\mathbf{v} + B\mathbf{w})$ $\forall_{A\ \text{and}\ B \in R}$.

Or verbally:

Proposition I.4.10. *If a vector is perpendicular to each of two other vectors, then it is also perpendicular to every linear combination of those two vectors.*

Of course, points cannot be perpendicular, so another term is used to describe points, whose dot product is zero:

Definition. Two coordinate vectors (or points), are *orthogonal*, if their dot product is zero.

Proposition I.4.11. *If a point is orthogonal to each of two other points, then it is also orthogonal to every linear combination of those two points.*

Translate proposition into equations: Let $\mathbf{q}$ and $\mathbf{r}$ be the two "other" points, and let $\mathbf{p}$ be the point orthogonal to them. The linear combinations of $\mathbf{q}$ and $\mathbf{r}$ are the points described by the formula: $A\mathbf{q} + B\mathbf{r}$, $\forall_{A\ \text{and}\ B \in R}$.

Given: Point $\mathbf{p}$ orthogonal to $\mathbf{q}$ and $\mathbf{r}$ translates into $\mathbf{p}\cdot\mathbf{q}=0$ and $\mathbf{p}\cdot\mathbf{r}=0$.

To prove: Point $\mathbf{p}$ orthogonal to the linear combination $A\mathbf{q}+B\mathbf{r}$, translates into $\mathbf{p}\cdot(A\mathbf{q}+B\mathbf{r})=0$.

Proof. (by calculation) We use the distributive rule for dot products of coordinate vectors:

$$\mathbf{p}\cdot(A\mathbf{q}+B\mathbf{r}) = A\mathbf{p}\cdot\mathbf{q}+B\mathbf{p}\cdot\mathbf{r} = A\times 0 + B\times 0 = 0,\ \forall_A \text{ and } _{B\in R}. \qquad \surd$$

Remark. This one line calculation proof for the last proposition is identical to the one for its predessor, just change the labels $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ to $\mathbf{p},\mathbf{q}$ and $\mathbf{r}$. This is an example of the calculations for geometric vectors being analogous to the calculations for coordinate vectors.

Definition. Geometric vectors, with length one, are called *unit vectors*; they are said to have *unit length.* Coordinate vectors with norm (or magnitude) one are called *unit vectors*; they are said to have *unit norm.*

Notation. The symbol for the length of a geometric vector, $\mathbf{v}$ is $||\mathbf{v}||$. The symbol for the length of a coordinate vector, $\mathbf{p}$ is $||\mathbf{p}||$.

Formulas : $||a\mathbf{i}+b\mathbf{j}+c\mathbf{k}|| = \sqrt{a^2+b^2+c^2} = ||(a,b,c)||,\ \forall_{a,b}$ and $_{c\in R}$

Observation I.4.12.

$$||c\mathbf{v}|| = c||\mathbf{v}||,\ \forall_{c\in R}\ \textit{and}\ \forall_{geometric\ vector,\mathbf{v}}\ \textit{and}\ ||c\mathbf{p}|| = c||\mathbf{p}||,\ \forall_{c\in R}\ \textit{and}\ \forall_{point\ \mathbf{p}\in\mathbf{R}^\mathbf{n}}$$

Example I.4.13. *Find a geometric vector* $\mathbf{u}\in R^3$, *such that* $\mathbf{u}$ *is parallel to* $\mathbf{v} = 2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ *and* $\mathbf{u}$ *has length one.*

Calculations. The vector $\mathbf{u}$ being parallel to $\mathbf{v} = 2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$, translates into $\mathbf{u} = A(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$, for an unknown, to-be-found constant, A.

Length of $\mathbf{u}$ being one implies:

$$1 = ||\mathbf{u}|| = ||A(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})|| = \sqrt{29A^2} \Longrightarrow A = \pm\frac{1}{\sqrt{29.}}$$

Thus: $\boxed{\mathbf{u} = \pm\frac{1}{\sqrt{29}}A(2\mathbf{i}+3\mathbf{j}+4\mathbf{k}).}$

Alternate Calculation. The length of $\mathbf{v} = 2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ is $\sqrt{2^2+3^2+4^2} = \sqrt{29}$.

Since $||A\mathbf{v}|| = A||\mathbf{v}||,\ \forall_{A\in R}$ and $\forall_{geometric\ vector,\mathbf{v}}$, set $\mathbf{u} = \frac{1}{\sqrt{29}}(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$. Clearly $\mathbf{u}$ is parallel to $\mathbf{v}$, and the length of $\mathbf{u}$ is $\frac{1}{\sqrt{29}}\{\text{length of } \mathbf{v}\} = 1$.

Remark. Another correct answer is $\mathbf{u} = -\frac{1}{\sqrt{29}}(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$.

Remark. Either method of this example may be used to calculate a unit vector which is parallel to a given vector.

Remark. We redo this example for coordinate vectors, using the same numbers:

Example I.4.14. *Find two points, which are multiples of the point,* $(1, 2, 3)$ *and have norm one.*

Calculations. A coordinate vector $\mathbf{u}$ being a multiple of the point, $(2, 3, 4)$, translates into $\mathbf{u} = A(2, 3, 4)$, for an unknown, to-be-found constant, A.

$$1 = ||\mathbf{u}|| = ||(2A, 3A, 4A)|| = \sqrt{29A^2} \Longrightarrow A = \pm\frac{1}{\sqrt{29.}}$$

Thus: $\boxed{\mathbf{u} = \pm\frac{1}{\sqrt{29}}(2, 3, 4).}$

Example I.4.15. *Find all vectors* $\mathbf{v} \in R^4$*, such that* $\mathbf{v}\cdot(0, 1, 2, 3) = 0$ *and* $\mathbf{v}\cdot(4, 5, 6, 7) = 0$.

Calculations. Set $\mathbf{v} = (w, x, y, z)$, where w, x, y and z are unknown, to-be-found constants.

$$0 = \mathbf{v}\cdot(0, 1, 2, 3) = x + 2y + 3z \text{ and } 0 = \mathbf{v}\cdot(4, 5, 6, 7) = 4w + 5x + 6y + 7z.$$

These equations may be written in echelon (upper-staircase) form:

$$\boxed{\text{4w}} + 5x + 6y + 7z = 0$$

$$\boxed{\text{x}} + 2y + 3z = 0$$

Place a box or circle around the left-most variables in each equation; here they are w and x; they will be the *dependent* variables. The remaining variables y and z, will be the *<u>in</u>dependent* variables.

Warning. If any single variable is boxed *twice* [1], then the equations are NOT in echelon (upper-staircase) form. Do not back solve! This will likely lead to wrong answers.

Back-solving for the boxed variables, w and x in terms of y and z yields:

$$x = -2y - 3z \text{ and then } w = y + 2z$$

Hence

$$\boxed{\mathbf{v} = (w, x, y, z) = (y + 2z, -2y - 3z, y, z) = y(1, -2, 1, 0) + z(2, -3, 0, 1), \ \forall_{y \text{ and } z}.}$$

Set $\mathbf{v}_1 = (1, -2, 1, 0)$ and $\mathbf{v}_2 = (2, -3, 0, 1)$. Then $v = y\mathbf{v}_1 + z\mathbf{v}_2$, $\forall_{y \text{ and } z}$. That is, the solution set is all the linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Check. You should check that:

$$\mathbf{v}_1\cdot(0, 1, 2, 3) = 0 \text{ and } \mathbf{v}_2\cdot(0, 1, 2, 3) = 0$$

and

$$\mathbf{v}_1\cdot(4, 5, 6, 7) = 0 \text{ and } \mathbf{v}_2\cdot(4, 5, 6, 7) = 0$$

[1] Here "twice" means "[at least] twice".

Remark. From Example I.4.13., we learned that when a vector is perpendicular to two other vectors, then it is also perpendicular to every linear combination of those vectors. For this reason, this check is sufficient

Example I.4.16. *Find many points* p*, which are linear combinations of* $(1,0,0)$ *and* $(1,1,1)$*, such that* $p{\cdot}(1,0,0)=0$.

Calculations. The point, p being a linear combination of $(1,0,0)$ and $(1,1,1)$, translates into $p=A(1,0,0)+B(1,1,1)=(A+B,B,B)$, for unknown, to-be-found constants, A and B.

Thus
$$0=p{\cdot}(1,0,0)=(A+B,B,B){\cdot}(1,0,0)=A+B.$$

Thus $A=-B$. Hence $\boxed{p=-B(1,0,0)+B(1,1,1)=B(0,1,1),\ \forall_{B\in R}.}$

Check. $p{\cdot}(1,0,0)=B(0,1,1){\cdot}(1,0,0)=0.$ $\surd$

Remark. In this manner, one can find a linear combination of two vectors, which is orthogonal to the first vector.

Additional Exercises for Ch. 1 Sec. 4.

Rules for exercises. Place final answers inside a box.

Check answers.

Exercise I.4.11. *Given that geometric vector* $\mathbf{v} \in R^4$ *is perpendicular to geometric vectors* $\mathbf{w}_1, \mathbf{w}_2$ *and* $\mathbf{w}_3$*, prove that* $\mathbf{v}$ *is also perpendicular to the geometric vector* $(3\mathbf{w}_1 - 7\mathbf{w}_2 + 5\mathbf{w}_3)$*.*

Exercise I.4.12. *Given that geometric vector* $\mathbf{v} \in R^7$ *is perpendicular to geometric vectors* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ *and* $\mathbf{w}_4$*, prove that* $\mathbf{v}$ *is also perpendicular to every linear combination of these geometric vectors* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ *and* $\mathbf{w}_4$*.*

Exercise I.4.13. *Find a geometric vector* $\mathbf{u} \in R^3$*, such that* $\mathbf{u}$ *is parallel to* $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ *and* $\mathbf{u}$ *has length one.*

Exercise I.4.14. *Find two points, which are multiples of the point,* $(3, 4, 12)$ *and have norm one.*

Exercise I.4.15. *Find all geometric vectors* $\mathbf{v} \in R^3$*, which are perpendicular to both* $(\mathbf{j} - 2\mathbf{k})$ *and* $(2\mathbf{i} + 5\mathbf{j} - \mathbf{k})$*. Check your answers as in Example I.4.15..*

Exercise I.4.16. *Find all coordinate vectors* $\mathbf{v} \in R^4$*, such that* $\mathbf{v}\cdot(0, 1, -2, 6) = 0$ *and* $\mathbf{v}\cdot(2, 5, -1, -1) = 0$*. Check your answers as in Example I.4.15..*

Exercise I.4.17. *Find a geometric vector* $\mathbf{v}$*, which is a linear combination of* $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ *and* $(\mathbf{i} + \mathbf{j} + \mathbf{k})$*, such that* $\mathbf{v}$ *is perpendicular to* $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$*. Check your answer as in Example I.4.16..*

Exercise I.4.18. *Find a point* p*, which is a linear combination of* $(1, 2, 3)$ *and* $(1, 1, 1)$*, such that* $p\cdot(1, 2, 3) = 0$*. Check your answer as in Example I.4.16..*

Exercise I.4.19. *Find a point* p*, which is a linear combination of* $(1, 2, 3, 4)$ *and* $(1, 1, 1, 1)$*, such that* $p\cdot(1, 2, 3, 4) = 0$*. Check your answer as in Example I.4.16..*

Exercise I.4.20. *Use the Triangle Inequality for two and/or three coordinate vectors to prove the Triangle Inequality for four coordinate vectors, that is prove that*

$$||p + q + r + s|| \le ||p|| + ||q|| + ||r|| + ||s||, \ \forall_{p,q,r,s \in R^n}.$$

Exercise I.4.21. *State and prove the Triangle Inequality for five coordinate vectors.*

Additional Example and Exercise for Ch. III Sec. 1A.

Example III.1.12. *Suppose that there are a batch of objects divided into two groups or states. Suppose that each day half the objects in State 1 go to State 2 and half remain, also each day 0.7 of the objects in State 2 go to State 1 and the rest remain in State 2. Find the transition matrix and its steady-state vector.*

Since there are two states, the transition matrix will be a 2×2-matrix, $M = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$. Using the definition and description of the $p'_{ij}s$, we see that $p_{11} = .5, p_{12} = .7, p_{21} = .5$ and $p_{22} = .3$. Hence $M = \begin{pmatrix} .5 & .7 \\ .5 & .3 \end{pmatrix}$.

Let $v = \binom{x}{y}$ be a candidate for a steady-state vector of this matrix, M. Then $Mv = v$, that is

$$\begin{pmatrix} .5 & .7 \\ .5 & .3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$.5x + .7y = x \Rightarrow 7y = 5x$$

$$.5x + .3y = y \Rightarrow 5x = 7y$$

The two equations are the same, and $y = \frac{5}{7}x$. The solutions to $Mv = v$ are $(x, y) = (x, \frac{5}{7}x)$.

But a steady-state vector is a probability vector, hence its coordinates must sum to one. Here $x + y = x + \frac{5}{7}x = \frac{12}{7}x$. We make $x + y = 1$ by choosing $x = \frac{7}{12}$. Thus the steady-state vector is $v = (x, y) = (\frac{7}{12}, \frac{5}{12})$.

Exercise III.1.5. *Given* $M = \begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix}$. *Find the steady state vector of* M, *call it* p_∞.

Check that your solution is a steady state vector.

Let $p_1 = \binom{1}{0}$ *and let* $p_1 = Mp_0$, $p_2 = Mp_1$, *and* $p_3 = Mp_2$, . *Find* p_1, p_2, *and* p_3.

Then calculate $||p_\infty - p_0||$, $||p_\infty - p_1||$ *and* $||p_\infty - p_2||$.

Present your solutions in a table.

Comment on the results.

Additional Example for Ch. III Sec. 1B.

Matrix check

Now we will use this Proposition III. 1B.5 to explain the check for the answers to Exercise 1.55 of Ch. 1. Following Slogan III.1.8, we write Exercise 1.55 of Ch. 1 in matrix vector form and then label the matrices and vectors:

$$\begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The solution was

$$(u, v, w, x, y, z) = (-4, -2, -1, 0, 0, 0) + x(4, 2, 1, 1, 0, 0) + y(4, 2, 1, 0, 1, 0) + z(4, 2, 1, 0, 0, 1), \ \forall\ x, y \text{ and } z.$$

Here

$v_0 = (-4, -2, -1, 0, 0, 0)$, $v_1 = (4, 2, 1, 1, 0, 0)$, $v_2 = (4, 2, 1, 0, 1, 0)$, $v_3 = (4, 2, 1, 0, 0, 1)$ and $w_0 = (-1, -1, -1)$

$$\text{and } \ M = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix}.$$

Matrix check. Check that

$$Mv_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad Mv_1 = \mathbf{0} = Mv_2 = Mv_3.$$

Then Proposition III. 1B.5 will confirm that *all* the solutions

$$(u, v, w, x, y, z) = (-4, -2, -1, 0, 0, 0) + x(4, 2, 1, 1, 0, 0) + y(4, 2, 1, 0, 1, 0) + z(4, 2, 1, 0, 0, 1), \ \forall_{x,y \text{ and } z}$$

$$= v_0 + Sp\{v_1, v_2, v_3\} = v_0 + Av_1 + Bv_2 + Cv_3, \forall_{A,B \text{ and } C},$$

are correct.

That is, after solving a matrix-vector equation, $Mv = w_0$, and writing the solution in vector form:

$$v = v_0 + Sp\{v_1, v_2, v_3\} = v_0 + Av_1 + Bv_2 + Cv_3, \forall_{A,B \text{ and } C}$$

the check for this infinite set of solutions, is to simply just check that:

$$Mv_0 = w_0 \quad \text{and} \quad Mv_1 = \mathbf{0} = Mv_2 = Mv_3.$$

Then Proposition III. 1B.5 will confirm that *all* the solutions are correct.

Additional Exercises for Ch. III Sec. 2.

Exercise III.2.37. *The question is: Which matrices commute with a diagonal matrix?*

Given

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

(i) *Calculate* DM *and* MD. *When is* $DM = MD$? *That is, list or describe the matrices,* M, *which satisfy the equation:* $DM = MD$?

(ii) *What is special about the numbers* $1, 2$ *and* 3 *in the matrix* D? *State a general rule.*

(iii) *Calculate* D^2, D^3 *and* D^4. *Predict a general rule.*

Exercise III.2.38. *Which matrices commute with this diagonal matrix:* $D_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$?

That is, list or describe all the matrices, M, *which satisfy the equation:* $D_2M = MD_2$? *Comment!*

Exercise III.2.39. (Complexity) *You need to calculate* $w = MNv$, *when* M *and* N *are* 10×10 *matrices and* v *is a vector in* R^{10}. *The two ways to do this are:*

Method 1. First calculate $A = MN$, *and then calculate* $w = Av$.

Method 2. First calculate $u = Nv$, *and then calculate* $w = Mu$.

For each of Methods 1 and 2, calculate how many multiplications (or dot products) are required to obtain w. *Which is much faster? Which method will you be using?*

Exercise III.2.40. *Do these calculations twice. First by hand; second using MATLAB.*

Given $M = \begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix}$. *Find the steady state vector of* M, *call it* p_∞.

Check that your solution is a steady state vector.

Let $p_1 = \binom{1}{0}$ *and let* $p_1 = Mp_0$, $p_2 = Mp_1$, *and* $p_3 = Mp_2$, . *Find* p_1, p_2, *and* p_3.

Then calculate $||p_\infty - p_0||$, $||p_\infty - p_1||$, $||p_\infty - p_2||$ *and* $||p_\infty - p_3||$.

Check that the MATLAB and hand calculations produce the same or equivalent answers. This will check that your MATLAB code is correct.

Present your solutions in a table.

Comment on the results.

Exercise III.2.41. *Let* M *be a square matrix, and* v_0 *a vector. Let* $v_1 = Mv_0$, $v_2 = Mv_1$, $v_3 = Mv_2$, $v_4 = Mv_3$. *Prove that* $v_4 = M^4 v_0$.

In this manner, (using math induction), it can be shown:

Proposition. Given a square matrix, M and a sequence of vectors $p_0, p_1, \cdots$, such that $p_n = Mp_{n-1}, \forall_{n \in Z^+}$, then $p_n = M^n p_0, \forall_{n \in Z^+}$.

Use MATLAB, together with this proposition, to do the following calculations:

Exercise III.2.42. *Given* $M = \begin{pmatrix} .9 & .1 \\ .1 & .9 \end{pmatrix}$. *Find the steady state vector of* M. *Let* $p_1 = \binom{1}{0}$ *and let* $p_n = Mp_{n-1}, \forall_{n \in Z^+}$. *This time use the formula* $p_n = M^n p_0$ *to calculate* $p_2, p_3, p_4,$ *and* p_{100}.

Check that this MATLAB calculation produced the same or equivalent answers for p_2, p_3 *and* p_4 *as MATLAB did in Exercise III.2.43.. This will check that your MATLAB code for* M^n *is probably correct.*

Calculate $||p_\infty - p_{100}||$. *Comment!*

Exercise III.2.43. *Use MATLAB to do the calculations for this exercise.*

Given $M_2 = \begin{pmatrix} .01 & .99 \\ .99 & .01 \end{pmatrix}$. *Find the steady state vector of* M_2.

Check this result.

Let $p_1 = \binom{1}{0}$ *and let* $p_n = Mp_{n-1}, \forall_{n \in Z^+}$. *Calculate* p_2, p_3, p_4, p_{10} *and* p_{1000}. *Comment!*

Calculate $||p_\infty - p_{1000}||$. *Comment!*

Calculate M_2^{1000}. *Comment!*

Exercise III.2.44. *Use MATLAB to do the calculations for this exercise.*

Given $M_3 = \begin{pmatrix} .9 & .1 & .1 \\ .1 & .9 & 0 \\ 0 & 0 & .9 \end{pmatrix}$. *Check that* M_3 *is a stochactic matrix. Find the steady state vector for* M_3.

Check this result.

Let $p_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *and let* $p_n = Mp_{n-1}, \forall_{n \in Z^+}$. *Calculate* p_2, p_3, p_4, p_{10} *and* p_{100}.

Calculate $||p_\infty - p_0||$, $||p_\infty - p_1||$, $||p_\infty - p_2||$, $||p_\infty - p_3||$, $||p_\infty - p_{10}||$ *and* $||p_\infty - p_{1000}||$.

Present your solutions in a table. Comment!

Calculate M_3^{10} *and* M_3^{1000}. *Comment!*

Some Quickie questions for Ch. 3

One and two minute problems.

Ch. III Sec. 1

Can write a system of linear equations in matrix-vector form.

Can do matrix-vector multiplication.

What does multiplication by a diagonal matrix do to a vector?

What does multiplication by the vector, $\binom{1}{0}$ do to a 2×2-matrix?

What does multiplication by the vector, $\binom{0}{1}$ do to a $n \times 2$-matrix?

What does multiplication by the one vector do to a matrix?

State the 2×2 identity matrix, the projection-of xy-plane-onto x-axis matrix, the 2×2 switch matrix. What does each do?

What does the rotation matrix do?

Ch. III Sec. 1A

Can state definitions of probability vector and of stochastic matrix.

What is the connection between the census vector and the fraction vector?

Stochastic matrix times probability vector yields another probability vector.

Ch. III Sec. 1B

Write this linear combination of vectors, $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$ as a matrix-vector product?

What does it mean that matrix transformations are linear?

State the column-vector form of the matrix vector product.

Can use linearity of matrix transformations, to prove Prop. III.1B. 3,4,5

Exer. III.1.14-19

Ch. III Sec. 2

State the reason/rational for the definition of matrix mult. (One equation will suffice.)

Prove that $R_\theta \times R_\phi = R_{\theta+\phi}$. Do not mult. out

What does multiplication by a diagonal matrix do to another matrix?

State two specific 2x2 matrices which do not commute.

Exer. III.2.11, 30,

Prove the Associate Law for matrix. mult. Do not mult. out

Ch. III Sec. 3

What is $(R_\theta)^{-1}$? Why?

Know statements of Examples III.3.12, 13,14

State and prove the Product Rule for inverse matrices.

Exer. III.3.4, the Product Rule for inverse of four matrices.

Exer. III.3.7 each part.

Can prove statements of Examples III.3.12 and Exercise II.3.8.

Exer. III.3.10,

Ch. III Sec. 4B

Can prove superposition for specific DC resistance circuit like exercises on handout.

Ch. III Sec. 5

Can state the Distributive rules for matrices.

III.5.4(a) and 5

Quickie questions

Note "$**$" denotes two-minute items; the others are one -minute items

$**$ Ch. III Sec. 4A. Can do a Superposition problem

Ch. III Sec. 5. Can state Distributive Rules.

Can use Distributive Rules for 3 matrices to prove one for four matrices (Rule III.5.3).

Exer. III.5.4(a), 5

Ch. III Sec. 5B. $**$ Can find a single solution " formally"/"sybolically", for a matrix-vector differential equation. as in Exer. III.5B.7

Ch. III Sec. 6. State the Product Rule for transposes.

$**$ Assuming Product Rule for transpose of two matrices, can prove the Product Rule for transpose of three matrices.

$**$ State and prove the Rule $(A^T)^{-1} = (A^{-1})^T$.

$**$ Exer. III.6.19, 20(a),(b) (See Prop. III.6A.2)

Ch. III Sec. 6A.

State the defining equations for real orthogonal and symmetric matrices.

$**$ Example III.6A.4, Exercises III.6A.6,7,8,10,11

Additional Exercises for Ch. IV Sec. 3.

Exercise IV.3.11. *Let* $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ *and* $\mathbf{1}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. *State/find a specific coordinate vector,* $\mathbf{v}$, *such that* $|\mathbf{v}| < \mathbf{1}_2$, *but* $|A \times \mathbf{1}_2| < |A \times \mathbf{v}|$.

Exercise IV.3.12. (Non-comparibility) *State an example of two specific coordinate vectors* $\mathbf{v}$ *and* $\mathbf{w} \in R^2$, *such that* $\mathbf{v} \neq \mathbf{w}$ *and* $\mathbf{v} \not< \mathbf{w}$ *and* $\mathbf{v} \not> \mathbf{w}$,

Exercise IV.3.13. ($\leq$ **is not {** $=$ *or* $<$ **})** *State an example of two specific coordinate vectors* $\mathbf{v}$ *and* $\mathbf{w} \in R^2$, *such that* $|\mathbf{v}+\mathbf{w}| \leq |\mathbf{v}|+|\mathbf{w}|$, *but* $|\mathbf{v}+\mathbf{w}| \neq |\mathbf{v}|+|\mathbf{w}|$ *and* $|\mathbf{v}+\mathbf{w}| \not< |\mathbf{v}| + |\mathbf{w}|$.

Exercise IV.3.14. *Use the Triangle Inequality for the absolute value of the sum of two and/or three coordinate vectors to prove the Triangle Inequality for four coordinate vectors, that is prove that*

$$|p+q+r+s| \leq |p| + |q| + |r| + |s|, \ \forall_{p,q,r,s \in R^n}.$$

Exercise IV.3.15. *State and prove the Triangle Inequality for the absolute value of the sum of five coordinate vectors.*

Additional Exercises and Warning for Ch. IV Sec. 3A.

Exercise IV.3.11. *Use the Triangle Inequality for the infinity norm of the sum of two coordinate vectors, to prove the Triangle Inequality for this linear combination of coordinate vectors:*

$$||Ap + Bq||_\infty \leq |A| \; ||p||_\infty + |B| \; ||q||_\infty, \; \forall_{p,q \in R^n} \quad \textit{and} \quad \forall_{A,B \in R}.$$

Exercise IV.3.12. *Use the Triangle Inequality for the infinity norm of the sum of two coordinate vectors, to state and prove the Triangle Inequality for the infinity norm of the sum of four coordinate vectors.*

Notation. $J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and in general, the matrix, J_n is the $n \times n$-matrix, in which each entrie is one. These "J" matrices are called *One* matrices.

Exercise IV.3.13. (Reversal of the inequality) *Let* $A = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$ *and* $J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. *State/find a specific* 2×2*-matrix,* E*, such that* $|E| < J_2$*, but* $|A \times E| > |A \times J_2|$.

Check that $|A| \times |E| \leq |A| \times |J_2|$.

Remark. The next exercise extends the result of Exercise IV.3A.3.

Exercise IV.3.14. ($(M + E)$ not invertible.) *Given* $M = \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}$.

(a) *Check that* $|M^{-1} \frac{1}{10} J_2 \; \underline{\mathbf{1}}_2| < \underline{\mathbf{1}}_2$.

(b) *Find a specific matrix,* E*, such that* $|E| \leq \frac{1}{10} J_2$ *and* $(M + E)$ *is not an invertible matrix.*

(c) *Check that* $|M^{-1}| \frac{1}{10} J_2 \; \underline{\mathbf{1}}_2$ *is not* $< \underline{\mathbf{1}}_2$.

(d) *Note that these results are consistent with Lemma IV.3A.4.*

Warning. Usually the error matrix, E is *unknown*. Hence

$$M^{-1}E, \;\; M^{-1}E\underline{\mathbf{1}}_2, \;\; |M^{-1}E|, \;\; |M^{-1}E\mathbf{v}| \;\text{ and }\; ||M^{-1}E\mathbf{v}||_\infty$$

are all *unknown* and *cannot be calculated*. Calculating them will produce *wrong* answers. Perhaps concluding that a perturbed matrix is invertible, when it is not; or calculating error bounds that are considerably smaller than the actual error.

What is usually known are just *bounds* on the matrix, E, usually *bounds* on $|E|$. For example, when the entries in matrix, M, have tolerences of .01, then $|E| \leq .01J_2$. This, can be used with Rules IV.3.4, to calculate error bounds, as follows:

$$|M^{-1}E\mathbf{v}| \;\leq\; |M^{-1}| \times |E| \times |\mathbf{v}| \;\leq\; |M^{-1}| \times .01J_2 \times |\mathbf{v}|;$$

$$||M^{-1}E\mathbf{v}||_\infty \;\le\; ||\, |M^{-1}| \times |E| \times |\mathbf{v}|\, ||_\infty \;\le\; ||\, |M^{-1}| \times .01J_2 \times |\mathbf{v}|\, ||_\infty.$$

This is what is happening in Equation (IV.3B.4). (Sorry, all the extra absolute value signs are necessary in order to avoid wrong answers.) As the two exercises above demonstrate, it is *crucial* to take the absolute values first. Calculating $|M^{-1} \times J_2|$ or $|M^{-1} \times .01J_2|$ will often produce *wrong* amswers.

Additional Exercises for Ch. II Sec. 3.

Exercise II.3.26. *(a) Prove that* $L(x(t)) = t^2\ddot{x} - 2t\dot{x} + 2x$ *is a linear transformation. Use the two defining equations or the single equation of Proposition II.2.1*

(b) Find many solutions to $t^2\ddot{x} - 2t\dot{x} + 2x = 0$*; the educated guess is* $x = x(t) = t^r$*, where* r *is an unknown, to-be-found constant. Indicate which and where the theorems on linear equations are used.*

Check the "basic" solutions.

Exercise II.3.27. (Change of variables) *Find many solutions to each of these equations (short-cuts encouraged):*

(a) $\ddot{y} - 4\dot{y} - 5y = 0$*, the educated guess is* $y = y(t) = e^{rt}$.

(b) $y'' - 4y' - 5y = 0$*, the educated guess is* $y = y(x) = e^{rx}$.

(c) $X'' - 4X' - 5X(x) = 0$*, the educated guess is* $X = X(x) = e^{rx}$.

(d) Comments.

Exercise II.3.28. (Change of variables) *Find many solutions to each of these equations (short-cuts encouraged):*

Write your answer in terms of hyperbolic functions, like $\sinh u = (e^u - e^{-u})$.

Indicate which and where the theorems on linear equations are used.

(a) $\ddot{x} - x = 0$*, with single condition,* $x(0) = 0$*. The educated guess is* $x = x(t) = e^{rt}$.

(b) $y'' - 4y = 0$*, with single condition,* $y(0) = 0$*. The educated guess is* $y = y(x) = e^{rx}$.

(c) $X'' - 4X(x) = 0$*, with single condition,* $X(0) = 0$*. The educated guess is* $X = X(x) = e^{rx}$.

(d) Comments.

Questions and Answers on linear equations

(a) What is a main use of the proposition on basic linear transformations?

Answer. It makes it easy to spot and verify many linear transformations.

(b) Why is it useful to know that a transformation is a linear transformation?

Answer. It is used for quickly identifying linear equations.

(c) Why is it useful to know that an equation is a homogeneous linear equation?

Answer. Then the theorem on homogeneous linear equations is applicable.

(d) Why is the theorem on homogeneous linear equations useful?

Answer. It provides an algorithm for solving all homogeneous linear equations.

(e) State the algorithm implied by the theorem on homogeneous linear equations.

(f) Why is it useful to know that an equation is a non-homogeneous linear equation?

Answer. Then the theorem on non-homogeneous linear equations is applicable.

(g) Why is the theorem on non-homogeneous linear equations useful?

Answer. It provides a formula and an algorithm for solving all non-homogeneous linear equations.

(h) State the formula provided by the theorem on non-homogeneous linear equations.

(i) State the algorithm implied by the theorem on non-homogeneous linear equations.

Additional Exercises for Ch. 5 Sec. 1B

Exercise V.1.26. *State a set of three vectors which is linearly dependent, but no vector is a multiple of another. For an* **answer** *Review Example V.1.2.*

Exercise V.1.27. *Given*

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

A. *Find a basis for the row space of* M*.*

Set-up *Let* r_1, r_2 *and* r_3 *be the three rows of* M*. Find a basis for* $Span\{r_1, r_2$ *and* $r_3\}$*.*

B. *Find a basis for the column space of* M*.*

Set-up *Let* w_1, w_2 *and* w_3 *be the three columns of* M*. Find a basis for* $Span\{w_1, w_2, w_3\}$*.*

C. *Find a basis for the kernel of* M*.*

D. *Find a basis for the kernel of* M^T*.*