EDS: Fundamental Theorem on the Maurer-Cartan form point (3) Spring 2020

Theorem 1. *Let M be a manifold, and G a Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form ω^G. Let $\omega \in \Omega^1(M, \mathfrak{g})$.*

1. *For all $p \in M$, there is a neighborhood $U \subseteq M$ of p and a map $f : U \to G$ such that $f^*\omega = \omega^G$ if and only if*
$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$
2. *A map f as above is unique up to post-composition with L_g for $g \in G$.*
3. *If M is simply connected (and connected), then f extends over M.*

We proved (1) and (2) in class. The graph of f lies in an integral leaf in $M \times G$ for the differential ideal $\langle\theta\rangle = \langle \pi_M^*\omega - \pi_G^*\omega_G\rangle$. Here is a proof of (3). Let $p \in M$, and let $\mathcal{L}$ be a leaf of the foliation for $\langle\theta\rangle$ through (p, e), where $e \in G$ is the identity. Assume that $\mathcal{L}$ is a maximal leaf—that is, a maximal connected, integral manifold through (p, e). This can be defined as the set of all points which can be conneced to (p, e) by a piecewise smooth path, which at all points of differentiability is tangent to $\ker\theta$. It is straightforward to find coordinate charts at each point of this set. There is a technical point, that it admits a second countable atlas, the verification of which can be found in Warner's book, Theorem 1.64.

The first point is that $\pi_M|_{\mathcal{L}}$ is a surjection onto M. Let γ be a piecewise smooth path from p to $q \in M$ and let

$$T = \sup_t\{\gamma[0, t) \subset \pi_M(\mathcal{L})\}$$

The desired lift of γ has the form $(\gamma(t), \delta(t))$, where $\delta(0) = e$ and $\omega^G(\delta'(t)) = \omega(\gamma'(t))$. We can assume γ' is defined on some interval $(T - \epsilon, T)$, and it is necessarily bounded as $t \to T$. Then δ solves a linear ODE with bounded coefficients, which means that δ extends continuously to $[0, T]$, by a basic result on solutions of linear ODEs. We conclude γ lifts to $\mathcal{L}$ on its full domain, and thus that all $q \in M$ are in $\pi_M(\mathcal{L})$ because M is path connected.

The next point is to verify that for $q \in M$, there is a neighborhood U of q such that $\pi_M|_{\mathcal{L}}^{-1}(U)$ is a disjoint union of homeomorphic copies of U, in bijection with $\pi_M|_{\mathcal{L}}^{-1}(q)$. The set $\pi_M^{-1}(q) \cap \mathcal{L}$ has the form $\{(q, g_i)\}_i$ where

$\{g_i\}_i$ is a closed, discrete subset of G. This means there is a neighborhood V_0 of g_0 such that $V_i = L_{g_i g_0^{-1}}(V_0)$ are disjoint neighborhoods of the g_i in G.

Let $\widetilde{U}_0$ be a compact neighborhood of (q, g_0) in $\mathcal{L}$ mapping diffeomorphically to $U = \pi_M(\widetilde{U}_0)$. If $\widetilde{U}_0$ is the graph of $f_0 : U \to G$, then $L_{g_i g_0^{-1}} \circ f_0 = f_i$ also pulls back ω^G to $\omega|_U$. Therefore $\widetilde{U}_i = (\mathrm{Id}_M \times L_{g_i g_0^{-1}})(\widetilde{U}_0)$ is a neighborhood of (q, g_i) in $\mathcal{L}$ projecting diffeomorphically to U.

Now replace intersect $\widetilde{U}_i$ with $M \times V_i$ to obtain the desired disjoint "stack of pancakes."