Solutions to some problems

6(b). First dose is 4. During the next 24 hours, 1/4 of it disappears, leaving 4(3/4). After the second dose, there is $4 + 4(3/4)$. In the next 24 hours, 1/4 of it disappears, leaving $(3/4)(4 + 4(4/3)) = 4(3/4) + 4(3/4)^2$ remaining. After the thrid dose, there is $4 + 4(3/4) + 4(3/4)^2$. After many weeks, there is $4 + 4(3/4) + 4(3/4)^2 + 4(3/4)^3 + \cdots = 16$ (from part (a)) in the bloodstream immediately after a dose is given.

8.(a) The derivative of the cdf is the pdf, so $f(x) = F'(x) = \frac{1}{2}kx^{-1/2}$. For this to be a pdf, we need

$$1 = \int_1^4 f(x)\,dx = \int_1^4 \frac{1}{2}kx^{-1/2}\,dx = \frac{k}{2}\frac{x^{1/2}}{1/2}\bigg|_1^4 = kx^{1/2}\bigg|_1^4 = k(4^{1/2} - 1^{1/2}) = k.$$

Therefore, $k = 1$ and the pdf is $f(x) = \frac{1}{2}x^{-1/2}$. Since $F(x) = \int_1^x f(x)\,dx$, we have $F(1) = \int_1^1 f(x)\,dx = 0$. Therefore, $0 = F(1) = \sqrt{1} + C = 1 + C$. This means that $C = -1$.

(b)

$$E(X) = \int_1^4 xf(x)\,dx = \int_1^4 x\frac{1}{2}x^{-1/2}\,dx = \int_1^4 \frac{1}{2}x^{1/2}\,dx = \frac{1}{2}\frac{x^{3/2}}{3/2}\bigg|_1^4 = \frac{7}{3}.$$

10. We are told that $p_0 = .5$. Since $p_0 = e^{-\lambda}$, we have $e^{-\lambda} = .5$, so $-\lambda = \ln(.5)$ and $\lambda = -\ln(.5)$.

(a) The probability that a page has more than one error is $p_2 + p_3 + p_4 + \cdots = 1 - (p_0 + p_1)$. It is given that $p_0 = .5$. Also,

$$p_1 = \lambda e^{-\lambda} = -\ln(.5)e^{-\lambda} = -\ln(.5)(.5),$$

where we used the fact that $e^{-\lambda} = .5$ from earlier. Putting this together, we find that

$$1 - (p_0 + p_1) = 1 - p_0 - p_1 = 1 - .5 - (-\ln(.5)(.5)) = .5 + .5\ln(.5)$$

(b) The average number of errors is $E(X) = \lambda = -\ln(.5)$.